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CHAOTIC BEHAVIOUR OF CHAIN COMPONENTS IN BISHADOWING SYSTEMS

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ABSTRACT. In this paper we show that if a dynamical system φ has bishadowing and cyclically bishadowing properties on the chain recurrent set $CR(\varphi)$ then all nearby continuous perturbations of φ behave chaotically on a neighborhood of each chain component of φ whenever it has a fixed point. This is a generalization of the results obtained by Diamond et al.([3]).

1. Introduction

Fundamental results in the theory of differentiable dynamics due to Anosov, Palis, Smale and others concern the occurence of complicated dynamical behaviour enjoyed by invariant hyperbolic sets and transverse homoclinic trajectories. Basic technical tools are symbolic dynamics with a conjugacy between the original system and a shift operator on sequences of symbols being sought, and shadowing.

The concepts of chaos are various, and we can find the various definitions of chaos([6], [7], [8]). Especially Daalderop and Fokkink showed that chaotic maps in the sense of Devaney are dense in the space of measure-preserving homeomorphisms([1]). Important attributes of chaotic behaviour include sensitive dependence on initial conditions, an abundance of unstable periodic trajectories and an irregular mixing effect. Symbolic dynamics allows a more exact formulation of this last characteristic.

Recently, Diamond et al.([3]) introduced a new concept of chaotic behaviour that captures the essential features of the classical definition

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of Li and Yorke, Smale and others which are based on symbolic dynamics([2], [6], [8]), and showed that all nearby continuous perturbations of a bishadowing and cyclically bishadowing system with a homoclinic trajectory behave chaotically in the sence of this definiton([2], [3]).

In this paper we show that if a dynamical system φ has bishadowing and cyclically bishadowing properties on the chain recurrent set $CR(\varphi)$ then all nearby continuous perturbations of φ behave chaotically, in the sense of Diamond et al., on a neighborhood of each chain component of φ whenever it has a fixed point. This is a generalization of the results given in [3].

2. Preliminaries

Let M be a compact metric space with a metric d. Consider a homeomorphism $\varphi : M \to M$ and the corresponding discrete-time dynamical system generated by φ. Let $\mathcal{H}(M)$ be the space of all discrete-time dynamical systems on M with the C^0 topology induced by the C^0 metric ϱ_0 ; for any $\varphi, \psi \in \mathcal{H}(M)$,

$$\varrho_0\ (\varphi,\ \psi)\ =\ \max_{x\in M}\ \{\ d(\varphi(x),\ \psi(x))\ ,\ d((\varphi^{-1}(x),\ \psi^{-1}(x))\ \}$$

A trajectory of this system is an infinite sequence $\mathbf{x} = \{x_n \in M : n \in \mathbb{Z}\}$ satisfying

$$x_{n+1} = \varphi(x_n), \quad n \in \mathbb{Z}.$$

Let $\mathrm{Tr}(\varphi)$ denote the totality of trajectories of the dynamical system generated by φ.

For any fixed positive integer m, we let $B(m)$ denote the set of all bi-infinite sequences $\mathbf{b}$ where $\mathbf{b} = \{b_n : n \in \mathbb{Z}\}$ with $b_n \in \{1, 2, \cdots, m\}$. First we recall the definition of chaos which is given in [3].

Definition 1. For any $\varepsilon > 0$ and a positive integer k, we say that a dynamical system φ on M is (ε, k) *-chaotic on a neighborhood of a compact subset* Y *of* M if for each finite subset $\{w_1, \cdots, w_m\}$ of Y with $d(w_i, w_j) \geq 2\varepsilon$ for all $i \neq j$ there exists a mapping $T : B(m) \to \mathrm{Tr}(\varphi)$ such that

S1. For each $\mathbf{b} \in B(m)$ the trajectory $\mathbf{x} = T(\mathbf{b})$ of φ satisfies $x_{kj} \in U_\varepsilon(w_{b_j})$ for all $i \in \mathbb{Z}$ where $U_\varepsilon(w_{b_j}) = \{y \in M : d(y, w_{b_i}) < \varepsilon\}$;

S2. The mapping $\mathbf{b} \longmapsto T(\mathbf{b})$ is a shift invariant in the sense that a 1-shift Sh^1 of a sequence $\mathbf{b} \in B(m)$ induces a k-shift Sh^k of the corresponding trajectory $T(\mathbf{b})$;

S3. If a sequence $\mathbf{b} \in B(m)$ is periodic with minimum period p, then the corresponding trajectory $T(\mathbf{b})$ is periodic with minimum period kp ;

S4. For each $\delta > 0$ there exists an uncountable subset $B_0(\delta)$ of $B(m)$ such that

$$\lim_{n\to\infty} \sup d(T(\mathbf{a})_n, T(\mathbf{b})_n) \ \geq \ \frac{1}{2}\varepsilon \qquad \text{and}$$
$$\lim_{n\to\infty} \inf d(T(\mathbf{a})_n, T(\mathbf{b})_n) \ < \ \delta \ ,$$

for all $\mathbf{a}, \mathbf{b} \in B(m)$ with $\mathbf{a} \neq \mathbf{b}$.

For any $\delta > 0$ a δ-pseudotrajectory of a dynamical system φ on M is a sequence $\mathbf{x} = \{x_n \in M : n \in \mathbb{Z}\}$ satisfying

$$d(x_{n+1}, \varphi(x_n)) < \delta, \quad n \in \mathbb{Z}.$$

Usually, a pseudotrajectory is considered as the result of application of a numerical method to our dynamical system φ. In this case, the value δ measures one-step errors of the method and round-off errors. The notion of a pseudotrajectory plays an important role in the general qualitative theory of dynamical systems. It is used to define some types of invariant sets such as chain recurrent set([8], [9]).

We say that a pseudotrajectory $\mathbf{x} = \{x_n\}$ is *ε-shadowed by a point* $x \in M$ if

$$d(\varphi^n(x), x_n) < \varepsilon, \quad n \in \mathbb{Z}$$

holds. A dynamical system φ on M is said to be *shadowing* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-pseudotrajectory of φ is ε-shadowed by a point in M.

Definition 2. For any $\alpha > 0$ and $\beta > 0$, a dynamical system φ on M is said to be *(α, β)-bishadowing on a subset Y of M* if for any δ-pseudotrajectory $\mathbf{y} = \{y_n\}$ in Y of φ with $0 \leq \delta \leq \beta$ and any $\psi \in \mathcal{H}(M)$ with

$$\varrho_0 \ (\varphi, \ \psi) + \delta \leq \beta$$

there exists a trajectory $\mathbf{x} = \{x_n\}$ of ψ such that

$$d(x_n, y_n) \leq \alpha(\delta + \varrho_0 \ (\varphi, \ \psi))$$

for all $n \in \mathbb{Z}$.

REMARK 3. By inverse shadowing we mean that for every sufficiently small continuous perturbations of the original system and for every trajectory of the original system there exists a trajectory of the perturbed system which approximates this particular trajectory of the original system([5]).

We can easily see that if a dynamical system φ has bishadowing property then it has both shadowing property and inverse shadowing property.

A trajectory $\mathbf{x} = \{x_n\}$ of ψ is said to be *a cycle of period* p if $x_{n+p} = x_n$ for all $n \in \mathbb{Z}$; and *a δ-pseudocycle of period* p if

$$d(x_{n+1}, \varphi(x_n)) \leq \delta \quad \text{and} \quad d(\varphi(x_{p-1}), x_0) \leq \delta$$

for all $n \in \mathbb{Z}$.

DEFINITION 4. A dynamical system φ on M is said to be (α, β)-*cyclically bishadowing on a subset* Y *of* M if for any δ-pseudocycle $\mathbf{y} = \{y_n\}$ in Y of φ with $0 \leq \delta \leq \beta$ and any $\psi \in \mathcal{H}(M)$ with

$$\varrho_0\ (\varphi,\ \psi) + \delta \leq \beta$$

there exists a proper cycle $\mathbf{x} = \{x_n\}$ of ψ of period N equal to that of $\mathbf{y}$ such that

$$d(x_n, y_n) \leq \alpha(\delta + \varrho_0\ (\varphi,\ \psi))$$

for $n = 0, \cdots, p-1$.

DEFINITION 5. A trajectory $\mathbf{x} = \{x_n\}$ of a dynamical system ψ on M is called *homoclinic* if its elements are not all identical and there exists a point $x_* \in M$ such that

$$\lim_{n \to \infty} x_n = \lim_{n \to -\infty} x_n = x_* \ .$$

3. Main results

The following theorem obtained by Diamond et al. in [3] means that all nearby continuous perturbations of a bishadowing and cyclically bishadowing system with a homoclinic trajectory behave chaotically.

THEOREM 6. *Let* $\mathbf{x} = \{x_n\}$ *be a homoclinic trajectory of a dynamical system* φ *on* M. *Suppose* φ *is both* (α, β)*-bishadowing and* (α, β)*-cyclically bishadowing on the set* $\mathbf{x} = \{x_n\}$, *and let* $\varepsilon > 0$ *be arbitrary. Define*

$$r(\varepsilon) = \frac{1}{2}\min\{\beta, \ \frac{\varepsilon}{\alpha}\} \quad \text{and}$$
$$k(\varepsilon) = 2\max\{m : \exists i_0 \in \mathbb{Z} \text{ with } d(x_i, x_*) \geq r(\varepsilon), \ i = i_0, \cdots, i_0 + m\}.$$

Then every dynamical system ψ *with* $\varrho_0(\varphi, \psi) < r(\varepsilon)$ *is* (ε, k)*-chaotic on a neighborhood of* $\mathbf{x}$ *for any* $k \geq k(\varepsilon)$.

A point $x \in M$ is said to be ε*-chain recurrent for* φ if there exists a finite ε-pseudotrajectory $\{x_0, x_1, \cdots, x_n\}$ of φ with $x_0 = x_n = x$, that is connecting x with itself. A point $x \in M$ is called *chain recurrent for* φ if for any $\varepsilon > 0$ there exists a finite ε-pseudotrajectory connecting x with itself. Let $CR(\varphi)$ denote the set of all chain recurrent points of φ. Note that the chain recurrent set $CR(\varphi)$ is compact and φ-invariant.

We define a relation $\sim$ on $CR(\varphi)$ by $x \sim y$ if for any $\varepsilon > 0$ there exist two finite ε-pseudotrajectories $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x}$ is connecting x with y and $\mathbf{y}$ is connecting y with x. Two such points are called *chain equivalent.* It is clear that this is an equivalence relation. The equivalence classes are called *the chain components of* φ.

Now we show that if a dynamical system φ has bishadowing and cyclically bishadowing properties on the chain recurrent set $CR(\varphi)$ then all nearby continuous perturbations of φ behave chaotically on a neighborhood of each chain component of φ whenever it has a fixed point. The proof relies on techniques elaborated by Diamond et al.([4]).

Let C be a chain component of φ, and x and y be any two points in C. For any $\delta > 0$, we denote $\mathrm{Tr}(x, y, \delta)$ the set of all finite δ-pseudotrajectories in $CR(\varphi)$ from x to y. Note that we can choose δ-pseudotrajectory from x to y which belongs to $CR(\varphi)$. For any $\mathbf{x} \in \mathrm{Tr}(x, y, \delta)$, we let

$$l(\mathbf{x}) = \text{ the cardinal number of the set } \mathbf{x}, \quad \text{and}$$
$$L_\delta(x, y) = \inf \ \{l(\mathbf{x}) : \mathbf{x} \in \mathrm{Tr}(x, y, \delta)\}.$$

LEMMA 7. *Let* $C \subset M$ *be a chain component of* φ. *For any* $\delta > 0$, *we can choose a positive integer* $N = N(\delta)$ *such that*

$$\sup\{L_\delta(x, y) : x, y \in C\} \leq N.$$

Proof. Suppose not. Then there exists $\delta_0 > 0$ such that for any $N > 0$,

$$\sup\{L_\delta(x, y) : x, y \in C\} > N.$$

Hence we can select $(x_n, y_n) \in C \times C$ satisfying

$$L_{\delta_0}(x_n, y_n) \geq n \text{ and } L_{\delta_0}(x_n, y_n) \leq L_{\delta_0}(x_{n+1}, y_{n+1})$$

for all $n = 1, 2, \cdots$. Since C is compact, we may choose subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, which are convergent ; say

$$\lim_{k\to\infty} x_{n_k} = x \text{ and } \lim_{k\to\infty} y_{n_k} = y.$$

Since C is compact, x and y belong to C. To complete the proof, it is enough to show that $L_{\delta_0}(x, y) = \infty$. Suppose that $L_{\delta_0}(x, y)$ is finite ; say $L_{\delta_0}(x, y) = L$. Then there exists a δ_0-pseudotrajectory $\mathbf{x} = \{w_1, \cdots, w_L\}$ in $CR(\varphi)$ satisfying $w_1 = x$ and $w_L = y$. Then we can choose a sufficiently large integer $\alpha > L$ such that

$$\tilde{\mathbf{x}} = \{x_\alpha, w_2, \cdots, w_{L-1}, y_\alpha\}$$

becomes a δ_0-pseudotrajectory in $CR(\varphi)$. Hence we have

$$\tilde{\mathbf{x}} \in \mathrm{Tr}(x_\alpha, y_\alpha, \delta_0), \quad l(\tilde{\mathbf{x}}) = L \text{ and } L_{\delta_0}(x_\alpha, y_\alpha) \leq L.$$

However this contradicts to the fact that

$$L_{\delta_0}(x_\alpha, y_\alpha) \geq \alpha > L. \qquad \square$$

THEOREM 8. *Let C be a chain component of a dynamical system φ on M. Suppose φ is both (α, β)-bishadowing and (α, β)-cyclically bishadowing on the chain recurrent set $CR(\varphi)$, and let $\varepsilon > 0$ be arbitrary with $\varepsilon <$ diam C. Let $r(\varepsilon) = \min\{\varepsilon, \frac{\varepsilon}{2\alpha}, \frac{\beta}{2}\}$. Then every dynamical system ψ on M with $\varrho_0(\varphi, \psi) < r(\varepsilon)$ is (ε, k)-chaotic on a neighborhood of C for any $k \geq 2N(r(\varepsilon))$ if C has a fixed point, where $N(r(\varepsilon))$ is an integer corresponding to the number $r(\varepsilon)$ as in Lemma 7.*

Proof. Let $\varepsilon > 0$ be arbitrary with $\varepsilon <$ diam C, and let ψ be a dynamical system on M satisfying

$$\varrho_0(\varphi, \psi) < r(\varepsilon),$$

where $r(\varepsilon) = \min\{\varepsilon, \frac{\varepsilon}{2\alpha}, \frac{\beta}{2}\}$. Let $N(r(\varepsilon))$ be an integer corresponding to the number $r(\varepsilon)$ as in Lemma 7, and let $k > 0$ be a fixed integer with $k \geq 2N(r(\varepsilon))$. Let $\{w_1, \cdots, w_m\}$ be a finite subset of C with

$$d(w_i, w_j) \geq 2\varepsilon , \quad i \neq j .$$

Now we are going to construct a mapping $T : B(m) \to \mathrm{Tr}(\psi)$ which have the properties S1 $\sim$ S4 in Definition 1. For each sequence $\mathbf{b} = (b_n)$ in $B(m)$, we can associate a sequence

$$W(\mathbf{b}) = \{\cdots, w_{b_{-1}}, w_{b_0}, w_{b_1}, \cdots\}$$

in the product space $W^{\mathbb{Z}}$.

For each integer $i \in \mathbb{Z}$ we may find a $r(\varepsilon)$-pseudotrajectory $\mathbf{x}_i$ in $CR(\varphi)$ from w_{b_i} to $w_{b_{i+1}}$ with $l(\mathbf{x}_i) = k$. To show this, we let $p \in C$ be a fixed point of φ. Since

$$\sup\{L_{r(\varepsilon)}(x, y) : x, y \in C\} \leq N(r(\varepsilon)),$$

we can choose a $r(\varepsilon)$-pseudotrajectory $\mathbf{y}_0$ in $CR(\varphi)$ from w_{b_i} to p with $l(\mathbf{y}_0) \leq N(r(\varepsilon))$, and a $r(\varepsilon)$-pseudotrajectory $\mathbf{y}_1$ in $CR(\varphi)$ from p to $w_{b_{i+1}}$ with $l(\mathbf{y}_1) \leq N(r(\varepsilon))$. By connecting two pseudotrajectories $\mathbf{y}_0$ and $\mathbf{y}_1$, we can construct a $r(\varepsilon)$-pseudotrajectory $\mathbf{x}_i$ in $CR(\varphi)$ from w_{b_i} to $w_{b_{i+1}}$ with $l(\mathbf{x}_i) = k$.

Let $\mathbf{x}(\mathbf{b}) = \{\cdots, \mathbf{x}_{-1}, \mathbf{x}_0, \mathbf{x}_1, \cdots\} \overset{\text{say}}{=} \{x_n : n \in \mathbb{Z}\}$. Then $\mathbf{x}(\mathbf{b})$ is a $r(\varepsilon)$-pseudotrajectory of φ in $CR(\varphi)$. Since φ is (α, β)-bishadowing on $CR(\varphi)$ and

$$r(\varepsilon) + \varrho_0(\varphi, \psi) \leq 2r(\varepsilon) \leq \beta ,$$

there exists a trajectory $\mathbf{z}(\mathbf{b}) = \{z_n\}$ of ψ such that

$$d(x_n, z_n) < \alpha(r(\varepsilon) + \varrho_0(\varphi, \psi)) < 2\alpha r(\varepsilon) < \varepsilon$$

for all $n \in \mathbb{Z}$.

Let $\mathbf{b} = (b_n)$ be a periodic sequence in $B(m)$ with period p. Since φ is (α, β)-cyclically bishadowing on $CR(\varphi)$, by the same techniques as above, we can select a $r(\varepsilon)$-pseudotrajectory $\mathbf{x}(\mathbf{b})$ of φ in $CR(\varphi)$ with period pk and a proper cycle $\mathbf{z}(\mathbf{b}) = \{z_n\}$ of ψ with period pk and

$$d(x_n, z_n) < \varepsilon$$

for all $n = 0, 1, \cdots, pk-1$.

For each $\mathbf{b} \in B(m)$ the set $\mathcal{S}(\mathbf{b})$ of all trajectories $\mathbf{z}(\mathbf{b}) = \{z_n\}$ of ψ satisfying

$$d(x_{nk}, w_{b_n}) < \varepsilon\ , \quad n \in \mathbb{Z}$$

is not empty. Moreover the set $\mathcal{S}(\mathbf{b})$ contains a trajectory of ψ with period pk if $\mathbf{b}$ is a periodic sequence in $B(m)$ with period p.

By our construction, we can consider the totality $\mathcal{T}$ of the single valued mappings $\mathcal{S} : D(\mathcal{S}) \to \mathrm{Tr}(\psi)$ satisfying the conditions S1 $\sim$ S3 in Definition 1, where $D(\mathcal{S})$ is a subset of $B(m)$ such that $\mathcal{S}$ is defined on it. Consider the set $\mathcal{T}$ as being partially ordered by inclusion of the set corresponding graphs :

$$\mathrm{Gr}(\mathcal{S}) = \{(\mathbf{b},\ \mathcal{S}(\mathbf{b})) : \mathbf{b} \in D(\mathcal{S})\}.$$

Clearly every chain $\hat{\mathcal{S}}$ of $\mathcal{T}$ has an upper bound, and the graph of which is defined as the union $\cup_{\mathcal{S}\in\hat{\mathcal{S}}}\mathrm{Gr}(\mathcal{S})$. By the Zorn's Lemma, there exists a maximal element T in $\mathcal{T}$. Then we can see that $D(T) = B(m)$.

Suppose not. Then there exists $\mathbf{a} \in B(m) - D(T)$. If $\mathbf{a} = \mathrm{Sh}^i(\mathbf{c})$ for some positive integer i and some $\mathbf{c} \in D(T)$ then the mapping $\mathcal{S}_0$: $D(T) \cup \{\mathbf{a}\} \to \mathrm{Tr}(\psi)$ defined by

$$\mathcal{S}_0(\mathbf{b}) = \begin{cases} T(\mathbf{b}), & \text{if } \mathbf{b} \in D(T); \\ \mathrm{Sh}^{-ik}T(\mathbf{c}), & \text{if } \mathbf{b} = \mathbf{a} \end{cases}$$

satisfies conditions S1 $\sim$ S3 and $D(T) \subsetneqq D(\mathcal{S}_0)$, which contradicts the definition of T. Similarly if the sequence $\mathbf{a}$ cannot be represented as a shift of a sequence in $D(T)$ then we can construct a mapping $\mathcal{S}_0$: $D(T) \cup \{\mathbf{a}\} \to \mathrm{Tr}(\psi)$ satisfying conditions S1 $\sim$ S3. This means that $D(T) = B(m)$.

The fact that the mapping $T : B(m) \to \mathrm{Tr}(\psi)$ satisfies the condition S4 in Definition 1 follows from Lemma 8.1 in [3]. □

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