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INJECTIVE COVERS UNDER CHANGE OF RINGS

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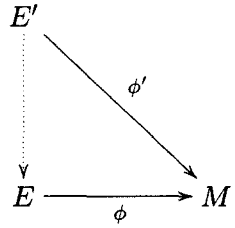
Abstract. In [8], Würful gave a characterization of those rings R which satisfy that for every ring extension $R \subset S$, $Hom_R(S, -)$ preserves injective envelopes. In this note, we consider an analogous problem concerning injective covers.

1. Introduction

Let R be a ring with identity 1 and let every module be unitary. We will use the terminology of Enochs [2].

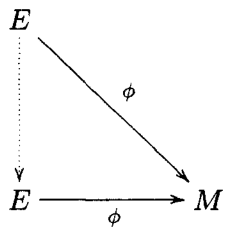
An *injective cover* of an R-module M is a linear map $\phi : E \to M$ with an injective R-module E such that

(1) for any injective R-module E' and any linear map $\phi' : E' \to M$, the diagram



can be completed to a commutative diagram.

(2) the diagram



can only be completed by automorphism of E.

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Hence if an injective cover exists, it is unique up to isomorphism. If $\phi : E \to M$ satisfies (1), and perhaps not (2), it is called an *injective precover.* We will sometimes simply say E is an injective cover (or precover).

The existence of an injective cover is not guaranteed for all cases but every R-module has an injective cover if and only if the ring R is Noetherian (see [2, Theorem 2.1]). However, examples of injective covers are hard to come by. The first nontrivial example was constructed by Cheatham, Enochs, and Jenda [1] when $R = \kappa[x_1, x_2, \cdots, x_n], n \geq 2$, where κ is a field. In this case, let $\mathcal{P} = (x_1, x_2, \cdots, x_n), R/\mathcal{P} = \kappa$ (with $x_i\kappa = 0$ for $i = 1, 2, \cdots, n$) and let $E(\kappa)$ denote the injective envelope of κ. Then the natural map $E(\kappa) \to E(\kappa)/\kappa$ is an injective cover. This used Northcott's description [5] of $E(\kappa)$ as the inverse polynomial ring $\kappa[x_1^{-1}, x_2^{-1}, \cdots, x_n^{-1}]$. Another example is when R is an n-dimensional regular local ring with residue field κ. If $n \geq 2$, then again the natural map $E(\kappa) \to E(\kappa)/\kappa$ is an injective cover (see [3, Corollary 4.2]).

LEMMA 1.1. (Wakamatsu, [7] ; [9, Lemma 2.1.1]) *Let $\phi : E \to M$ be an injective cover of an R-module M. Then $ker\phi$ has the property that $Ext^1_R(\bar{E}, ker\phi) = 0$ for any injective R-module $\bar{E}$.*

DEFINITION 1.2. A *special injective precover* is defined to be a precover $\phi : E \to M$ such that $ker\phi$ has the property that $Ext^1_R(\bar{E}, ker\phi) = 0$ for any injective R-module $\bar{E}$.

PROPOSITION 1.3. (Kim, Park, Song [4, Proposition 1.3]) *If an R-module M has an injective cover and $\phi : E \to M$ is an injective precover of M, then the followings are equivalent;*

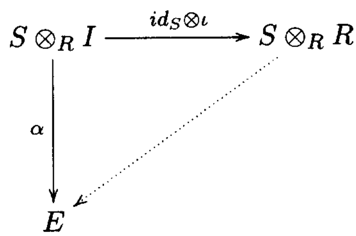
(a) *ϕ is an injective cover of M*
(b) *There is no nonzero direct summand of E contained in $ker\phi$*
(c) *Any linear map $f : E \to E$ with $\phi \circ f = \phi$ is a surjection.*

2. Ring extensions and injective precovers

In [8], Würful gave a characterization of those rings R such that for every ring extension $R \subset S$, $Hom_R(S, -)$ converts injective envelopes of R-modules into injective envelopes of S-modules. In this section, we will consider an analogous problem concerning injective covers.

LEMMA 2.1. *Let $f : R \to S$ be a ring homomorphism and S_R flat. If ${}_SE$ is injective, then ${}_RE$ is also injective.*

PROOF. Let $g : I \to_R E$ be an R-linear map for an ideal I of R. Define $\alpha : S \otimes_R I \to E$ by $\alpha(s \otimes x) = sg(x)$. Then α is S-linear. Also $0 \to S \otimes_R I \to S \otimes_R R$ is exact since S_R is flat. So the diagram



can be completed to a commutative diagram, where $\iota : I \to R$ is the inclusion map. But the composition map $I \to S \otimes_R I \to E$ with $\beta : I \to S \otimes_R I$ defined by $\beta(a) = 1 \otimes a$ is equal to the original g. So $R \to S \otimes_R R \to E$ gives an R-linear extension. □

REMARK 2.2. Let $f : R \to S$ be a ring homomorphism and let E be an injective R-module. Then for any S-module M, Ext^1_S $(M, Hom_R$ $(S, E)) \cong Ext^1_R$ $(S \otimes M, E) = 0$, and thus $Hom_R(S, E)$ is an injective S-module.

THEOREM 2.3. *Let $f : R \to S$ be a ring homomorphism, S_R flat and $\phi : E \to M$ be an injective precover of an R-module M. Then $Hom_R(S, E) \to Hom_R(S, M)$ is a special injective precover.*

PROOF. To show that $Hom_R(S, E) \to Hom_R(S, M)$ is an injective precover, it suffices to show that

$$Hom_S(\bar{E}, Hom_R(S, E)) \to Hom_S(\bar{E}, Hom_R(S, M)) \to 0$$

is exact for any injective S-module $\bar{E}$, or equivalently to show that $Hom_R(S \otimes_S \bar{E}, E) \to Hom_R(S \otimes_S \bar{E}, M) \to 0$ is exact. Since $\phi : E \to M$ is an injective precover of M and $S \otimes_S \bar{E} \cong \bar{E}$ is R-injective by Lemma 2.1, therefore $Hom_R(S \otimes_S \bar{E}, E) \to Hom_R(S \otimes_S \bar{E}, M) \to 0$ is exact.

Next we need to show that $Hom_R(S, Ker\phi)$ has the property that for any injective S-module E', $Ext^1_S(E', Hom_R(S, Ker\phi)) = 0$.

Since $Ker(Hom_R(S,E) \to Hom_R(S,M)) \cong Hom_R(S, Ker\phi)$,

$$Ext^1_S(E', Hom_R(S, Ker\phi)) \cong Ext^1_R(S \otimes_S E', Ker\phi)$$
$$\cong Ext^1_R(E', Ker\phi) = 0. \qquad \square$$

COROLLARY 2.4. *With the above situations, the followings are equivalent;*

(1) $\psi : Hom_R(S,E) \to Hom_R(S,M)$ *is an injective cover*
(2) (a) $\phi : E \to M$ *is an injective precover*
 (b) $Hom_R(S, Ker\phi)$ *has no nonzero injective submodules in* $Hom_R(S,E)$
(3) ψ *is an injective precover and* $Hom_R(S, Ker\phi)$ *has no nonzero injective submodules in* $Hom_R(S,E)$.

EXAMPLE 2.5. Let $S = R[x]$. Given an injective cover $\phi : E \to M$, $Hom_R(R[x],E) \to Hom_R(R[x],M)$ is a special injective precover since $R[x]$ is a flat R-module. Note that $Hom_R(R[x],E) \cong E[[x^{-1}]]$ and $Hom_R(R[x],M) \cong M[[x^{-1}]]$. Since $\phi : E \to M$ is an injective cover, $K = Ker\phi$ has no nonzero injective submodules. So $E[[x^{-1}]] \to M[[x^{-1}]]$ is an injective cover if $K[[x^{-1}]]$ has no nonzero injective submodule as $R[x]$-module. But any injective $R[x]$-module is injective as an R-module. So $E[[x^{-1}]] \to M[[x^{-1}]]$ is an injective cover if $K[[x^{-1}]] \cong K \times K \times K \times \cdots$ has no nonzero injective submodule as an R-module.

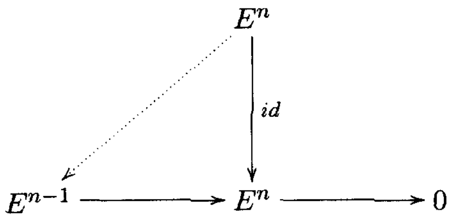
PROPOSITION 2.6. *Let R be a semi-local ring and $\phi : E \to M$ an injective cover. Then*

(1) $Ext^n_R(\bar{E}, Ker\phi) = 0$ *for all* $n > 1$ *and injective* $\bar{E}$.
(2) $Ext^n_R(\bar{E}, E) \cong Ext^n_R(\bar{E}, M)$ *for all injective* $\bar{E}$ *and* $n \geq 1$.

PROOF. (1) For any injective R-module $\bar{E}$, let $0 \to K \to F \to \bar{E} \to 0$ be an exact sequence with F free. Since $Ext^n_R(F, Ker\phi) = 0$ for all $n \geq 1$, $Ext^n_R(K, Ker\phi) \cong Ext^{n+1}_R(\bar{E}, Ker\phi)$ for all $n \geq 1$. And since K is injective, $Ext^1_R(K, Ker\phi) = 0$. So $Ext^2_R(\bar{E}, Ker\phi) = 0$. Proceeding in this manner, $Ext^n_R(\bar{E}, Ker\phi) = 0$ for all $n \geq 1$.

(2) It follows from $Hom_R(\bar{E}, E) \to Hom_R(\bar{E}, M) \to 0$ is exact for all injective $\bar{E}$ and $Ext^n_R(\bar{E}, Ker\phi) = 0$ for all $n \geq 1$. $\square$

REMARK 2.7. Suppose that for any module M over a ring R, Ext^1_R $(E,M)=0$ for all injective R-module E implies that Ext^i_R $(E,M)=0$ for all injective R-module E and $i\geq 1$. If $inj.dim_R\ M=n<\infty$, then $n=0$, i.e. M is injective. For if $0\to M\to E^0\to E^1\to\cdots\to E^n\to 0$ is an injective resolution of M with $n\geq 1$, then $Ext^n_R(E^n,M)=0$. This means



can be completed to a commutative diagram. But then $E^{n-1}\cong E\oplus E^n$ for some injective E and we have an injective resolution $0\to M\to E^0\to\cdots\to E^{n-2}\to E\to 0$ of M of length $n-1$. If $n-1\geq 1$, then we can repeat the procedure.

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References

[1] T. Cheatham, E. Enochs, and O. Jenda, *The structure of injective covers of special modules*, Israel J. Math. **63** (1988), 237–242.
[2] E. Enochs, *Injective and flat covers, envelopes and resolvents*, Israel J. Math. **39** (1981), 189–209.
[3] O. Jenda, *The dual of the grade of a module*, Arch. Math. **51** (1988), 297–308.
[4] H. Kim, Y. S. Park, and Y. M. Song, *S-torsion free covers of modules*, to appear in Comm. Algebra.
[5] D. Northcott, *Injective envelopes and inverse polynomials*, J. London Math. Soc. **68** (1974), 290–296.
[6] J. Rotman, *An introduction to Homological Algebra*, Academic Press, 1979.
[7] T. Wakamatsu, *Stable equivalence of self-injective algebras and a generalization of tilting modules*, J. Algebra **134** (1990), 298–325.
[8] T. Würful, *Ring extensions and essential monomorphisms*, Proc. Amer. Math. Soc. **69** (1978), 1–7.
[9] J. Xu, *Flat covers of modules*, Lecture Notes in Math. 1634, Springer-Verlag, 1996.

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