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A STATISTICAL INTERPOLATION METHOD: LINEAR PREDICTION IN A STOCK PRICE PROCESS

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Abstract. We propose a statistical interpolation approximate solution for a nonlinear stochastic integral equation of a stock price process. The proposed method has the order $O(h^2)$ of local error under the weaker conditions of μ and σ than those of Milstein's scheme.

1. Introduction

The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, F, P)$ on which is a one dimensional Brownian motions W. The filtration $F = \{\mathcal{F}_t; t \geq 0\}$ is the augmentation under P of the filtration generated by W. We assume that $\mathcal{F} = \mathcal{F}_T, t \in [0, T]$ or that the true state of nature is completely determined by the sample path of W on $[0, t]$.

Throughout we fix a finite time horizon $T \in (0, \infty)$. We interpret the σ-field as representing the information available at time $t \in [0, T]$ and the probability measure P as representing the agent's common beliefs. All the stochastic processes to appear in the sequel are progressively measurable with respective to $\mathcal{F}$ and all equations, equalities or inequalities involving variables are understood to hold P-a.s. It is assumed that the price process of a security is strictly positive P-a.s. Let X_t be the price process of a security/asset at time t. The security of asset is risky because of the presence of its diffusion term σ. In valuing contingent claims, it is convenient to represent the underlying state variables as a

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continuous-time diffusion process, satisfying a nonlinear stochastic differential equations;

$$(1.1)\qquad \begin{cases} dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \\ X_0 \text{ is given.} \end{cases}$$

Examples include the interest rate models of Cox, Ingersoll and Ross[2] and many others [5, 6, 3]. Here W_t is a standard Brownian motion and μ and σ are the drift and diffusion functions of the price process X_t respectively. The initial value X_0 is assumed to be independent of W_t and $E|X_0|^2 < \infty$. It is further assumed that the functions $\mu, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ are Borel measurable and satisfy a Lipschitz condition and a linear growth condition:

$$(1.2)\qquad \|f(t,x) - f(t,y)\| \le C\|x - y\|,$$

$$(1.3)\qquad \|f(t,x)\| \le C(1 + \|x\|).$$

Then it is well-known that the equation (1.1) has a unique solution X_t [7, 11, 13]. Throughout we further assumes that $\mu, \sigma, \mu_x, \sigma_x, \mu_t, \sigma_t$ are continuous and satisfy the growth condition (1.3).

The generalization to a multidimensional price process X_t is avoided only for simplicity. One of course wants to obtain the exact solution X_t of (1.1). However, this can be accomplished generally only for simple(linear) equations. Thus numerical methods for (1.1) is inevitable and tremendously important in application sides. The common numerical method employed include the binomial scheme, finite difference method, Euler type iteration [12] and Monte Carlo simulation [8]. The binomial schemes are most widely used in the finance community for valuation of a wide variety of option models, due primarily to its ease of implementation and pedagogical appeal. The finite difference approach seeks discretization of the differential operators in the continuous Black-Scholes models. Monte Carlo methods simulates the stochastic movement of the price of securities/assets and provides a probabilistic solution. In most case, the Monte Carlo simulation can be applied in quite a straightforward manner, even without a deep understanding of the nature of the price model. Generally speaking each class of numerical methods has its merits and limitations. Unlike [9, 1] and [10], we propose statistical interpolation method for a nonlinear stochastic Volterra integral equation appearing in various financial models [11, 13]. This paper is organized as following . In section 2 statistical divided differences is discussed. In section 3 the statistical linear interpolation method is presented and analyzed. The rate of convergence is obtained, which is higher than that of the classical Euler type iteration method.

2. Euler method and Statistical Divied Differences

Let $[0, T]$ be a finite time horizon. consider a discretization $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$. The analogue of Euler's method is the Euler-Maruyama scheme defined by

$$(2.1)\quad \begin{aligned} \hat{X}_0 &= X(0) := X_0 \\ \hat{X}_{k+1} &= \hat{X}_k + \mu(t_k, \hat{X}_k)(t_{k+1} - t_k) + \sigma(t_k, \hat{X}_k)(W_{k+1} - W_k) \end{aligned}$$

where $\hat{X}_k := X(t_k), \quad W_k := W(t_k), \quad k = 0, 1, 2, \cdots, n-1.$

A corresponding continuous parameter process to (2.1) is given by the linear interpolation,

$$(2.2)\quad \begin{aligned} \hat{X}_t &= \hat{X}_k + \mu(t_k, \hat{X}_k)(t - t_k) + \sigma(t_k, \hat{X}_k)(W_t - W_k) \\ \hat{X}(t_{k-1}) &= \hat{X}_{k-1} \end{aligned}$$

The following theorem is due to Maruyama[16]. Note that, in addition to the Lipschitz and growth condition of μ and σ in the second variable which guarantee existence and uniqueness of the solution(strong) of the equation (1,1), a modulus of continuity condition in the t variable is required to obtain the similar order of convergence as in the deterministic case of Euler's scheme.

THEOREM 2.1. (Maruyama [16]) *Assume that*

$$\begin{aligned} &(A.1)\quad |\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K_1|x - y| \\ &(A.2)\quad |\mu(t,x)|^2 + |\sigma(t,x)|^2 \leq K_2(|1 + |x|^2) \\ &(A.3)\quad |\mu(s,x) - \mu(t,x)|^2 + |\sigma(s,x) - \sigma(t,x)|^2 \leq K_3|s - t|, \ s, t \in [0, T] \end{aligned}$$

Then $E(X_{t_k} - \hat{X}_{t_k})^2 = O(h^2), \ k = 1, 2, \cdots, n \quad t \in [0, T]$, *where* $h = t_k - t_{k-1}$ *is the constant step-size.*

It is clear that the linear interpolation process $\hat{X}_t$ has the same order of mean squence error, i.e. $E(\hat{X}_t - X_t)^2 \leq Ch^2, \quad t \in [t_k, t_k + h]$, where h is the equidistant step-size and C is a constant independent of h. It is a standard fact that the Euler-Maruyama scheme is numerically stable in the mean square sense[16].

Assume that

$$(A.4)\quad \mathrm{E}[\mu^2(\mathrm{t}, \mathrm{X_t})] < \infty \quad ; \mathrm{E}[\sigma^2(\mathrm{t}, \mathrm{X_t})] < \infty.$$

We then need the following Ito's isometry formula ;

$$E\left(\int_0^t \mu(s,X_s)dW_s\right)^2 = \int_0^t E[\mu^2(s,X_s)]ds,$$
$$E\left(\int_0^t \sigma(s,X_s)dW_s\right)^2 = \int_0^t E[\sigma^2(s,X_s)]ds.$$

We now introduce the Newton form of a polynomial interpolation at point to, $t_1, \cdots, t_n$. Let $p_n(t)$ be the polynomial of degree n interpolating $f(x)$ at the points to, $t_0, t_1, \cdots, t_n$. We can express the Newton form of $p_n(t)$ as follows ;

$$\begin{aligned} p_n(t) =& f(t_0) + f[t_0,t_1](t-t_0) + f[t_0,t_1,t_2](t-t_0)(t-t_1) \\ &+ \cdots + f[t_0,t_1,\cdots,t_n]\prod_{i=0}^{n-1}(t-t_j), \end{aligned}$$

and the error function $e(t)$ is given by

$$e(t) := f(t) - p_n(t) = f[t_0,t_1,\cdots,t_n,t]\prod_{j=0}^{n}(t-t_j), \tag{2.3}$$

where the divided differences $f[t_0,t_1,\cdots,t_k]$ is defined recurssively by

$$\begin{aligned} f[t_0] &= f(t_0) \\ f[t_0,t_1] &= \frac{f(t_1)-f(t_0)}{t_1-t_0} \\ &\cdots \\ f[t_0,\cdots,t_k] &= \frac{f[t_1,\cdots t_k]-f[t_0,\cdots,t_{k-1}]}{t_k-t_0} \end{aligned}$$

For details for algebraic properties of the divided differences, see[14]. To define statistical divided differences, assume that we are given by the market data for the stock price process X_t at the discrete time point $t_0, t_1, \cdots, t_n$, which means that the data of the stock price in the market are given by $(t_0, X_0), (t_1, X_1), \cdots, (t_n, X_n)$. Accordingly we have the corresponding market data for the drift term $\mu(t, X_t)$ and the diffusion term $\sigma(t, X_t)$ respectively ;

$$\mu(t_0,X_0), \quad \mu(t_1,X_1), \cdots, \mu(t_n,X_n);$$
$$\sigma(t_0,X_0), \quad \sigma(t_1,X_1), \cdots, \sigma(t_n,X_n).$$

We then define the statistical divided differences as follows ; for $k = 1, 2, \cdots, n$

(2.3)

$$\mu[t_0; X_0] = \mu(t_0, X_0), \quad \mu[t_0, t_1; X_0, X_1] = \frac{\mu[t_1; X_1] - \mu[t_0; X_0]}{t_1 - t_0}$$

$$\mu[t_0, \cdots, t_k; X_0, \cdots, X_k] = \frac{\mu[t_1, \cdots, t_k; X_1, \cdots, X_k] - \mu[t_0, \cdots, t_{k-1}; X_0, \cdots, X_{k-1}]}{t_k - t_0}$$

(2.4)

$$\sigma[t_0; X_0] = \sigma(t_0, X_0), \quad \sigma[t_0, t_1; X_0, X_1] = \frac{\sigma[t_1; X_1] - \sigma[t_0; X_0]}{t_1 - t_0}$$

$$\sigma[t_0, \cdots, t_k; X_0, \cdots, X_k] = \frac{\sigma[t_1, \cdots, t_k; X_1, \cdots, X_k] - \sigma[t_0, \cdots, t_{k-1}; X_0, \cdots, X_{k-1}]}{t_k - t_0}$$

Accordingly the statistical interpolating polynomials of for $\hat{\mu}_n$ and $\hat{\sigma}_n$ are defined by degree n

$$\hat{\mu}_n(t) = \sum_{k=0}^{n} \mu[t_0, \cdots, t_k; X_0, \cdots, X_k] \prod_{j=i}^{k-1} (t - t_j)$$

$$\hat{\sigma}_n(t) = \sum_{k=0}^{n} \sigma[t_0, \cdots, t_k; X_0, \cdots, X_k] \prod_{j=i}^{k-1} (t - t_j)$$

3. Construction of a linear statistical interpolation prediction process

Let $T \in [0, \infty)$ be a finite time horizon. For simplicity we consider the case in which the data (t_0, X_0) and (t_1, X_1) are given. Let $\hat{\mu}$ and $\hat{\sigma}$ be the linear interpolation of μ and σ at the points t_0 and t_1 respectively;

$$\hat{\mu}(t; X_t) = \mu(t_0, X_0) + \mu[t_0, t_1; X_0, X_1](t - t_0), \tag{3.1}$$

$$\hat{\sigma}(t; X_t) = \sigma(t_0, X_0) + \sigma[t_0, t_1; X_0, X_1](t - t_0), \tag{3.2}$$

where

$$\mu[t_0, t_1; X_0, X_1] = \frac{\mu(t_1, X_1) - \mu(t_0, X_0)}{t_1 - t_0},$$

$$\sigma[t_0, t_1; X_0, X_1] = \frac{\sigma(t_1, X_1) - \sigma(t_0, X_0)}{t_1 - t_0}$$

are the first order divided difference of μ and σ respectively. Note that all processes $\mu, \sigma, \hat{\mu}$ and $\hat{\sigma}$ are progressively measurable with respective to $\mathcal{F}_{t_1}$, $t_1 \in (0,T]$. Throughout it is assumed that μ and σ are L^2-processes;

$$E|X_t|^2 < \infty, \; E\mu(t,X_t)^2 < \infty, \quad E\sigma(t,X_t)^2 < \infty, \quad t \in [0,T].$$

For details of the algebraic properties of the divided differences, we refer to [14]. For $t \in (t_1, T]$, we define a linear interpolating prediction process $\hat{X}_t$ by

$$\hat{X}_t := X_1 + \int_{t_1}^{t} \hat{\mu}(s; X_s)ds + \int_{t_1}^{t} \hat{\sigma}(s; X_s)dW_s. \tag{3.3}$$

Then we have the following convergence result for $\hat{X}_t$. We are concerned with only the local mean square error. Throughout the constant C denotes a generic constant.

THEOREM 3.1. *Assume that*

(A1) $|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le K|x-y|$,
(A2) $|\mu(t,x)|^2 + |\sigma(t,x)|^2 \le K^2(1+|x|^2)$,
(A3) $|\mu(t,x) - \mu(s,x)|^2 + |\sigma(t,x) - \sigma(s,x)|^2 \le K|t-s|^\alpha, \quad 0 < \alpha \le 1.$

Then we have

$$\sup_{t \in (t_1, t_1+h)} E|X_t - \hat{X}_t|^2 \le Ch^{1+\alpha}, \quad 0 < \alpha \le 1$$

Proof. We can rewrite the stochastic differential equation (1.1) as

$$X_t = X_t + \int_{t_1}^{t} \mu(s, X_s)ds + \int_{t_1}^{t} \sigma(s, X_s)dW_s. \tag{3.4}$$

Subtraction (3.3) from the equation (3.4) gives

$$\begin{aligned} X_t - \hat{X}_t &= \int_{t_1}^{t} [\mu(s,X_s) - \hat{\mu}(s,X_s)]ds + \int_{t_1}^{t} [\sigma(s,X_s) - \hat{\sigma}(s,X_s)]dW_s \\ &= \int_{t_1}^{t} \mu[t_0,t_1,s;X_0,X_1](s-t_0)(s-t_1)ds \\ &\quad + \int_{t_1}^{t} \sigma[t_0,t_1,s;X_0,X_1](s-t_0)(s-t_1)dW_s \\ &= \int_{t_1}^{t} (\mu(s;X_s) - \mu(t_1;X_1))ds - \int_{t_1}^{t} \frac{\mu(t_1)-\mu(t_0)}{t_1-t_0}(s-t_1)ds \\ &\quad + \int_{t_1}^{t} (\sigma(s;X_s) - \sigma(t_1;X_1))dW_s \\ &\quad - \int_{t_1}^{t} \frac{\sigma(t_1;X_1)-\sigma(t_0;X_0)}{t_1-t_0}(s-t_1)dW_s. \end{aligned}$$

Squaring both sides yields

$$\begin{aligned} |X_t - \hat{X}_t|^2 \quad \leq \quad & 2\Big\{ \Big(\int_{t_1}^{t} [\mu[s;X_s] - \mu[t_1;X_1]]ds \Big)^2 \\ + \quad & \Big(\int_{t_1}^{t} \frac{\mu[t_1;X_1] - \mu[t_0;X_0]}{(t_1-t_0)}(s-t_1)ds \Big)^2 \\ + \quad & \Big(\int_{t_1}^{t} [\sigma[s;X_s] - \sigma[t_1;X_1]dW_s \Big)^2 \\ + \quad & \Big(\int_{t_1}^{t} \frac{\sigma[t_1;X_1] - \sigma[t_0;X_0]}{(t_1-t_0)}(s-t_1)dW_s \Big)^2 \Big\} \end{aligned}$$

Since $E(X_t - X_s)^2 \leq C|t-s|$, $\quad 0 < \alpha \leq 1$, and

$$|\mu[s;X_s] - \mu[t_1;X_1]| \leq |\mu[s;X_s] - \mu[t_1;X_s]| + |\mu[t_1;X_s] - \mu[t_1;X_1]|$$

and so by (A.1) and (A.3), we see that

$$\begin{aligned} & E(\mu[s;X_s] - \mu[t_1;X_1])^2 \\ \leq \ & \mathrm{CE}[(\mu[\mathrm{s};\mathrm{X_s}] - \mu[\mathrm{t_1};\mathrm{X_s}])^2 + (\mu[\mathrm{t_1};\mathrm{X_s}] - \mu[\mathrm{t_1};\mathrm{X_1}])^2] \\ \leq \ & C[(s-t_1)^\alpha + E(X_s - X_1)^2] \\ \leq \ & C(s-t_1)^\alpha. \end{aligned}$$

It follows that

$$\begin{aligned} E\left(\textstyle\int_{t_1}^{t} [\mu[s;X_s] - \mu[t_1;X_1]]ds \right)^2 \quad &\leq \quad \textstyle\int_{t_1}^{t} E(\mu[s;X_s] - \mu[t_1;X_1])^2 ds \int_{t_1}^{t} ds \\ &\leq \quad C(t-t_1)^{\alpha+2} \leq C(t_1-t_0)^{\alpha+2}. \end{aligned}$$

Similarly

$$
\begin{aligned}
& E\left(\int_{t_1}^t \tfrac{\mu[t_1;X_1]-\mu[t_0;X_0]}{(t_1-t_0)}(s-t_1)ds\right)^2 \\
\leq\ & C\int_{t_1}^t E\left(\frac{\mu[t_1;X_1]-\mu[t_0;X_0]}{t_1-t_0}\right)^2 (s-t_1)^2 ds\cdot\int_{t_1}^t ds \\
\leq\ & C(t_1-t_0)^{\alpha-2}(t-t_1)^4 \\
\leq\ & C(t_1-t_0)^{\alpha+2}.
\end{aligned}
$$

Following the same procedure as above we see that, by the It's Isometry formula,

$$
\begin{aligned}
& E\left(\int_{t_1}^t ([\sigma[s;X_s]-\sigma[t_1;X_1]]dw_s\right)^2 \\
=\ & \int_{t_1}^t E(\sigma[s;X_s]-\sigma[t_1;X_1])^2 ds \\
\leq\ & C\int_{t_1}^t (s-t_1)^\alpha ds \leq C(t-t_1)^{\alpha+1} \leq C(t_1-t_0)^{\alpha+1},
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left(\int_{t_1}^t \tfrac{\sigma[t_1;X_1]-\sigma[t_0;X_0]}{t_1-t_0}(s-t_1)dw_s\right)^2 \\
=\ & \int_{t_1}^t E\left(\frac{\sigma[t_1;X_1]-\sigma[t_0;X_0]}{t_1-t_0}\right)^2 (s-t_1)^2 ds \\
\leq\ & C(t_1-t_0)^{\alpha-2}\int_{t_1}^t (s-t_1)^2 ds \leq C(t_1-t_0)^{\alpha+1}
\end{aligned}
$$

Hence we have

$$E(X_t-\hat{X}_t)^2 \leq Ch^{\alpha+1}, \quad t\in(t_1,t_1+h), \quad h:=t_1-t_0.$$

This completes the proof. □

Now we consider a statistical linear prediction of X_t. Let us define $\tilde{X}_{t.}$ as follows;

(3.5)

$$
\begin{aligned}
\tilde{X}_t \ := \ & X_{t_1}+\mu[t_0,X_0]\triangle t+\mu[t_0,t_1;X_0,X_1](t_1-t_0)\triangle t \\
& +\sigma[t_0,t_1]\triangle W_t+\sigma[t_0,t_1;X_0,X_1](t_1-t_0)\triangle W_t,
\end{aligned}
$$

where $\triangle t = t-t_1$ and $\triangle W_t = W_t - W_{t_1}, \quad t\in(t_1,t_1+h)$.

Note that $\tilde{X}_t$ is obtained by discretization of $\int_{t_1}^t \hat{\mu}(s;X_s)ds$ and $\int_{t_1}^t \hat{\sigma}(s;X_s)dW_s$. Clearly $\tilde{X}_t$ is a L^2-process i.e. $E|\tilde{X}_t|^2<\infty$.

Lemma 1. *Assume (A.1), (A.2) and (A.3). Then*

$$\sup_{t_1<t<t_1+h} E(|\hat{X}_t-\tilde{X}_t|^2) \leq Ch^{\alpha+1}.$$

Proof. Write $\hat{X}_t$ and $\tilde{X}_t$ as

$$\hat{X}_t = X_{t_1} + \int_{t_1}^{t} \hat{\mu}(s, X_s)ds + \int_{t_1}^{t} \hat{\sigma}(s, X_s)dW_s, \tag{3.6}$$

$$\begin{aligned} \tilde{X}_t = X_{t_1} + \mu[t_0; X_0]\triangle t + \mu[t_0, t_1; X_0, X_1](t_1 - t_0)\triangle t \\ + \sigma[t_0, X_0]\triangle W_t + \sigma[t_0, t_1; X_0, X_1](t_1 - t_0)\triangle W_t. \end{aligned} \tag{3.7}$$

Subtraction (3.7) from (3.6) yields

$$\hat{X}_t - \tilde{X}_t = \int_{t_1}^{t} \mu[t_0, t_1; X_0, X_1](s - t_1)ds + \int_{t_1}^{t} \sigma[t_0, t_1; X_0, X_1](s - t_1)dW_s.$$

Squaring both sides yields

$$\begin{aligned} |\hat{X}_t - \tilde{X}_t|^2 \quad \leq \quad & 2\Big[\Big(\int_{t_1}^{t} \mu[t_0, t_1; X_0, X_1](s - t_1)ds\Big)^2 \\ + \quad & \Big(\int_{t_1}^{t} \sigma[t_0, t_1; X_0, X_1](s - t_1)dw_s\Big)^2\Big] \end{aligned}$$

Note that by (A.3)

$$\begin{aligned} E\int_{t_1}^{t} \mu[t_0, t_1; X_0, X_1]^2 ds \quad &= \quad \int_{t_1}^{t} E\left[\frac{\mu(t_1; X_1) - \mu[t_0; X_0]}{t_1 - t_0}\right]^2 ds \\ &\leq \quad C\int_{t_1}^{t} (t_1 - t_0)^{\alpha-2} ds \leq Ch^{\alpha-1}. \end{aligned}$$

According to Hölder's inequality,

$$\begin{aligned} & E\left(\int_{t_1}^{t} \mu[t_0, t_1; X_0, X_1](s - t_1)ds\right)^2 \\ \leq \quad & \int_{t_1}^{t} E\mu[t_0, t_1; X_0, X_1]^2 ds \int_{t_1}^{t} (s - t_1)^2 ds \\ \leq \quad & Ch^{\alpha+2}. \end{aligned}$$

By Ito isometry formula and (A.3)

$$\begin{aligned} & E\left(\int_{t_1}^{t} \sigma[t_0, t_1; X_0, X_1](s - t_1)dW_s\right)^2 \\ = \quad & \int_{t_1}^{t} E\sigma[t_0, t_1; X_0, X_1]^2 (s - t_1)^2 ds \\ = \quad & \int_{t_1}^{t} E\left(\frac{\sigma[t_1; X_1] - \sigma[t_0; X_0]}{t_1 - t_0}\right)^2 (s - t_1)^2 ds \\ \leq \quad & Ch^{\alpha+1}. \end{aligned}$$

This implies that

$$\sup_{t_1<t<t_1+h} E|\hat{X}_t - \tilde{X}_t|^2 \le Ch^{\alpha+1},$$

which completes the proof. □

THEOREM 3.2. *Assume (A.1), (A.2) and (A.3). Then*

$$\sup_{t_1<t<t_1+h} E|X_t - \tilde{X}_t|^2 \le Ch^{\alpha+1}, \quad 0<\alpha\le 1$$

Proof. It is straightforward from the inequality

$$|X_t - \tilde{X}_t| \le |X_t - \hat{X}_t| + |\hat{X}_t - \tilde{X}_t|$$

and the Lemma 1.

Under the strong condition $\mu, \sigma \in C^2$, Milstein scheme [15] has the order $O(h^3)$ of local error. The proposed method has the order $O(h^{\alpha+1})$, $0<\alpha\le 1$, of local error under the weaker conditions than those of the Milstein scheme. □

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