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A GENERALIZATION OF THE KRASNOSELSKII–PETRYSHYN COMPRESSION AND EXPANSION THEOREM: AN ESSENTIAL MAP APPROACH

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Abstract. This paper introduces the notions of essential and inessential maps for countably k–set contractive maps. These ideas enable us to establish a Krasnoselskii–Petryshyn compression and expansion theorem in a cone for countably k–set contractive maps.

1. Introduction

This paper presents new fixed point results for countably condensing maps. We begin with a fixed point result for self maps which follows from a multivalued version of Mönch's theorem [5] established recently by O'Regan and Precup [7]. Next we introduce the notions of essential and inessential maps for countably condensing maps, and some properties of these maps will be established. In particular we show if a map G is essential and $G \cong F$ then F is essential. In Section 3 we present a generalization of the Krasnoselskii–Petryshyn compression and expansion theorem in a cone [3].

For the remainder of this section we present some preliminary results which will be needed in Section 2 and Section 3. Let E be a Banach space and $P_B(E)$ the bounded subsets of E. The Kuratowskii measure of noncompactness is the map $\alpha : P_B(E) \to [0, \infty)$ defined by

$$\alpha(X) = \inf \{\epsilon > 0 : \; X \subseteq \cup_{i=1}^{n} X_i \; \text{ and } \; diam\,(X_i) \leq \epsilon\}\,;$$

here $X \in P_B(E)$. Let Z be a nonempty subset of E and $F : Z \to 2^E$ (the nonempty subsets of E). F is called (i). countably k–set

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contractive ($k \geq 0$) if $F(Z)$ is bounded and $\alpha(F(Y)) \leq k\,\alpha(Y)$ for all countably bounded sets Y of Z; (ii). countably condensing if F is countably 1–set contractive and $\alpha(F(Y)) < \alpha(Y)$ for all countably bounded sets Y of Z with $\alpha(Y) \neq 0$.

In [7] O'Regan and Precup established the following generalization of Mönch's fixed point theorem.

THEOREM 1.1. *Let D be a closed, convex subset of a Banach space E and $F : D \to CK(D)$ a upper semicontinuous map; here $CK(D)$ denotes the family of nonempty, convex, compact subsets of D. Assume there is an $x_0 \in D$ with the following property holding:*

$$(1.1) \qquad \begin{cases} M \subseteq D,\ M = co\,(\{x_0\} \cup F(M)) \ \text{ and } \ \overline{M} = \overline{C} \ \text{ with} \\ C \subseteq M \ \text{ countable, implies } \ \overline{M} \ \text{ is compact.} \end{cases}$$

Then there exists $x \in D$ with $x \in F(x)$.

REMARK 1.1. In Theorem 1.1 (and also in many of the results in Sections 2 and 3) it is possible to replace the condition $F : D \to CK(D)$ is upper semicontinuous with

$$\begin{cases} F : D \to C(D) \ \text{ is a closed map and } \ F\cdot \ \text{ maps} \\ \text{compact sets into relatively compact sets;} \end{cases}$$

here $C(D)$ denotes the family of nonempty, convex subsets of D.

Theorem 1.1 immediately guarantees a new result for countably condensing maps (this is a multivalued version of Daher's theorem [2]).

THEOREM 1.2. *Let D be a nonempty, closed, convex subset of a Banach space E and $F : D \to CK(D)$ a upper semicontinuous, countably condensing map. Then there exists $x \in D$ with $x \in F(x)$.*

Proof. Let $x_0 \in D$, $M \subseteq D$, $M = co\,(\{x_0\} \cup F(M))$ and $\overline{M} = \overline{C}$ with $C \subseteq M$ countable. If we show $\overline{M}$ is compact, then (1.1) will hold, so the result follows from Theorem 1.1.

Now since

$$C \subseteq co\,(\{x_0\} \cup F(M)),$$

each $x \in C$ can be written as a finite convex combination of points from $\{x_0\} \cup F(M)$. Thus there exists a <u>countable</u> set $M_C \subseteq M$ with

$$(1.2) \qquad C \subseteq co\,(\{x_0\} \cup F(M_C)).$$

If $\alpha(M_C) \neq 0$ then

$$\begin{aligned} &\alpha(C) \leq \alpha(co\,(\{x_0\} \cup F(M_C))) \\ =&\alpha(F(M_C)) < \alpha(M_C) \leq \alpha(M) \\ =&\alpha(\overline{M}) = \alpha(\overline{C}) = \alpha(C), \end{aligned}$$

a contradiction. Thus $\alpha(M_C) = 0$ so from (1.2) we obtain,

$$\alpha(C) \leq \alpha(F(M_C)) \leq \alpha(M_C) = 0.$$

Thus $\overline{C}$ is compact. □

COROLLARY 1.3. *Let D be a nonempty, closed, convex subset of a Banach space E and $F : D \to CK(D)$ a upper semicontinuous, countably k–set contractive ($0 \leq k < 1$) map. Then there exists $x \in D$ with $x \in F(x)$.*

2. Essential and inessential maps

Throughout this section, we will assume E is a Banach space, C is a closed, convex subset of E, and U is an open subset of C. We note that some of the ideas in this section were motivated by a recent paper of the authors [1].

DEFINITION 2.1. $D(\overline{U}, C)$ denotes the set of all upper semicontinuous, countably k–set contractive ($0 \leq k < 1$) maps $F : \overline{U} \to CK(C)$; here $\overline{U}$ denotes the closure of U in C.

REMARK 2.1. It is also possible to consider the set of all upper semicontinuous, countably condensing maps in this section. We leave the minor adjustments necessary to the reader.

DEFINITION 2.2. We let $D_{\partial U}(\overline{U}, C)$ denote the set of all maps $F \in D(\overline{U}, C)$ with $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in C.

DEFINITION 2.3. A map $F \in D_{\partial U}(\overline{U}, C)$ is essential in $D_{\partial U}(\overline{U}, C)$ if for every map $G \in D_{\partial U}(\overline{U}, C)$ with $G|_{\partial U} = F|_{\partial U}$ we have that there exists $x \in U$ with $x \in G(x)$. Otherwise F is inessential in

$D_{\partial U}(\overline{U},C)$ i.e. there exists a map $G \in D_{\partial U}(\overline{U},C)$ with $G|_{\partial U} = F|_{\partial U}$ and $x \notin G(x)$ for $x \in \overline{U}$.

We begin with a simple example of an essential map, which is all we need from an application viewpoint.

THEOREM 2.1. *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Then the zero map is essential in $D_{\partial U}(\overline{U},C)$.*

Proof. Let $\theta \in D_{\partial U}(\overline{U},C)$ with $\theta|_{\partial U} = \{0\}$. We must show there exists $x \in U$ with $x \in \theta(x)$. Let $D = \overline{co}\,(\theta(\overline{U}))$ and consider the map $J : D \to CK(D)$ given by

$$J(x) = \begin{cases} \theta(x), & x \in \overline{U} \\ \{0\}, & \text{otherwise.} \end{cases}$$

Notice $J : D \to CK(D)$ is an upper semicontinuous, countably k–set contractive map. To see that J is countably k–set contractive let $\Omega \subseteq D$ be countable. Then since

$$J(\Omega) \subseteq co\,(\theta(\Omega \cap \overline{U}) \cup \{0\}),$$

we have

$$\alpha(J(\Omega)) \leq \alpha(\theta(\Omega \cap \overline{U})) \leq k\,\alpha(\Omega \cap \overline{U}) \leq k\,\alpha(\Omega),$$

since $\Omega \cap \overline{U}$ is countable.

Now Corollary 1.3 guarantees that there exists $x \in D$ with $x \in J(x)$. Notice $x \in U$ since $0 \in U$, so $x \in \theta(x)$. □

Next we establish a homotopy property for essential maps.

THEOREM 2.2. *Let E be a Banach space, C a closed, convex subset of E and U an open subset of C. Suppose F, $G \in D(\overline{U},C)$ and assume the following hold:*

(2.1) $$G \text{ is essential in } D_{\partial U}(\overline{U},C)$$

(2.2) $$x \notin \lambda\,F(x) + (1-\lambda)\,G(x) \quad \text{for } x \in \partial U \text{ and } \lambda \in (0,1].$$

Then F is essential in $D_{\partial U}(\overline{U},C)$.

Proof. Let $H \in D_{\partial U}(\overline{U}, C)$ with $H|_{\partial U} = F|_{\partial U}$. We must show H has a fixed point in U. Consider

$$B = \{x \in \overline{U} : \ x \in t\, H(x) + (1-t)\, G(x) \ \text{ for some } \ t \in [0,1]\}.$$

Notice (2.1) guarantees that $B \neq \emptyset$. It is also immediate that B is closed (in C). In addition (2.2) together with $H|_{\partial U} = F|_{\partial U}$ guarantees that $B \cap \partial U = \emptyset$. Thus there exists a continuous $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map $R_\mu : \overline{U} \to CK(C)$ by

$$R_\mu(x) = \mu(x)\, H(x) + (1 - \mu(x))\, G(x).$$

Notice $R_\mu \in D(\overline{U}, C)$ since if $\Omega \subseteq \overline{U}$ is countable then since

$$R_\mu(\Omega) \subseteq co\,(H(\Omega) \cup G(\Omega))$$

we have

$$\alpha(R_\mu(\Omega)) \leq \alpha(H(\Omega) \cup G(\Omega)) = \max\{\alpha(H(\Omega))\,,\, \alpha(G(\Omega))\} \leq k\, \alpha(\Omega).$$

Also for $x \in \partial U$ we have $R_\mu(x) = G(x)$ so $x \notin R_\mu(x)$ for $x \in \partial U$. Consequently $R_\mu \in D_{\partial U}(\overline{U}, C)$ with $R_\mu|_{\partial U} = G|_{\partial U}$. Now (2.1) guarantees that there exists $x \in U$ with $x \in R_\mu(x)$ (i.e. $x \in \mu(x)\, H(x) + (1 - \mu(x))\, G(x)$). Thus $x \in B$ and so $\mu(x) = 1$. As a result we have $x \in U$ with $x \in H(x)$. □

Next we will establish some "inessential" map type results. These results will be needed in Section 3 to establish the generalization of the Krasnoselskii–Petryshyn compression and expansion theorem.

THEOREM 2.3. *Let E be a Banach space, C a closed, convex subset of E with $\alpha\, u + \beta\, v \in C$ for all $\alpha \geq 0$, $\beta \geq 0$ and u, $v \in C$, and U an open subset of C. Suppose $F \in D(\overline{U}, C)$ with the following holding:*
(2.3)
there exists $v \in C$ with $x \notin F(x) + \delta\, v$ for all $\delta \geq 0$ and $x \in \partial U$

and

(2.4) *there exists $\delta_0 > 0$ with $x \notin F(x) + \delta_0\, v$ for $x \in \overline{U}$.*

Then F is inessential in $D_{\partial U}(\overline{U}, C)$.

Proof. Let the map J be defined by $J(x) = F(x) + \delta_0\, v$. Clearly $J \in D_{\partial U}(\overline{U}, C)$ (note if $\Omega \subseteq \overline{U}$ is countable then $J(\Omega) \subseteq F(\Omega) + \{\delta_0\, v\}$). To show F is inessential in $D_{\partial U}(\overline{U}, C)$ we must show that there exists a fixed point free $\theta \in D_{\partial U}(\overline{U}, C)$ with $\theta|_{\partial U} = F|_{\partial U}$. Consider

$$B = \{x \in \overline{U} : \ x \in F(x) + (1-t)\, \delta_0\, v \ \text{ for some } \ t \in [0,1]\}.$$

If $B = \emptyset$ then in particular F has no fixed points in $\overline{U}$ and thus F is inessential in $D_{\partial U}(\overline{U}, C)$. It remains to consider the case when $B \neq \emptyset$. Notice B is closed. Also (2.3) guarantees that $B \cap \partial U = \emptyset$. Thus there exists a continuous $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 1$ and $\mu(B) = 0$. Define a map $N_\mu : \overline{U} \to CK(C)$ by

$$N_\mu(x) = F(x) + (1 - \mu(x))\,\delta_0\, v.$$

Notice $N_\mu \in D(\overline{U}, C)$ (note if $\Omega \subseteq \overline{U}$ is countable then $N_\mu(\Omega) \subseteq F(\Omega) + co\,(\{\delta_0\, v\} \cup \{0\}))$. Also if $x \in \partial U$ then $\mu(x) = 1$ so $N_\mu|_{\partial U} = F|_{\partial U}$. Consequently $N_\mu \in D_{\partial U}(\overline{U}, C)$ with $N_\mu|_{\partial U} = F|_{\partial U}$. We claim N_μ is fixed point free on $\overline{U}$. If this is true then F is inessential in $D_{\partial U}(\overline{U}, C)$. It remains to prove the claim. If there exists $x \in \overline{U}$ with $x \in N_\mu(x) = F(x) + (1 - \mu(x))\,\delta_0\, v$ then $x \in B$ and so $\mu(x) = 0$ i.e. $x \in F(x) + \delta_0\, v$ (this contradicts (2.4)). □

Our next result generalizes Theorem 2.3.

THEOREM 2.4. *Let E be a Banach space, C a closed, convex subset of E with $\alpha\, u + \beta\, v \in C$ for all $\alpha \geq 0$, $\beta \geq 0$ and $u, v \in C$, and U an open subset of C. Suppose $F \in D(\overline{U}, C)$ satisfies* (2.3), *and assume the following condition holds:*

(2.5)
$$\begin{cases} \text{there exists } \delta_0 > 0 \text{ with } J, \text{ defined by } J(x) = F(x) + \delta_0\, v, \\ \text{inessential in } D_{\partial U}(\overline{U}, C). \end{cases}$$

Then F is inessential in $D_{\partial U}(\overline{U}, C)$.

Proof. Now (2.5) implies that there exists $\tau \in D_{\partial U}(\overline{U}, C)$ with $\tau|_{\partial U} = (F + \delta_0\, v)|_{\partial U}$ and $x \notin \tau(x)$ for $x \in \overline{U}$. Let $\gamma : \overline{U} \to CK(C)$ be given by $\gamma(x) = \tau(x) - \delta_0\, v$. Notice $\gamma|_{\partial U} = F|_{\partial U}$. Consider

$$B = \{x \in \overline{U} : \ x \in \gamma(x) + (1 - t)\,\delta_0\, v \ \text{ for some } \ t \in [0,1]\}.$$

If $B = \emptyset$ then γ has no fixed points in $\overline{U}$, and since $\gamma|_{\partial U} = F|_{\partial U}$ we have that F is inessential in $D_{\partial U}(\overline{U}, C)$. It remains to consider the case when $B \neq \emptyset$. Notice $B \cap \partial U = \emptyset$ so there exists a continuous $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 1$ and $\mu(B) = 0$. Let $T_\mu : \overline{U} \to CK(C)$ be defined by

$$T_\mu(x) = \gamma(x) + (1 - \mu(x))\,\delta_0\, v.$$

Clearly $T_\mu \in D_{\partial U}(\overline{U}, C)$ with $T_\mu|_{\partial U} = \gamma|_{\partial U} = F|_{\partial U}$. Also if $x \in \overline{U}$ and $x \in T_\mu(x)$ then $x \in \gamma(x) + (1 - \mu(x))\,\delta_0\, v$, so $x \in B$ and $\mu(x) = 0$ i.e.

$x \in \gamma(x) + \delta_0 v = \tau(x)$ (a contradiction). Thus $T_\mu \in D_{\partial U}(\overline{U}, C)$ is a fixed point free map with $T_\mu|_{\partial U} = F|_{\partial U}$. Consequently F is inessential in $D_{\partial U}(\overline{U}, C)$. □

THEOREM 2.5. *Let E be a Banach space, C a closed, convex subset of E with $\alpha u + \beta v \in C$ for all $\alpha \geq 0$, $\beta \geq 0$ and $u, v \in C$, U an open subset of C and $0 \in U$. Suppose $F \in D(\overline{U}, C)$ and assume the following conditions are satisfied:*

$$\text{there exists } \delta_0 > 0 \text{ and } v \in C \setminus \{0\} \text{ with } \delta_0 v \in C \setminus \overline{U} \tag{2.6}$$

and

$$x \notin \lambda F(x) + \delta_0 v \text{ for all } \lambda \in (0,1] \text{ and } x \in \partial U. \tag{2.7}$$

Then ϕ $(= F + \delta_0 v)$ is inessential in $D_{\partial U}(\overline{U}, C)$.

Proof. Let ψ be given by $\psi(x) = \{\delta_0 v\}$. Now since $\delta_0 v \in C \setminus \overline{U}$ we have $\psi \in D_{\partial U}(\overline{U}, C)$. Consider

$$B = \{x \in \overline{U} : \ x \in t F(x) + \delta_0 v \text{ for some } t \in [0,1]\}.$$

If $B = \emptyset$ then ϕ has no fixed points in $\overline{U}$ so ϕ is inessential in $D_{\partial U}(\overline{U}, C)$. It remains to consider the case when $B \neq \emptyset$. Now B is closed and $B \cap \partial U = \emptyset$ from (2.7). Thus there exists a continuous $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 1$ and $\mu(B) = 0$. Let $L_\mu : \overline{U} \to CK(C)$ be given by

$$L_\mu(x) = \mu(x)\, F(x) + \delta_0 v.$$

Notice $L_\mu \in D(\overline{U}, C)$ and $L_\mu|_{\partial U} = \phi|_{\partial U}$, so $L_\mu \in D_{\partial U}(\overline{U}, C)$. If there exists $x \in \overline{U}$ with $x \in L_\mu(x)$ then $x \in B$ so $\mu(x) = 0$ i.e. $x \in \{\delta_0 v\}$ (a contradiction since $x \in \overline{U}$ and $\delta_0 v \in C \setminus \overline{U}$). Thus L_μ is fixed point free on $\overline{U}$ and as a result ϕ is inessential in $D_{\partial U}(\overline{U}, C)$. □

3. The Krasnoselskii–Petryshyn compression and expansion theorem for countably k–set contractive maps

In this section $E = (E, \|\,.\,\|)$ will be a Banach space and C will be a closed, convex subset of E with $\alpha u + \beta v \in C$ for all $\alpha \geq 0$, $\beta \geq 0$ and u, $v \in C$. Let $\rho > 0$ with

$$B_\rho = \{x : \ x \in C \text{ and } \|x\| < \rho\}, \qquad S_\rho = \{x : \ x \in C \text{ and } \|x\| = \rho\}$$

and of course $\overline{B_\rho} = B_\rho \cup S_\rho$.

THEOREM 3.1. *Let E and C be as above and let r, R be constants with $0 < r < R$. Suppose $F \in D(\overline{B_R}, C)$ and assume the following conditions hold:*

$$x \notin F(x) \quad \text{for } x \in S_R \cup S_r \tag{3.1}$$

$$\begin{cases} F : \overline{B_r} \to CK(C) \text{ is inessential in } D_{S_r}(\overline{B_r}, C) \\ (\text{i.e. } F|_{\overline{B_r}} \text{ is inessential in } D_{S_r}(\overline{B_r}, C)) \end{cases} \tag{3.2}$$

$$F : \overline{B_R} \to CK(C) \text{ is essential in } D_{S_R}(\overline{B_R}, C). \tag{3.3}$$

Then F has a fixed point in $\Omega = \{x \in C : r < \|x\| < R\}$.

Proof. Suppose F has no fixed points in Ω. Now (3.2) implies that there exists $\theta \in D(\overline{B_r}, C)$ with $\theta|_{S_r} = F|_{S_r}$ and $x \notin \theta(x)$ for $x \in \overline{B_r}$. Let $\Phi : \overline{B_R} \to CK(C)$ be defined by

$$\Phi(x) = \begin{cases} F(x), & r < \|x\| \le R \\ \theta(x), & \|x\| \le r. \end{cases}$$

Notice $\Phi \in D(\overline{B_R}, C)$ and Φ has no fixed points in $\overline{B_R}$ (since θ has no fixed points in $\overline{B_r}$ and F has no fixed points in Ω). This contradicts (3.3). $\square$

Our next result is a generalization of the Krasnoselskii–Petryshyn compression theorem [3, 8].

THEOREM 3.2. *Let E and C be as above and let r, R be constants with $0 < r < R$. Suppose $F \in D(\overline{B_R}, C)$ and assume the following conditions hold:*

(3.4)

there exists $v \in C \setminus \{0\}$ with $x \notin F(x) + \delta\, v$ for all $\delta > 0$ and $x \in S_r$

and

$$x \notin \lambda F(x) \quad \text{for } x \in S_R \text{ and } \lambda \in (0,1). \tag{3.5}$$

Then F has a fixed point in $\{x \in C : r \le \|x\| \le R\}$.

Proof. Suppose $x \notin F(x)$ for $x \in S_r \cup S_R$ (otherwise we are finished). Then

$$x \notin F(x) + \delta\, v \quad \text{for } \delta \ge 0 \text{ and } x \in S_r \tag{3.6}$$

and

$$x \notin \lambda F(x) \ \text{ for } \ x \in S_R \ \text{ and } \ \lambda \in [0,1]. \tag{3.7}$$

Choose $M > 0$ such that

$$\|y\| \leq M \ \text{ for all } \ y \in F(x) \ \text{ and } \ x \in \overline{B_r}.$$

Now choose $\delta_0 > 0$ such that

$$\|\delta_0 \, v\| > M + r. \tag{3.8}$$

Let $J : \overline{B_r} \to CK(C)$ be given by $J(x) = F(x) + \delta_0 \, v$. Notice $J \in D_{S_r}(\overline{B_r}, C)$ and also (3.8) guarantees that

$$x \notin J(x) \ \text{ for } \ x \in \overline{B_r}.$$

This together with Theorem 2.3 implies

$$F : \overline{B_r} \to CK(C) \ \text{ is inessential in } \ D_{S_r}(\overline{B_r}, C). \tag{3.9}$$

Theorem 2.1 guarantees that the zero map is essential in $D_{S_R}(\overline{B_R}, C)$. This together with Theorem 2.2 and (3.7) implies

$$F : \overline{B_R} \to CK(C) \ \text{ is essential in } \ D_{S_R}(\overline{B_R}, C). \tag{3.10}$$

Finally (3.9), (3.10) and Theorem 3.1 imply that F has a fixed point in Ω; here $\Omega = \{x \in C : \ r < \|x\| < R\}$. □

In our next theorem $C \subseteq E$ will be a cone. Let $\rho > 0$ with

$$\Omega_\rho = \{x \in E : \ \|x\| < \rho\},$$
$$\partial_E \, \Omega_\rho = \{x \in E : \ \|x\| = \rho\},$$
$$B_\rho = \{x : \ x \in C \ \text{ and } \ \|x\| < \rho\},$$

and

$$S_\rho = \{x : \ x \in C \ \text{ and } \ \|x\| = \rho\}.$$

Notice

$$B_\rho = \Omega_\rho \cap C \quad \text{and} \quad S_\rho = \partial_C \, (\Omega_\rho \cap C) = \partial_E \, \Omega_\rho \cap C.$$

THEOREM 3.3. *Let $E = (E, \|\,.\,\|)$ be a Banach space, $C \subseteq E$ a cone and let $\|\,.\,\|$ be increasing with respect to C. Also r, R are constants with $0 < r < R$. Suppose $F : \overline{\Omega_R} \cap C \to CK(C)$ is a upper semicontinuous, countable k–set contractive (here $0 \leq k < 1$) map and assume the following conditions hold:*

$$\|y\| \leq \|x\| \ \textit{ for all } \ y \in F(x) \ \textit{ and } \ x \in \partial_E \, \Omega_R \cap C \tag{3.11}$$

and

$$\|y\| > \|x\| \quad \text{for all } y \in F(x) \text{ and } x \in \partial_E \Omega_r \cap C. \tag{3.12}$$

Then F has a fixed point in $C \cap \{x \in E:\ r \leq \|x\| \leq R\}$.

Proof. First we show (3.11) implies (3.5). Suppose there exists $x \in S_R$ and $\lambda \in (0,1)$ with $x \in \lambda F(x)$. Then there exists a $y \in F(x)$ with $x = \lambda y$ and so

$$R = \|x\| = |\lambda|\,\|y\| < \|y\| \leq \|x\| = R,$$

a contradiction. Next notice (3.12) implies (3.4). Suppose there exists $v \in C \setminus \{0\}$ with $x \in F(x) + \delta v$ for some $\delta > 0$ and $x \in S_r$. Then there exists a $y \in F(x)$ with $x = y + \delta v$. Now since $\|\,.\,\|$ is increasing with respect to C we have

$$\|x\| = \|y + \delta v\| \geq \|y\| > \|x\|,$$

a contradiction. The result now follows from Theorem 3.2. □

For our next two results we again assume $C \subseteq E$ is a closed, convex nonempty set with $\alpha u + \beta v \in C$ for all $\alpha \geq 0$, $\beta \geq 0$ and $u, v \in C$.

THEOREM 3.4. *Let E and C be as above and let r, R be constants with $0 < r < R$. Suppose $F \in D(\overline{B_R}, C)$ and assume the following conditions hold:*

$$x \notin F(x) \quad \text{for } x \in S_R \cup S_r \tag{3.13}$$

$$F : \overline{B_r} \to CK(C) \text{ is essential in } D_{S_r}(\overline{B_r}, C) \tag{3.14}$$

and

$$F : \overline{B_R} \to CK(C) \text{ is inessential in } D_{S_R}(\overline{B_R}, C). \tag{3.15}$$

Then F has at least two fixed points x_0 and x_1 with $x_0 \in B_r$ and $x_1 \in \Omega$; here $\Omega = \{x \in C:\ r < \|x\| < R\}$.

Proof. From (3.14) we know that F has a fixed point in B_r. Let $\Psi = F|_{\overline{\Omega}}$ and suppose $\Psi : \overline{\Omega} \to CK(C)$ has no fixed points. Now (3.15) guarantees that there exists an upper semicontinuous, countable k–set contractive map $\theta : \overline{B_R} \to CK(C)$ with

$$\theta|_{S_R} = F|_{S_R} \quad \text{and} \quad x \notin \theta(x) \quad \text{for } x \in \overline{B_R}.$$

Fix $\rho \in (0,r)$ and consider the map Φ given by

$$\Phi(x)=\begin{cases} \frac{\rho}{R}\;\theta\left(\frac{R}{\rho}\,x\right), & \|x\|\le\rho \\ \frac{(r-\rho)\,\|x\|}{(R-\rho)\,r-(R-r)\,\|x\|}\;\Psi\left(\frac{(R-\rho)\,r-(R-r)\,\|x\|}{(r-\rho)\,\|x\|}\,x\right), & \rho\le\|x\|\le r \\ \Psi(x), \quad r\le\|x\|\le R. \end{cases}$$

Notice $\Phi:\overline{B_R}\to CK(C)$ is well defined since if $\rho\le\|x\|\le r$ then

$$r\le\left\|\frac{(R-\rho)\,r-(R-r)\,\|x\|}{(r-\rho)\,\|x\|}\,x\right\|\le R.$$

Note $\Phi:\overline{B_R}\to CK(C)$ is a upper semicontinuous, countable k–set contractive map (the proof of countably k–set contractive is essentially contained in [6]). In addition

$$\Phi|_{S_R}=\Psi|_{S_R}=F|_{S_R}=\theta|_{S_R} \quad\text{and}\quad \Phi|_{\overline{\Omega}}=\Psi|_{\overline{\Omega}}=F|_{\overline{\Omega}}$$

and Φ has no fixed point in $\overline{B_R}$ (since θ has no fixed points in $\overline{B_R}$ and F has no fixed points in $\overline{\Omega}$).

Lets concentrate on $\Phi:\overline{B_r}\to CK(C)$ (i.e. $\Phi|_{\overline{B_r}}$). Now

$$\Phi|_{S_r}=\Psi|_{S_r}=F|_{S_r}$$

so $\Phi\in D(\overline{B_r},C)$ with $\Phi|_{S_r}=F|_{S_r}$ and Φ has no fixed points in $\overline{B_r}$. This of course contradicts (3.14). □

Our next result is a generalization of Krasnoselskii–Petryshyn expansion theorem [3].

THEOREM 3.5. *Let E and C be as above and let r, R be constants with $0<r<R$. Suppose $F\in D(\overline{B_R},C)$ and assume the following conditions hold:*

(3.16) $\qquad x\notin\lambda F(x)$ *for* $x\in S_r$ *and* $\lambda\in(0,1)$

and

(3.17) $\exists\, v\in C\setminus\{0\}$ *with* $x\notin F(x)+\delta\, v$ *for all* $\delta>0$ *and* $x\in S_R$.

Then F has a fixed point in $\{x\in C:\ r\le\|x\|\le R\}$.

Proof. Assume $x\notin F(x)$ for $x\in S_r\cup S_R$ (otherwise we are finished). Then

(3.18) $\qquad x\notin\lambda F(x)$ for $x\in S_r$ and $\lambda\in[0,1]$

and

$$x \notin F(x) + \delta\, v \quad \text{for} \quad \delta \geq 0 \quad \text{and} \quad x \in S_R. \tag{3.19}$$

Theorem 2.1 guarantees that the zero map is essential in $D_{S_r}(\overline{B_r}, C)$, and this together with Theorem 2.2 and (3.18) implies

$$F : \overline{B_r} \to CK(C) \ \text{ is essential in } \ D_{S_r}(\overline{B_r}, C). \tag{3.20}$$

Let $\delta_0 > 0$ be such that

$$\|\delta_0\, v\| > \sup_{x \in S_R} \|y\| + R \quad \text{for all} \quad y \in F(x). \tag{3.21}$$

Note $\delta_0\, v \in C \setminus \overline{B_R}$. Let $\phi : \overline{B_r} \to CK(C)$ be given by $\phi(x) = F(x) + \delta_0\, v$. Note (3.21) implies for $\lambda \in [0,1]$ and $x \in S_R$ that

$$\|\delta_0\, v + \lambda\, y\| > R = \|x\| \quad \text{for all} \quad y \in F(x),$$

and as a result

$$x \notin \lambda\, F(x) + \delta_0\, v \quad \text{for all} \quad \lambda \in (0,1] \quad \text{and} \quad x \in S_R. \tag{3.22}$$

Now (3.22) together with Theorem 2.5 guarantees that

$$\phi \ (= F + \delta_0\, v) : \overline{B_R} \to CK(C) \ \text{ is inessential in } \ D_{S_R}(\overline{B_R}, C). \tag{3.23}$$

Also notice (3.19), (3.23) and Theorem 2.4 imply that

$$F : \overline{B_R} \to CK(C) \ \text{ is inessential in } \ D_{S_R}(\overline{B_R}, C). \tag{3.24}$$

Finally (3.20), (3.24) and Theorem 3.4 guarantees that F has a fixed point in Ω; here $\Omega = \{x \in C : \ r < \|x\| < R\}$. □

In our final theorem $C \subseteq E$ will be a cone, and Ω_ρ, $\partial_E\, \Omega_\rho$ $(\rho > 0)$ are as discussed before Theorem 3.3.

THEOREM 3.6. *Let $E = (E, \|\,.\,\|)$ be a Banach space, $C \subseteq E$ a cone and let $\|\,.\,\|$ be increasing with repect to C. Also r, R are constants with $0 < r < R$. Suppose $F : \overline{\Omega_R} \cap C \to CK(C)$ is a upper semicontinuous, countable k–set contractive (here $0 \leq k < 1$) map and assume the following conditions hold:*

$$\|y\| > \|x\| \quad \textit{for all} \quad y \in F(x) \quad \textit{and} \quad x \in \partial_E\, \Omega_R \cap C \tag{3.25}$$

and

$$\|y\| \leq \|x\| \quad \textit{for all} \quad y \in F(x) \quad \textit{and} \quad x \in \partial_E\, \Omega_r \cap C. \tag{3.26}$$

Then F has a fixed point in $C \cap \{x \in E : \ r \leq \|x\| \leq R\}$.

Proof. Notice (3.16) and (3.17) are true so the result follows from Theorem 3.5. □

REMARK 3.1. It is easy to combine the ideas in Theorem 3.3 with those in Theorem 3.6 to obtain multiplicity results for F. We leave the details to the reader.

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