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ON THE LARGE AND SMALL INCREMENTS OF GAUSSIAN RANDOM FIELDS

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ABSTRACT. In this paper we establish limit theorems on the large and small increments of a two-parameter Gaussian random process on rectangles in the Euclidean plane via estimating upper bounds of large deviation probabilities on suprema of the two-parameter Gaussian random process.

1. Introduction and Results

The limit theorems on the increments of Wiener processes and Gaussian processes have been investigated in various directions by many authors [1∼3, 7∼10, 13∼17, 19, 20, 23∼29, 31]. Furthermore, the moduli of continuity for Gaussian processes and Ornstein-Uhlenbeck processes have been intensively studied recently by Csőrgö and Shao [11], Csáki and Csőrgö [5], Csáki et al. [6] and Csőrgö et al. [12]. Concerning the limit theorems for the increments of two-parameter Wiener processes and Gaussian processes on rectangles, we refer to Csőrgö and Révész [8], Lin [22], Kong [18], Csőrgö et al. [13] and Zhang [30], etc.

We are interested in studying limiting behaviors for the increments of a two-parameter Gaussian random process on rectangles in the Euclidean plane, whose increments are, specially, composed of mixed types that one side of the rectangle increases to infinity and the other side decreases to zero as time passes by infinitely. Thus, as time goes to infinity, each increment brings to both large and small increment for one-parameter Gaussian random process, respectively.

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The starting point of this work was some limit theorems on the increments of a two-parameter Wiener process in the books of Csőrgö and Révész [10], Lin and Lu [24]. Our results for the two-parameter Gaussian random process in this paper are different from those results in the previous books because the structures of the above two processes differ from each other.

To show the difference more concretely, we shall first quote one of the results in the books of Csőrgö and Révész [10], Lin and Lu [24]: Let a_T be a nondecreasing function of $T (0 < T < \infty)$ for which

(a) $0 < a_T \le T$,

(b) T/a_T is nondecreasing on T.

Let $R_T = R(a_T)$ be the set of rectangles

$$R = [x_1, x_2] \times [y_1, y_2] \quad (0 \le x_1 < x_2 \le \sqrt{T}, 0 \le y_1 < y_2 \le \sqrt{T})$$

for which $\lambda(R) = (x_2 - x_1)(y_2 - y_1) \le a_T$. Let $R_T^* = R^*(a_T) \subset R_T$ be the set of those elements R of R_T for which $\lambda(R) = a_T$. For a two-parameter Wiener process $\{W(x, y),\ 0 \le x, y < \infty\}$, define the Wiener measure of a rectangle $R = [x_1, x_2] \times [y_1, y_2]$ by

$$W(R) = W(x_2, y_2) - W(x_1, y_2) - W(x_2, y_1) + W(x_1, y_1).$$

THEOREM A. ([10], [24]) *Let $\{W(x, y),\ 0 \le x, y < \infty\}$ be a two-parameter Wiener process, and let a_T be a nondecreasing function of T satisfying above conditions* (a) *and* (b). *Then*

$$\limsup_{T\to\infty} \sup_{R\in R_T} \gamma_T |W(R)| = \limsup_{T\to\infty} \sup_{R\in R_T^*} \gamma_T |W(R)| = 1 \qquad \text{a.s.},$$

where $\gamma_T = \left(2a_T(\log(T/a_T) + \log\log T)\right)^{-1/2}$.

If a_T also satisfies the condition

(c) $\lim_{T\to\infty} \log(T/a_T)/\log\log T = \infty$,

then

$$\lim_{T\to\infty} \sup_{R\in R_T} \gamma_T |W(R)| = \lim_{T\to\infty} \sup_{R\in R_T^*} \gamma_T |W(R)| = 1 \qquad \text{a.s.}$$

For our purpose, let us introduce a limit theorem for the large increments of one-parameter Gaussian process and a modulus of continuity (i.e. a limit theorem for the small increments) of the Gaussian process:

Let $\{X(t),\ 0 \leq t < \infty\}$ be an almost surely continuous centered, one-parameter Gaussian process with $X(0) = 0$ and stationary increments

$$\sigma^2(|t-s|) := E\{X(t) - X(s)\}^2.$$

Assume that σ is a nondecreasing continuous, regularly varying function with exponent α at 0 and ∞ for some $0 < \alpha < 1$ and that σ satisfies $\int_0^\infty \sigma(e^{-y^2})\, dy < \infty$. Let a_T be a nondecreasing function of $T (0 < T < \infty)$ for which

(a) $0 < a_T \leq T$,
(b) T/a_T is nondecreasing on T,
(c) $\lim_{T\to\infty} \log(T/a_T)/\log\log T = \infty$.

THEOREM B. (large increment theorem) ([1], [7]) *Let $X(t)$ and a_T be as above statements. Assume also that for $t > 0$, either*

(d) *$\sigma^2(t)$ is concave*

or

(e) *$\sigma^2(t)$ is twice continuously differentiable which satisfies*

$$\left|\frac{d^2}{dt^2}\sigma^2(t)\right| \leq c\frac{\sigma^2(t)}{t^2} \quad \textit{for some} \quad c > 0.$$

Then

$$\begin{aligned}
&\lim_{T\to\infty} \sup_{0\leq t\leq T-a_T} \frac{|X(t+a_T) - X(t)|}{\{2(\log(T/a_T) + \log\log T)\}^{1/2}\sigma(a_T)} \\
&= \lim_{T\to\infty} \sup_{0\leq t\leq T-a_T} \sup_{0\leq s\leq a_T} \frac{|X(t+s) - X(t)|}{\{2(\log(T/a_T) + \log\log T)\}^{1/2}\sigma(a_T)} \\
&= 1 \qquad \text{a.s.}
\end{aligned}$$

THEOREM C. (modulus of continuity) [7] *Let $\{X(t),\ 0 \leq t \leq 1\}$ be an almost surely continuous, centered Gaussian process with $X(0) = 0$ and $E\{X(t+h) - X(t)\}^2 = C_0 h^{2\alpha} > 0$ for some $C_0 > 0$ and $0 < \alpha < 1$. Then*

$$\begin{aligned}
&\lim_{h\to 0} \sup_{0\leq t\leq 1-h} \frac{|X(t+h) - X(t)|}{\{2\log(1/h)\}^{1/2}\sigma(h)} \\
&= \lim_{h\to 0} \sup_{0\leq t\leq 1-h} \sup_{0<s\leq h} \frac{|X(t+s) - X(t)|}{\{2\log(1/h)\}^{1/2}\sigma(h)} = 1 \qquad \text{a.s.}
\end{aligned}$$

In this paper we are interested in combining Theorems B and C to get a mixed type's result on the two-parameter Gaussian process whose increments are defined on the rectangles of which one side increases to infinity and the other side decreases (to zero) at time passes by infinitely, whose results are different from Theorem A.

Throughout the paper we shall always assume the following statements: Let $\{X(x,y),\ (x,y)\in\mathbb{R}_+^2\}$ be an almost surely continuous, centered two-parameter Gaussian process with $X(0,0)=0$ on the probability space $(\Omega,\mathcal{S},P)$, where $\mathbb{R}_+^2=[0,\infty)\times[0,\infty)$. For two distinct points (x_1,y_1) and (x_2,y_2) in $\mathbb{R}_+^2$, let $X(x,y)$ have the stationary increments

$$E\{X(x_1,y_1)-X(x_2,y_2)\}^2=\sigma^2\big(\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\big),$$

where $\sigma(t), t>0$, is a nondecreasing continuous and regularly varying function with exponent α at 0 and ∞ for some $0<\alpha<1$, i.e.

$$\sigma(t)=t^\alpha l(t),\quad t>0,\tag{1.1}$$

where $l(t)$ is a slowly varying function with exponent α at 0 and ∞. Further assume that there exists $c_0>0$ such that

$$\frac{d\sigma^2(x)}{dx}\le c_0\frac{\sigma^2(x)}{x},\quad x>0.\tag{1.2}$$

Then the process $\{X(x,y),\ (x,y)\in\mathbb{R}_+^2\}$ is a generalization of the two-parameter fractional Lévy Brownian motion with $\sigma^2(t)=t^{2\alpha}$ (cf. [21]).

Let us consider the rectangle $R:=R(s,t,u,v):=[s,s+t]\times[u,u+v]\subset\mathbb{R}_+^2$ for all $s,u\ge 0$ and $t,v>0$. Define the increment $X(R)$ on R by

$$\begin{aligned}X(R)&:=X(R(s,t,u,v))\\&:=X(s+t,u+v)-X(s,u+v)-X(s+t,u)+X(s,u).\end{aligned}$$

Using the relation $ab=\frac{1}{2}\big(a^2+b^2-(a-b)^2\big)$, it is easy to see that the standard deviation of $X(R)$ has the translation invariant with respect to s and u, that is,

$$\begin{aligned}S(t,v)&:=\big\{E(X(R(s,t,u,v))^2\big\}^{1/2}\\&=\big\{E\big(X(R(s+h_1,t,u+h_2,v))\big)^2\big\}^{1/2}\\&=\{2(\sigma^2(t)+\sigma^2(v)-\sigma^2(\sqrt{t^2+v^2}))\}^{1/2}>0\end{aligned}\tag{1.3}$$

for all $h_1, h_2 \geq 0$.

For $0 < T < \infty$, let A_T and a_T be nondecreasing functions, and B_T and b_T be nonincreasing functions of T for which

(i) $0 < a_T \leq A_T$ and $0 < b_T \leq B_T$,

(ii) $\limsup_{T\to\infty} a_T b_T < \infty$,

(iii) $\lim_{T\to\infty} a_T = \infty$.

For convenience, we denote:

$$G_T = \frac{A_T B_T}{{b_T}^4}, \quad \beta_T = \left\{2(\log G_T + \log\log A_T + \log|\log B_T|)\right\}^{1/2},$$

where $\log x = \ln(x \vee 1)$ and $m \vee n = \max\{m, n\}$. Set

$$D(a_T, b_T) = \sup_{0\leq s\leq A_T} \sup_{0\leq u\leq B_T} \frac{|X(R(s, a_T, u, b_T))|}{S(a_T, b_T)\beta_T},$$

$$D^*(a_T, b_T) = \sup_{0\leq s\leq A_T} \sup_{0\leq t\leq a_T} \sup_{0\leq u\leq B_T} \sup_{0\leq v\leq b_T} \frac{|X(R(s, t, u, v))|}{S(a_T, b_T)\beta_T}.$$

The main results are as follows:

THEOREM 1.1. *Let a_T and b_T satisfy above conditions* (i)~(iii) *and*
(iv) $\lim_{T\to\infty} \log(G_T b_T^2)/(\log\log A_T + \log|\log B_T|) = \infty$.
Then we have

$$\limsup_{T\to\infty} D^*(a_T, b_T) \leq 1 \qquad \text{a.s.} \tag{1.4}$$

THEOREM 1.2. *In addition, let the following conditions be satisfied:*
(v) *there exists a positive constant c_1 such that*

$$\left|\frac{d^2\sigma^2(x)}{dx^2}\right| \leq c_1 \frac{\sigma^2(x)}{x^2}, \qquad x > 0,$$

(vi) $|\log(A_T B_T)/\log b_T| \to \infty$ *and* $B_T/b_T \to \infty$ *as* $T \to \infty$.
Then we have

$$\liminf_{T\to\infty} D(a_T, b_T) \geq 1 \qquad \text{a.s.} \tag{1.5}$$

Combining Theorems 1.1 and 1.2, we immediately obtain the following limit theorem:

COROLLARY 1.3. *Under the assumptions of Theorem 1.2, we have*

$$\lim_{T\to\infty} D^*(a_T, b_T) = \lim_{T\to\infty} D(a_T, b_T) = 1 \qquad \text{a.s.}$$

REMARK. From the proofs of Theorems 1.1 and 1.2 we shall see that conditions (1.2) and (v) can be repalced by

$$(1.2)' \qquad \frac{\sigma^2(x_2) - \sigma^2(x_1)}{x_2 - x_1} \le c_0 \frac{\sigma^2(x_2)}{x_1} \qquad \text{for all large } x_1 < x_2$$

and

(v)′

$$\left|\frac{\sigma^2(x_3) - 2\sigma^2(x_2) + \sigma^2(x_1)}{(x_3 - x_2)(x_2 - x_1)}\right| \le c_1 \frac{\sigma^2(x_3)}{x_1^2} \qquad \text{for all large } x_1 < x_2 < x_3,$$

respectively.

EXAMPLE 1.4. Let $\{X(x,y) : 0 \le x < \infty, 0 \le y < \infty\}$ be a two-parameter Lévy Brownian motion with $\alpha = 1/2$. For $T > e$, let $A_T = T, a_T = 2(\log T)^2, B_T = (\log T)^{-1}$ and $b_T = (\log T)^{-2}$. Then all conditions of Corollary 1.3 are satisfied. Thus, noting Lemma 2.1 below, we have

$$\lim_{T\to\infty} \sup_{0\le s\le T} \sup_{0\le u\le(\log T)^{-1}} (\log T)^{1/2} \times |X(R(s, 2(\log T)^2, u, (\log T)^{-2}))| = 2 \quad \text{a.s.}$$

2. Proofs

The following Lemmas 2.1~2.5 are essential to prove the theorems.

LEMMA 2.1. *Let $\sigma(\cdot)$ be a nondecreasing continuous regularly varying function with exponent α $(0 < \alpha < 1)$ at ∞. Suppose that condition (1.2) is satisfied. Then*

$$(2.1) \qquad \lim_{\substack{t\to\infty \\ v/t\to 0}} \frac{S^2(t,v)}{\sigma^2(v)} = 2.$$

Proof.

$$\sigma^2\big(\sqrt{t^2+v^2}\big)-\sigma^2(t)=\int_t^{\sqrt{t^2+v^2}}\frac{d\sigma^2(x)}{dx}\,dx\le c_0\int_t^{\sqrt{t^2+v^2}}\frac{\sigma^2(x)}{x}\,dx$$
$$\le c_0\frac{\sigma^2\big(\sqrt{t^2+v^2}\big)}{t}\big(\sqrt{t^2+v^2}-t\big)$$
$$\le\frac{c_0}{2}\Big(\frac{v}{t}\Big)^2\sigma^2\big(\sqrt{t^2+v^2}\big),$$

which implies that

$$\frac{\sigma^2\big(\sqrt{t^2+v^2}\big)-\sigma^2(t)}{\sigma^2(v)}$$
$$\le\frac{c_0}{2}\Big(\frac{v}{t}\Big)^2\frac{(t^2+v^2)^\alpha}{v^{2\alpha}}\Big(\frac{l\big(\sqrt{t^2+v^2}\big)}{l\big(\sqrt{t^2+v^2}\big(v/\sqrt{t^2+v^2}\big)\big)}\Big)^2$$
$$\to 0\qquad\text{as}\quad t\to\infty\ \text{and}\ v/t\to 0.$$

Here we have used the following well known fact for a slowly varying function: for any function $h=h(x)\to 0$ as $x\to\infty$,

$$\lim_{x\to\infty}\frac{h^\varepsilon l(x)}{l(hx)}=\lim_{x\to\infty}\frac{h^\varepsilon l(hx)}{l(x)}=0,\quad\forall\,\varepsilon>0. \tag{2.2}$$ □

Let $\mathbb{D}=\{\mathbf{t}:\mathbf{t}=(t_1,\dots,t_N),a_i\le t_i\le b_i,i=1,2,\dots,N\}$ be a real N-dimensional time parameter space. We assume that the space $\mathbb{D}$ has the usual Euclidean norm $\|\cdot\|$, that is,

$$\|\mathbf{t}-\mathbf{s}\|^2=\sum_{i=1}^N(t_i-s_i)^2.$$

Let $\{X(\mathbf{t}):\mathbf{t}\in\mathbb{D}\}$ be a real-valued separable Gaussian process with $EX(\mathbf{t})=0$. Suppose that

$$0<\sup_{\mathbf{t}\in\mathbb{D}}EX(\mathbf{t})^2=:\Gamma^2<\infty$$

and

$$E\{X(\mathbf{t})-X(\mathbf{s})\}^2\le\varphi^2(\|\mathbf{t}-\mathbf{s}\|),$$

where $\varphi(\cdot)$ is a nondecreasing continuous function which satisfies $\int_0^\infty\varphi(e^{-y^2})dy<\infty$.

The following Lemma 2.2 is a version of Fernique's inequality ([14]), which is proved in [4].

LEMMA 2.2. *Let $\{X(\mathbf{t}) : \mathbf{t} \in \mathbb{D}\}$ be given as the above statements. Then, for $\lambda > 0$, $x \geq 1$ and $A > \sqrt{2N \log 2}$, we have*

$$P\Big\{\sup_{\mathbf{t}\in\mathbb{D}} X(\mathbf{t}) \geq x\Big(\Gamma + 2(\sqrt{2}+1)A\int_0^\infty \varphi(\sqrt{N}\,\lambda 2^{-y^2})\,dy\Big)\Big\}$$
$$\leq (2^N + B)\Big(\prod_{i=1}^{N}\Big(\frac{b_i - a_i}{\lambda} \vee 1\Big)\Big)e^{-x^2/2},$$

where $B = \sum_{n=1}^{\infty} \exp\Big\{-2^{n-1}\big(A^2 - 2N\log 2\big)\Big\}$.

LEMMA 2.3. *Let the condition* (v) *of Theorem 1.2 be given. Let a be a nonzero real number. For positive numbers N, m, r and θ, let $b = Nm\theta^{-r} - \theta^{-r} > 0$, $c = Nm\theta^{-r}$ and $d = Nm\theta^{-r} + \theta^{-r}$. Then, for some $C > 0$,*

$$\Big|\int_{\sqrt{a^2+c^2}}^{\sqrt{a^2+d^2}} d\sigma^2(x) - \int_{\sqrt{a^2+b^2}}^{\sqrt{a^2+c^2}} d\sigma^2(x)\Big| \leq C\frac{\sigma^2\big(\sqrt{a^2+d^2}\big)}{a^2+b^2}\theta^{-2r}.$$

The proof is similar to that of Lemma 2.5 in [3], and hence, is omitted.

The following lemma is a two-parametric modification of Corollary 4.2.4 in [20].

LEMMA 2.4. *Let $\{\xi_{ij} : i, j = 1, 2, \cdots, n\}$ be jointly standardized normal random variables with $\mathrm{Cov}(\xi_{ij}, \xi_{i'j'}) = \Lambda_{ij}^{i'j'}$ such that*

$$\delta := \max_{(i,j)\neq(i',j')} |\Lambda_{ij}^{i'j'}| < 1.$$

Then, for any real number u and integers $1 \leq l_1 < l_2 < \cdots < l_f \leq n$ and $1 \leq l_1 < l_2 < \cdots < l_g \leq n$ with $f, g \leq n$,

(2.3)
$$P\Big\{\max_{1\leq i\leq f}\max_{1\leq j\leq g} \xi_{l_i l_j} \leq u\Big\} \leq \big\{\Phi(u)\big\}^{fg}$$
$$+ c\sum_{(i,j)\neq(i',j')} |\lambda_{ij}^{i'j'}|\exp\Big(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\Big),$$

where $\lambda_{ij}^{i'j'} = \Lambda_{l_i l_j}^{l_{i'} l_{j'}}$ *and* $c = c(\delta)$ *is a constant independent of* n, u, f *and* g, *and we denote* $\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2) dy$.

Estimating the upper bound for the second term of the right hand side of (2.3), we have the following lemma, whose proof can be found in [3].

LEMMA 2.5. *Let* $\{\xi_{ij}\}, \delta, f, g$ *and* $\lambda_{ij}^{i'j'}$ *be as in Lemma 2.4. Assume that*

$$|\lambda_{ij}^{i'j'}| < (|i-i'||j-j'|)^{-\nu}, \qquad i \neq i',\ j \neq j',$$

and set $u = \sqrt{(2-\eta)\log(fg)}$, *where* ν *and* η *are positive constants such that* $0 < \eta < (1-\delta)\nu/\ (1+\nu+\delta)$. Then we have

$$\sum := \sum_{(i,j)\neq(i',j')} |\lambda_{ij}^{i'j'}| \exp\Big(-\frac{u^2}{1+|\lambda_{ij}^{i'j'}|}\Big) \leq c\,(fg)^{-\delta_0},$$

where $\delta_0 = \{\nu(1-\delta) - \eta(1+\delta+\nu)\}/\{(1+\nu)(1+\delta)\} > 0$, *and* c *is a positive constant independent of* n, f, *and* g.

In the sequel, c and $c_i (i = 1, 2, \cdots)$ denote positive constants, whose values are irrelevant.

Proof of Theorem 1.1. Without loss of generality, assume that $0 < b_T \leq B_T \leq 1$. For positive integers k, j, l and r, let

$$\mathbb{A}_{kjlr} = \{T : \theta^{k-1} \leq A_T < \theta^k,\ \theta^{j-1} \leq a_T < \theta^j,$$
$$\theta^{-l-1} \leq B_T < \theta^{-l},\ \theta^{-r-1} \leq b_T < \theta^{-r}\}$$

for any fixed $\theta > 1$. We always consider such k, j, l and r that $\mathbb{A}_{kjlr}$ is non-empty. By condition (ii), we have

$$\theta^{j-r} \leq c_2 \text{ and equivalently, } \ j - r \leq c_3. \tag{2.4}$$

The condition (iv) and (2.4) imply that for any $\mathcal{B} > 0$

$$\theta^{k-l+4r} > c(kl)^{\mathcal{B}} \theta^{r+j}, \tag{2.5}$$

provided k is large enough. Moreover,

$$\begin{aligned}\inf_{T\in \mathbb{A}_{kjlr}} \beta_T &\geq \Big\{2\log\Big(\frac{\theta^{k-l-2}}{\theta^{-4r}}\log\theta^{k-1}|\log\theta^{-l}|\Big)\Big\}^{1/2} \\ &\geq \theta^{-2}\big\{2\log\big(\theta^{k-l+4r}\log\theta^{k}|\log\theta^{-l}|\big)\big\}^{1/2} \\ &=: \theta^{-2}\beta_{klr}\end{aligned} \tag{2.6}$$

for all large k. And, by Lemma 2.1, for any $0<\varepsilon<1/2$,

$$\begin{aligned}\inf_{\substack{\theta^{j-1}\leq t<\theta^{j}\\ \theta^{-r-1}\leq v<\theta^{-r}}} S^2(t,v) &\geq (2-\varepsilon)\sigma^2(\theta^{-r-1}) \\ &\geq 2(1-\varepsilon)\sigma^2(\theta^{-r}) \geq (1-2\varepsilon)S^2(\theta^{j},\theta^{-r}),\end{aligned} \tag{2.7}$$

provided j and r are large enough. By conditions (i)$\sim$(iii), A_T and a_T are unbounded while B_T and b_T are bounded. Without loss of generality, assume $a_T\geq 1$. We have $1\leq j\leq k+1$ and $-r\leq -l+1\leq 1$. Using (2.6) and (2.7), we can write

(2.8)

$$\begin{aligned}&\limsup_{T\to\infty} D^*(a_T,b_T) \\ &\leq \limsup_{\substack{k\to\infty\\ l\to\infty}} \sup_{\substack{1\leq j\leq k+1\\ r\geq l-1, j-r\leq c_3}} \sup_{T\in\mathbb{A}_{kjlr}} \sup_{0\leq s\leq A_T} \sup_{0\leq t\leq a_T} \\ &\qquad\qquad \sup_{0\leq u\leq B_T} \sup_{0\leq v\leq b_T} \frac{|X(R(s,t,u,v))|}{S(a_T,b_T)\beta_T} \\ &\leq \limsup_{\substack{k\to\infty\\ l\to\infty}} \sup_{\substack{1\leq j\leq k+1\\ r\geq l-1, j-r\leq c_3}} \sup_{0\leq s\leq \theta^{k}} \sup_{0\leq t\leq \theta^{j}} \\ &\qquad\qquad \sup_{0\leq u\leq \theta^{-l}} \sup_{0\leq v\leq \theta^{-r}} \frac{|X(R(s,t,u,v))|\theta^2}{(1-2\varepsilon)S(\theta^{j},\theta^{-r})\beta_{klr}}.\end{aligned}$$

Let $\mathbb{C}_{kjlr}=\{(s,t,u,v):\,0\leq s\leq\theta^{k}, 0\leq t\leq\theta^{j}, 0\leq u\leq\theta^{-l}, 0\leq v\leq\theta^{-r}\}$ be a four-dimensional set. In order to apply Lemma 2.2, we put

$$\begin{aligned}Y_{jr}(s,t,u,v) &= \frac{X(R(s,t,u,v))}{S(\theta^{j},\theta^{-r})}, \quad (s,t,u,v)\in\mathbb{C}_{kjlr}, \\ \varphi(z) &= \frac{4\sigma\big(\sqrt{2}z\big)}{S(\theta^{j},\theta^{-r})}, \quad z>0.\end{aligned}$$

Clearly,

$$EY_{jr}(s,t,u,v) = 0,\ \Gamma^2 := \sup_{(s,t,u,v)\in\mathbb{C}_{kjlr}} E\{Y_{jr}(s,t,u,v)\}^2 = 1$$

and further

$$\begin{aligned}
&E\{X(R(s_1,t_1,u_1,v_1)) - X(R(s_2,t_2,u_2,v_2))\}^2 \\
&\le 2E\{([X(s_1+t_1,u_1+v_1) - X(s_2+t_2,u_2+v_2)] \\
&\quad - [X(s_1,u_1+v_1) - X(s_2,u_2+v_2)])^2 \\
&\quad + ([X(s_2+t_2,u_2) - X(s_1+t_1,u_1)] - [X(s_2,u_2) - X(s_1,u_1)])^2\} \\
&\le 4\{\sigma^2\big(\sqrt{(s_1+t_1-s_2-t_2)^2 + (u_1+v_1-u_2-v_2)^2}\,\big) \\
&\quad + \sigma^2\big(\sqrt{(s_1-s_2)^2 + (u_1+v_1-u_2-v_2)^2}\,\big) \\
&\quad + \sigma^2\big(\sqrt{(s_1+t_1-s_2-t_2)^2 + (u_1-u_2)^2}\,\big) \\
&\quad + \sigma^2\big(\sqrt{(s_1-s_2)^2 + (u_1-u_2)^2}\,\big)\} \\
&\le 16\sigma^2\big(\sqrt{2}\sqrt{(s_1-s_2)^2 + (t_1-t_2)^2 + (u_1-u_2)^2 + (v_1-v_2)^2}\,\big).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&E\{Y_{jr}(s_1,t_1,u_1,v_1) - Y_{jr}(s_2,t_2,u_2,v_2)\}^2 \\
&\quad le\varphi^2(\sqrt{(s_1-s_2)^2 + (t_1-t_2)^2 + (u_1-u_2)^2 + (v_1-v_2)^2}\,).
\end{aligned}$$

For any $y_0 > 0$ and $0 < \delta < 1$, write

$$\int_0^\infty \varphi(2\delta\theta^{-r}2^{-y^2})\,dy = 4\Big(\int_0^{y_0} + \int_{y_0}^\infty\Big)\frac{\sigma(2\sqrt{2}\delta\theta^{-r}2^{-y^2})}{S(\theta^j,\theta^{-r})}\,dy.$$

By the regularity of σ and Lemma 2.1, we have

$$\begin{aligned}
&\int_{y_0}^\infty \frac{\sigma(2\sqrt{2}\delta\theta^{-r}2^{-y^2})}{S(\theta^j,\theta^{-r})}\,dy \le \int_{y_0}^\infty \frac{\sigma(2\sqrt{2}\theta^{-r}2^{-y^2})}{S(\theta^j,\theta^{-r})}\,dy \\
&= \int_{y_0}^\infty (2\sqrt{2}\cdot 2^{-y^2})^\alpha \frac{l(2\sqrt{2}\theta^{-r}2^{-y^2})}{l(\theta^{-r})}\cdot\frac{\sigma(\theta^{-r})}{S(\theta^j,\theta^{-r})}\,dy \\
&\le \varepsilon_1
\end{aligned}$$

for any given $\varepsilon_1 > 0$, provided y_0, j and r are large enough. Moreover, for the fixed y_0, taking $\delta > 0$ to be small enough, we have

$$\begin{aligned}&\int_0^{y_0} \frac{\sigma(2\sqrt{2}\delta\theta^{-r}2^{-y^2})}{S(\theta^j,\theta^{-r})}\,dy\\&= \int_0^{y_0} (2\sqrt{2}\delta\cdot 2^{-y^2})^{\alpha}\cdot\frac{l(2\sqrt{2}\delta\theta^{-r}2^{-y^2})}{l(\theta^{-r})}\cdot\frac{\sigma(\theta^{-r})}{S(\theta^j,\theta^{-r})}\,dy\\&\le \varepsilon_1.\end{aligned}$$

Then, putting $\varepsilon' = 64(\sqrt{2}+1)A\varepsilon_1$, where A is defined in Lemma 2.2, we obtain

$$2(\sqrt{2}+1)A\int_0^{\infty}\varphi(2\delta\theta^{-r}2^{-y^2})\,dy \le \frac{\varepsilon'}{8}.$$

For any given $\varepsilon > 0$, take $0 < \varepsilon' < 2\varepsilon$. Then it follows from Lemma 2.2, (2.4) and (2.5) that

$$\begin{aligned}&P\Big\{\sup_{0\le s\le\theta^k}\sup_{0\le t\le\theta^j}\sup_{0\le u\le\theta^{-l}}\sup_{0\le v\le\theta^{-r}}\frac{|X(R(s,t,u,v))|}{S(\theta^j,\theta^{-r})\beta_{klr}}\ge 1+\varepsilon\Big\}\\&\le 2P\Big\{\sup_{0\le s\le\theta^k}\sup_{0\le t\le\theta^j}\sup_{0\le u\le\theta^{-l}}\sup_{0\le v\le\theta^{-r}} Y_{jr}(s,t,u,v)\\&\qquad\qquad \ge \sqrt{1+\varepsilon}\,\beta_{klr}\Big(1+\frac{\varepsilon'}{8}\Big)\Big\}\\&\le 2P\Big\{\sup_{0\le s\le\theta^k}\sup_{0\le t\le\theta^j}\sup_{0\le u\le\theta^{-l}}\sup_{0\le v\le\theta^{-r}} Y_{jr}(s,t,u,v)\\&\qquad\qquad \ge \sqrt{1+\varepsilon}\,\beta_{klr}\Big[1+2(\sqrt{2}+1)A\int_0^{\infty}\varphi(2\delta\theta^{-r}2^{-y^2})\,dy\Big]\Big\}\\&\le c\Big(\frac{\theta^k}{\delta\theta^{-r}}\Big)\Big(\frac{\theta^j}{\delta\theta^{-r}}\Big)\Big(\frac{\theta^{-l}}{\delta\theta^{-r}}\Big)\frac{1}{\delta}\exp\Big\{-\frac{1+\varepsilon}{2}\beta_{klr}^2\Big\}\\&\le c\theta^{j-r}\theta^{-\varepsilon(k-l+4r)}(kl)^{-1-\varepsilon}\le c\theta^{-\varepsilon(r+j)}(kl)^{-1-\varepsilon-\varepsilon\mathcal{B}},\end{aligned}$$

which implies that the sum

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=1}^{k+1}\sum_{r=l-1}^{\infty} P\Big\{&\sup_{0\le s\le\theta^k}\sup_{0\le t\le\theta^j}\sup_{0\le u\le\theta^{-l}}\sup_{0\le v\le\theta^{-r}}\frac{|X(R(s,t,u,v))|}{S(\theta^j,\theta^{-r})\beta_{klr}}\\&\ge 1+\varepsilon\Big\}\end{aligned}$$

is convergent. Then the Borel-Cantelli lemma and (2.8) yield (1.4) by the arbitrariness of θ and ε. □

Proof of Theorem 1.2. Similarly to (2.8), write

$$
\begin{aligned}
&\liminf_{T\to\infty} D(a_T, b_T)\\
&\geq \liminf_{\substack{k\to\infty\\ l\to\infty}} \inf_{\substack{1\leq j\leq k+1\\ r\geq l-1, j-r\leq c_3}} \sup_{0\leq s\leq \theta^{k-1}} \sup_{0\leq u\leq \theta^{-l-1}} \frac{|X(R(s,\theta^j,u,\theta^{-r}))|}{S(\theta^j,\theta^{-r})\theta\beta_{klr}}\\
&\qquad - \limsup_{\substack{k\to\infty\\ l\to\infty}} \sup_{\substack{1\leq j\leq k+1\\ r\geq l-1, j-r\leq c_3}} \sup_{0\leq s\leq \theta^k} \sup_{0\leq t\leq \theta^{j-1}(\theta-1)}\\
&\qquad\qquad \sup_{0\leq u\leq \theta^{-l}} \sup_{0\leq v\leq \theta^{-r-1}(\theta-1)} \frac{|X(R(s,t,u,v))|\theta^2}{(1-2\varepsilon)S(\theta^j,\theta^{-r})\beta_{klr}}\\
&=: J_1 - J_2.
\end{aligned}
\tag{2.9}
$$

Imitating the proof of Theorem 1.1 and comparing the ranges of t, v in the right hand sides of (2.8) and J_2, we have, for any $\varepsilon > 0$,

$$J_2 \leq \varepsilon \qquad \text{a.s.} \tag{2.10}$$

provided θ is one near enough. Consider J_1. The condition (iv) implies that for any $\mathcal{B} > 0$ there exists $c > 0$ such that

$$\theta^{k-l+2r} > c(kl)^{\mathcal{B}} \tag{2.11}$$

for all large k. For such given k, l, r and integer $j \geq 1$, we define positive integers f_{kj} and g_{lr} by

$$f_{kj} = \Big[\frac{\theta^{k-1-j}}{N} \vee 1\Big] \quad \text{and} \quad g_{lr} = \Big[\frac{\theta^{-l-1+r}}{N} \vee 1\Big],$$

where $N > 0$ is a large number specified later on and $[\cdot]$ denotes the integer part. Moreover, (2.11), (2.4) and condition (vi) together imply that for any $\tau > 2$

$$\beta_{klr}^2 \leq \tau \log(f_{kj} g_{lr}), \tag{2.12}$$

provided k is large enough. For $p = 0, 1, \cdots, f_{kj}$ and $q = 0, 1, \cdots, g_{lr}$, we also define incremental random variables

$$X(R_{pq}) := X(R(Np\theta^j, \theta^j, Nq\theta^{-r}, \theta^{-r})).$$

It follows from (2.12) that, for any $0 < \varepsilon' < \varepsilon < 1$,

$$
\begin{aligned}
&P\Big\{\sup_{0\le s\le\theta^{k-1}}\sup_{0\le u\le\theta^{-l-1}}\frac{X(R(s,\theta^j,u,\theta^{-r}))}{S(\theta^j,\theta^{-r})\beta_{klr}}<\sqrt{1-\varepsilon}\Big\}\\
&\le P\Big\{\sup_{0\le s\le\theta^{k-1}}\sup_{0\le u\le\theta^{-l-1}}\frac{X(R(s,\theta^j,u,\theta^{-r}))}{S(\theta^j,\theta^{-r})}\\
&\qquad\qquad<\{2(1-\varepsilon')\log(f_{kj}g_{lr})\}^{1/2}\Big\}\\
&\le P\Big\{\max_{0\le p\le f_{kj}}\max_{0\le q\le g_{lr}}\frac{X(R_{pq})}{S(\theta^j,\theta^{-r})}\\
&\qquad\qquad<\{2(1-\varepsilon')\log(f_{kj}g_{lr})\}^{1/2}\Big\},
\end{aligned}
\tag{2.13}
$$

provided k is large enough. Define the correlation function of $X(R_{pq})$ and $X(R_{p'q'})$:

$$
\lambda(p,q,p',q')=\mathrm{Corr}(X(R_{pq}),X(R_{p'q'})),\quad p\ne p',\ q\ne q',
$$

and let $h=p-p', m=q-q'$. By the relation $ab=\frac{1}{2}(a^2+b^2-(a-b)^2)$, we obtain

$$
\begin{aligned}
&|\mathrm{Cov}(X(R_{pq}),X(R_{p'q'}))|\\
&\le\Big|\Big\{\sigma^2\big(\sqrt{(Nh\theta^j)^2+(Nm\theta^{-r}+\theta^{-r})^2}\,\big)\\
&\qquad-\sigma^2\big(\sqrt{(Nh\theta^j)^2+(Nm\theta^{-r})^2}\,\big)\Big\}\\
&\qquad-\Big\{\sigma^2\big(\sqrt{(Nh\theta^j)^2+(Nm\theta^{-r})^2}\,\big)\\
&\qquad-\sigma^2\big(\sqrt{(Nh\theta^j)^2+(Nm\theta^{-r}-\theta^{-r})^2}\,\big)\Big\}\Big|\\
&\quad+\frac{1}{2}\Big|\Big\{\sigma^2\big(\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^{-r}+\theta^{-r})^2}\,\big)\\
&\qquad-\sigma^2\big(\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^{-r})^2}\,\big)\Big\}\\
&\qquad-\Big\{\sigma^2\big(\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^{-r})^2}\,\big)\\
&\qquad-\sigma^2\big(\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^{-r}-\theta^{-r})^2}\,\big)\Big\}\Big|\\
&\quad+\frac{1}{2}\Big|\Big\{\sigma^2\big(\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^{-r}+\theta^{-r})^2}\,\big)\\
&\qquad-\sigma^2\big(\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^{-r})^2}\,\big)\Big\}\\
&\qquad-\Big\{\sigma^2\big(\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^{-r})^2}\,\big)\\
&\qquad-\sigma^2\big(\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^{-r}-\theta^{-r})^2}\,\big)\Big\}\Big|
\end{aligned}
$$

$$= \left| \int_{\sqrt{(Nh\theta^j)^2+(Nm\theta^{-r})^2}}^{\sqrt{(Nh\theta^j)^2+(Nm\theta^{-r}+\theta^{-r})^2}} d\sigma^2(x) \right.$$
$$\left. - \int_{\sqrt{(Nh\theta^j)^2+(Nm\theta^{-r}-\theta^{-r})^2}}^{\sqrt{(Nh\theta^j)^2+(Nm\theta^{-r})^2}} d\sigma^2(x) \right|$$
$$+ \frac{1}{2} \left| \int_{\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^{-r})^2}}^{\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^{-r}+\theta^{-r})^2}} d\sigma^2(x) \right.$$
$$\left. - \int_{\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^{-r}-\theta^{-r})^2}}^{\sqrt{(Nh\theta^j-\theta^j)^2+(Nm\theta^{-r})^2}} d\sigma^2(x) \right|$$
$$+ \frac{1}{2} \left| \int_{\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^{-r})^2}}^{\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^{-r}+\theta^{-r})^2}} d\sigma^2(x) \right.$$
$$\left. - \int_{\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^{-r}-\theta^{-r})^2}}^{\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^{-r})^2}} d\sigma^2(x) \right|.$$

Without loss of generality, assume that $h > 0$ and $m > 0$. Applying Lemma 2.3 for $a = Nh\theta^j, Nh\theta^j - \theta^j$ and $Nh\theta^j + \theta^j$, respectively, we obtain

$$\begin{aligned} &|\mathrm{Cov}(X(R_{pq}), X(R_{p'q'}))| \\ &\le c\frac{\sigma^2\big(\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^{-r}+\theta^{-r})^2}\,\big)}{(Nh\theta^j-\theta^j)^2+(Nm\theta^{-r}-\theta^{-r})^2}\theta^{-2r}\,. \end{aligned}$$

It follows from the regularity of σ, (2.1) and (2.2) that, for some large N and all large k,

$$\begin{aligned} &|\lambda(p,q,p',q')| \\ &\le c\frac{\sigma^2\big(\sqrt{(Nh\theta^j+\theta^j)^2+(Nm\theta^{-r}+\theta^{-r})^2}\,\big)\theta^{-2r}}{\{(Nh\theta^j-\theta^j)^2+(Nm\theta^{-r}-\theta^{-r})^2\}S^2(\theta^j,\theta^{-r})} \\ &\le c\frac{\{(Nh+1)^2\theta^{2j}+(Nm+1)^2\theta^{-2r}\}^\alpha}{\{(Nh-1)^2\theta^{2j}+(Nm-1)^2\theta^{-2r}\}} \\ &\qquad \times \frac{l\big(\sqrt{(Nh+1)^2\theta^{2j}+(Nm+1)^2\theta^{-2r}}\,\big)^2\theta^{-2r}}{l(\theta^{-r})^2\theta^{-2\alpha r}} \\ &\le (h^2+m^2)^{(\alpha-1)/2} \le (2hm)^{(\alpha-1)/2} < (hm)^{-\nu}, \end{aligned}$$

where $\nu = (1-\alpha)/2 > 0$. In order to estimate an upper bound for the right hand side of (2.13), let us now apply Lemmas 2.4 and 2.5 for

$$\begin{aligned}
&f = f_{kj}, \quad g = g_{lr},\\
&\xi_{l_p l_q} = X(R_{pq})/S(\theta^j, \theta^{-r}), \quad p = 0, 1, \cdots, f_{kj}; \quad q = 0, 1, \cdots, g_{lr},\\
&\left|\lambda_{pq}^{p'q'}\right| = \left|\lambda(p, q, p', q')\right| < (|hm|)^{-\nu}, \quad h = p - p' \neq 0, \ m = q - q' \neq 0,\\
&u = u_{kjlr} = \{(2-\eta)\log(f_{kj} g_{lr})\}^{1/2}, \quad \eta = 2\varepsilon' < \frac{(1-\delta)\nu}{1+\nu+\delta}.
\end{aligned}$$

Then the right hand side of (2.13) is less than or equal to

$$\{\Phi(u_{kjlr})\}^{(f_{kj}+1)(g_{lr}+1)} + c(f_{kj} g_{lr})^{-\delta_0}$$

for some $\delta_0 > 0$ and large k. Thus we have, from (2.13), condition (vi) and (2.11),

$$\begin{aligned}
&P\Big\{\sup_{0 \le s \le \theta^{k-1}} \sup_{0 \le u \le \theta^{-l-1}} \frac{X(R(s, \theta^j, u, \theta^{-r}))}{S(\theta^j, \theta^{-r})\beta_{klr}} \le \sqrt{1-\varepsilon}\Big\}\\
&\le \exp\{-c((f_{kj}+1)(g_{lr}+1))^{\varepsilon'}\} + c(f_{kj} g_{lr})^{-\delta_0}\\
&\le c(f_{kj} g_{lr})^{-\delta_0} \le c\theta^{-\delta_0(k-l-j+r)}\\
&\le c\theta^{-\delta_0(k-l+4r)/2} \le c(kl)^{-\delta_0 \mathcal{B}/2} \theta^{-\delta_0 r}.
\end{aligned} \tag{2.14}$$

Taking $\mathcal{B} > 4/\delta_0$, we have

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{j=d_1}^{k+1} \sum_{r=l-1}^{\infty} (kl)^{-\delta_0 \mathcal{B}/2} \theta^{-\delta_0 r} < \infty.$$

Hence, by (2.14) and the Borel-Cantelli lemma, we obtain

$$J_1 \ge \sqrt{1-\varepsilon} \qquad \text{a.s.} \tag{2.15}$$

Combining (2.9), (2.10) with (2.15) yields (1.5). □

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