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FINANCIAL SYSTEM: INNOVATIONS AND PRICING OF RISKS

A. V. MELNIKOV

ABSTRACT. The paper studies the evolution of the financial markets and pays the basic attention to the role of financial innovations (derivative securities) in this process. A characterization of both complete and incomplete markets is given through an identification of the sets of contingent claims and terminal wealths of self-financing portfolios. The dynamics of the financial system is described as a movement of incomplete markets to a complete one when the volume of financial innovations is growing up and the spread tends to zero (the Merton financial innovation spiral). Namely in this context the paper deals with the problem of pricing risks in both field: finance and insurance.

1. Introduction

The space equipped with such a structure as financial markets, intermediaries (banks, insurance companies,...), clients (firms, individuals,...) operating their financial resources is called *financial system*.

The evolution of the World financial system has crossed at least several stages: "Gold Standard", "Bretton-Wood mechanism", "Floating exchange rates". Its deep changes for the last 25 years can be explained by the wide spectrum of *new financial instruments* and by the progress in *information* and *computer technologies*. A new scale of financial intermediation, extending bounds of credit mechanisms, new processes of internationalization in finance and insurance are comprehensive features of the financial system. To provide very important financial system functions in managing risks, payment flows, insurance, savings, borrowings,

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etc. it is necessary to take into account these new objects and reduce previous view points to financial markets ([5]).

On one hand there is a real process of financial innovations and problems of pricing risks in finance and insurance. On the other hand new financial instruments (forwards, futures, options, swaps,...) can be priced in framework of the comprehensive actuarial and financial mathematics.

This paper is an attempt to give an appropriate description of these convergent processes in practice and in the theory.

2. Innovative changes in financial system

During the "Bretton-Wood system" (1944-1971) the price of gold and exchange rates were fixed with respect to the US dollar. And therefore the institutional methods were attracted to managing such a system. For instance, The International Monetary Fund (IMF) was established with this basic aim. When the Bretton-Wood mechanism was destroyed the exchange rates became random. To control such system with "floating exchange rates" it was necessary to use functional methods based on financial innovations, or derivative securities like forwards, futures, options, swaps, etc. Financial intermediaries have gotten a new role too. They were shifted to credit instruments when capital market risks were diversified in the corresponding derivative securities markets. The problem of risk management became more complicated because the derivative securities pricing was more volatile.

The financial business practice replicated by opening of the new exchanges: Chicago Board of Options Exchange (CBOE), London International Financial Futures Exchange (LIFFE), etc. Moreover the old exchanges (CBOT, AMEX, NYSE, etc.) shifted their activity to different kinds of financial innovations too. The whole volume of using derivative securities was permanently growing up: from 500 billion of the US dollars in 1985 to 3500 billion in 1991. The same tendency was in 90-th as well ([31]).

Looking at the innovation processes in finance we should emphasize some features of these processes in industry. Industrial "know-how" sometimes leads to very important changes in all sections of the industry. So, development in computer science and information technology led in the last 20 years to a real revolution in the finance industry. New information, computer based technologies allow to store and use in real time a huge information. Therefore traders became new facilities for choosing arbitrage possibilities in the markets, for permanent monitoring of financial information, for managing huge portfolios, for reducing transactions, for transferring their activity to global financial market. As a result the structure of banks and investing companies was enriched by "front-office", "middle-office", "back-office".

Automatic financial "risk-management systems" became a reality.

The progress in information, computer based technologies allowed to study very deeply the real behavior of financial markets because the intra-day information could be taken into account. This is a huge information (changes in the rate USD/DM are registered 18000 times per day which is equivalent to a day-information for 72 years). Now, for instance, 90% of FOREX-market is operated by intra-day traders. Moreover analysis of intra-day information leads to a very important inference about the fractal structure of financial markets ([15], [24]).

These dynamic changes in real financial markets demanded an adequate theory. Recall, that there existed only one-step theories by Markowitz [21], Sharp [30], and Lintner [20] with their diversification inferences and ideas of the *efficient market.*

The new important theory was developed by Black, Scholes and Merton [3], [26]. Their *option pricing theory* (OPT) or *contingent claim analysis* (CCA) gave real possibilities for dynamic hedging and investing in connection with financial innovations. It meant the final transition from the financial arithmetics to stochastic financial and actuarial mathematics. *non-arbitrage, completeness* and *incompleteness* of the markets became systematically used as the key characteristics of pricing derivatives.

The next section gives a general description of notations, facts and perspectives of this theory.

3. Dynamics of financial system: from incomplete markets to a complete one through financial innovation expansions. Facts, models and methods

Consider a financial market model ((B,S)-market) as a pair of non-risky (bank account) B and risky (stock) S assets represented by their prices B_t and S_t, where $t = 0, 1, 2\dots$ (discrete-time model) or $t \geq 0$ (continuous-time model).

Let us define (for fixed T) a function $f_T \equiv f_T(S_0, \dots, S_T)$, which called a *contingent claim.* Taking a non-risky asset B_t and a risky asset S_t in amount β_t and γ_t respectively we form an *investment portfolio*

$\pi_t = (\beta_t, \gamma_t)$. The *capital* of such a portfolio π is equal to

$$X_t^\pi(x) = \beta_t B_t + \gamma_t S_t, X_0^\pi(x) = x.$$

Denote by SF the set of *self-financing* portfolios π:

$$\pi \in SF, \text{ if } X_t^\pi(x) - X_{t-1}^\pi(x) = \Delta X_t^\pi(x) = \beta_t \Delta B_t + \gamma_t \Delta S_t$$

($dX_t^\pi(x) = \beta_t dB_t + \gamma_t dS_t$ in continuous time if differentials dB_t and dS_t are well defined).

Arbitrage (at time T) means a possibility to make a positive capital at time T (with a positive probability) by some self-financing portfolio starting from zero initial capital.

Any *derivative security* on the given (B,S)–market can b e identified with a corresponding contingent claim. For instance, forward contract with forward price F and expiration date T is equivalent to $f_T = f_T(S_T) \equiv F$ and *call option* with the strike price K is equivalent to $f_T = (S_T - K)^+$, etc. So the set of derivative securities induces a set of graphs (CCG).

On the other hand, the set of self-financing portfolios forms a set of graphs of its terminal values $X_t^\pi(x)$ (TVG). The market is complete if TVG=CCG. Otherwise it is incomplete.

In other words, the market is called complete iff any contingent claim f_T can be *replicated*: there exist $x \geq 0$ and $\pi \in SF$ such that $X_T^\pi(x) = f_T$.

There is an important question: how to describe the risky asset S? The natural answer is to model the prices S_t as a *random process*. Therefore it is quite natural to have some fixed probability space, where the prices of all assets are stochastic processes.

Denote by V_t the price of f_T at time $t \leq T$. One of the basic problems is to describe this stochastic process in terms of the (B,S)–market.

The heuristic principle of obtaining the price V_t consists of two ideas: the value f_T should be discounted by the non-risky asset $B_t \frac{f_T}{B_T}$ and its conditional expectation $E[B_t B_T^{-1} f_T | F_t]$ one can take as a candidate for V_t.

The first idea is perfect, but the second one can be criticized: why should we look at the price V_t from the view point of the physical measure P? Any probability measure P^* on (Ω, F, P) defines its own "probability character" of the market prices B_t and S_t. It is clear that the "stable" character of the market should lead to more "natural" price of the given contingent claim. So, this heuristic principle should be adjusted by choosing such a character, or some probability measure P^* which is equivalent to P.

The *non-arbitrage principle* gives a perfect way for such an explanation, which is based on the following.

The (B, S)–market admits no arbitrage possibility if there exists a probability measure P^* such that P^* is equivalent to P and the process of discounted risky asset prices $B_t^{-1}S_t$ is a martingale with respect to P^*. It turns out that often this statement can be reversed ([16], [33]).

Therefore the measure P^* is called a *martingale* one and gives a a form of the market stability. That is why P^* is called *"risk-neutral" measure.*

According to Harrison and Pliska [17], the uniqueness of P^* leads to the statement: O n the non-arbitrage complete (B, S)–market the price of any contingent claim f can be defined uniquely iff the martingale measure is unique. Here is the sketch of the proof.

($\Rightarrow$) Let us assume there are two martingale measures $P_i^*, i = 1, 2$. Define two processes as price processes for given contingent claim f_T:

$$V_t^i = B_t E_i^*[B_T^{-1} f | F_t], i = 1, 2.$$

But $V_t^1 \equiv V_t^2$ and therefore $P_1^* \equiv P_2^*$.

($\Leftarrow$) In view of the previous statement there a martingale measure P^* such that $\frac{S_t}{B_t}$ and $\frac{V_t}{B_t}$ are local martingales with respect to P^*. But P^* is unique and therefore the price process V_t can be defined uniquely with $V_t = B_t E^*[B_T^{-1} f | F_t]$.

As a result we arrive to the following methodology of pricing contingent claims in *complete markets*:

Assume that the (B, S)–market is complete and P^* is a unique martingale measure. Define the price V_t of a given contingent claim f_T by the following

$$V_t = B_t E^*[B_T^{-1} f | F_t].$$

Then (B, S, V) is a unique system of prices for (basic and derivative) securities when the corresponding expanded market admits no arbitrage. Moreover, there exists such a *hedging* strategy $\pi \in SF$ that $X_t^\pi(V_0) = V_t, t \leq T$.

This statement can be interpreted as a possibility to reduce any risk (connected with any contingent claim) to zero. Below we give several classical examples of complete markets to illustrate this.

Binomial model or Cox, Ross, Rubinstein model [6].

Denote by $\rho_t = \frac{\Delta S_t}{S_{t-1}}$ for $t = 1, 2, \ldots$ the rate of return of the risky asset S. Assume that $\rho = (\rho_t)_{t/ge1}$ is a sequence of independent random variables with two values $b > a$ and corresponding probabilities p and $1 - p$, $p \in (0, 1)$. Rewriting $\rho_t = \mu + w_t$, where $\mu = bp + a(1 - p) = E\rho_t$

we can represent ρ_t as a random walk near the mean relative rate of return μ. If the interest rate of B is r, the unique martingale measure is defined by $p^* = \frac{r-a}{b-a}$ if $b > r > a > -1$.

Bachelier model [2].

This is a pure continuous model and the corresponding interest rate of S is defined by $dS_t = \mu + \sigma_t \dot{w}_t, S_0 > 0$, where $\dot{w}_t$ is the well-known Gaussian "white noise" and σ is a volatility parameter.

The non-arbitrage principle gives the following price for the call option $f_T = (S_T - K)^+$.

Define a martingale measure $dP_T^*/dP_T = \exp\left\{-\frac{\mu}{\sigma}w_T - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 T\right\}$. According to the methodology described above we obtaine the initial price of the call option:

$$\begin{aligned} C_T = V_0 &= E^*(S_T - K)^+ \\ &= (S_0 - K)\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right), \end{aligned}$$

where $\Phi(x) = \int_{-\infty}^{x} \phi(y)dy$, $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ and the interest rate of B is equal to zero.

Geometrical Brownian Motion of the Black-Scholes-Merton [3], [26].

Consider the following relative rates of return for B and S:

$$\frac{\dot{B}_t}{B_t} = r, \frac{\dot{S}_t}{S_t} = \mu + \sigma\dot{w}_t.$$

It is clear that the graphical realization of $\frac{\dot{S}_t}{S_t}$ will be the same as in the Bachelier model for $\dot{S}_t$.

This model can be rewritten in the form of the stochastic differential equations:

$$dB_t = B_t r\, dt, dS_t = S_t(\mu\, dt + \sigma\, dw_t),$$

where w is a Wiener process (Brownian motion), or in t he following "exponential form"

$$B_t = B_0 e^{rt}, S_t = S_0 e^{\mu t - \frac{\sigma^2 t}{2} + \sigma w_t}.$$

The non-arbitrage principle leads to the unique martingale measure $dP_T^* = Z_T^* dP$ with the density $Z_T^* = \exp\left\{-\left(\frac{\mu - r}{\sigma}\right)w_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T\right\}$. According to the non-arbitrage methodology the call option price equals to

$$C_T(T, r, \sigma) = E^* e^{-rT}(S_T - K)^+ = S_0\Phi(d_+(r)) - Ke^{-rT}\Phi(d_-(r)),$$

where $d_{\pm}(r) = \frac{\ln(\frac{S_0}{K}) + T(r \pm \frac{\sigma^2}{2})}{\sigma\sqrt{T}}$.

This is the famous Black-Scholes-Merton formula.

Note, the heuristic principle gives another price

$$C_{heuristic}(\mu) = e^{(\mu-r)T} S_0 \Phi(d_+(\mu)) - K e^{-rT} \Phi(d_-(\mu))$$

and therefore

$$C_{heuristic}(r) = C(T, r, \sigma).$$

The third method used by Black and Scholes is the method of differential equations ([10]). It briefly means the following. Let $f_T = g(S_T), g \geq 0$ be a measurable function. Consider portfolios with the capital of the form $V_t = \nu(S_t, t), t \leq T$, a smooth function, such that

$$\nu(S_t, T) = g(x), x > 0$$

$$\nu(x, t) \geq 0, x > 0, t \leq T.$$

Applying the Ito-Kolmogorov formula to $\nu(S_t, t)/B_t$ we have

$$\frac{\nu(S_t, t)}{B_t} = \frac{\nu(S_0, 0)}{B_0} + \int_0^t \frac{\partial \nu}{\partial x} d\left(\frac{S}{B}\right)_u + \int_0^t \left(\frac{\partial \nu}{\partial t} + L^0 \nu - r\nu\right) B_u^{-1}\, du,$$

where $L^0 = rx\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}$ is the generator of the diffusion process S_t.

It is clear that $V_t = \nu(S_t, t)$ is the capital of a self-financing strategy iff $\frac{\partial \nu}{\partial t} + L^0 \nu - r\nu = 0$ (Black-Scholes equation). It means ν is a harmonic function for the operator $L = \frac{\partial}{\partial t} + L^0 - r$.

So, it is necessary to solve this equation under the above boundary conditions. If $g(x)$ has a polinomial growth such a solution does exist and $\nu(x, t)$ has a form

$$\nu(x, t) = \int_0^\infty \beta(y, T-t, x) g(y) e^{-r(T-t)}\, dy,$$

where $\beta(y, t, x) = \frac{1}{y\sigma\sqrt{2\pi t}} \exp\left\{-\frac{\ln y - \ln x - (r - \frac{\sigma^2}{2})}{2\sigma^2 t}\right\}$ is a lognormal distribution density. What about the non-arbitrage principle for *incomplete markets*? There is a *set* $M(S/B)$ of martingale measures and it is impossible to *replicate* f_T by some self-financing strategy. Consider a richer set of portfolios $\{(\pi, C) : \pi = (\beta, \gamma), C$ is an increasing process$\}$ called portfolios (π) with consumption (C). For such a portfolio (π, C) the corresponding capital has a form

$$\frac{X_t^{\pi,C}}{B_t} = \frac{X_0^{\pi,C}}{B_0} + \int_0^t \gamma_u \, d\left(\frac{S}{B}\right)_u - \int_0^t B_u^{-1} \, dC_u,$$

which is a supermartingale w. r. to any $P^* \in M(S/B)$.

According to the optional decomposition ([18], [25]) any positive supermartingale can be represented in this form.

Consider the following supermartingale $\frac{V_t^*}{B_t} = \operatorname*{ess\,sup}_{P^* \in M(S/B)} E^* \left[B_T^{-1} f_T | F_t\right]$ (Snell envelope). It's clear that $\frac{V_T^*}{B_T} = \frac{f_T}{B_T}$ and therefore and therefore $V_T^* = f_T$ (a.s.). For any hedging strategy (π, C) with the capital $V_t = X_t^{\pi,C}$ we have according to the supermartingale property of V_t that for $P^* \in M(S/B)$ and for $t > 0$

$$\frac{V_t}{B_t} \geq E^* \left(\frac{V_T}{B_T} | F_t\right) \geq E^* \left(\frac{f_T}{B_T}\right) \text{ (a.s.)}.$$

Therefore the process V_t^* is minimal and according to the optional decomposition there exists such a minimal hedge (π^*, C^*).

So, in the case of incomplete markets the natural price for f_T should be $\operatorname*{ess\,sup}_{P^* \in M(S/B)} E^* \left[B_T^{-1} f_T | F_t\right] B_t$. This approach is called *superhedging.* Below we show this methodology for quite a representative incomplite market example. This is so-called (B, S)*–market with stochastic volatility*, see [36]:

$$dB_t = B_t r \, dt,$$

$$dS_t = S_t(\mu dt + \Sigma_t \, dw_t),$$

where r is a constant. $\Sigma_t^2 = \sigma^2 + (-1)^{N_t} \Delta\sigma^2, 0 \leq \Delta\sigma^2 < \sigma^2$, where N_t is a standard Poisson process with intensity $\lambda > 0$ independent of the Wiener process w_t. According to the Ito-Kolmogorov formula

$$d\left(\frac{S}{B}\right)_t = \left(\frac{S}{B}\right)_t \Sigma_t \, dw_t^*$$

where $dw_t^* = dw_t + \frac{\mu - r}{\Sigma_t} \, dt$.

It is clear that a measure $\tilde{P} \in M(S/B)$ if and only if the process w^* is a (local) martingale. But by virtue of the Girsanov theorem ([10],

[31], [25]) any $\tilde{P} \in M(S/B)$ has a local density (w.r. to P) of the form:

$$\tilde{Z}_t = \exp\left\{\int_0^t \left(\frac{r-\mu}{\Sigma_u}\,dw_u - \frac{(r-\mu)^2}{2\Sigma_u^2}\,du\right)\right\}\tilde{N}_t,$$

where $\tilde{N}$ is a (local) martingale which is orthogonal to w w.r. to P.

The (local) martingale

$$\bar{Z}_t = \exp\left\{\int_0^t \left(\frac{r-\mu}{\Sigma_u}\,dw_u - \frac{(r-\mu)^2}{2\Sigma_u^2}\,du\right)\right\}$$

is a uniformly integrable martingale in view of the *Novikov condition*:

$$E\exp\left\{\frac{1}{2}\int_0^T \left(\frac{r-\mu}{\Sigma_u}\right)^2 du\right\}$$
$$= E\exp\left\{\frac{1}{2}\int_0^T \frac{(r-\mu)}{\sigma^2+\Delta\sigma^2(-1)^{N_u}}\,du\right\} < \infty.$$

Therefore the set $M(S/B) \neq \emptyset$ and there exist at least two martingale measures defined as

$$\bar{Z}_T^{\pm} = \exp\left\{\int_0^T \left(\frac{r-\mu}{\sqrt{\sigma^2 \pm \Delta\sigma^2}}\,dw_u - \frac{(r-\mu)^2}{2(\sigma^2 \pm \Delta\sigma^2)}\,du\right)\right\}$$

and the market is incomplete.

Putting $\sigma_{\min} = \sqrt{\sigma^2 - \Delta\sigma^2}$ and $\sigma_{\max} = \sqrt{\sigma^2 + \Delta\sigma^2}$ we can represent Σ_t in such a way

$$d\Sigma_t = (\sigma_{\max} - \sigma_{\min})(I_{\{\Sigma_{t-}=\sigma_{\min}\}} - I_{\{\Sigma_{t-}=\sigma_{\max}\}})\,dN_t$$

and $\Sigma_{t-} = \sigma_{\max}$. Let us represent the0 capital of the minimal hedge $\hat{V}_t = \hat{v}(S_t, t)$ with $\hat{\gamma}_t = \frac{\partial \hat{v}}{\partial x}(S_t, t)$, where

$$\hat{v}(x,t) = e^{-r(T-t)}\sup_{\alpha} Eg(S_{T-t}^{(\alpha)}).$$

$S^{(\alpha)}$ is a controlled diffusion process which is a strong solution of the following stochastic differential equation

$$dS_u^{(\alpha)} = S_u^{(\alpha)}(r\,du + \alpha_u\,dw_u),\, S_0^{(\alpha)} = x,$$

where the "control" parameter $\alpha = (\alpha_t)_{t \le T}$ value is in the set $\{\sigma_{\min}, \sigma_{\max}\}$.

From the general theory of controlled diffusion processes ([19]) $\hat{v} \in C^{2,1}$ and satisfies the *Bellmann equation*:

$$\frac{\partial \hat{v}}{\partial t} + L^0\hat{v} - r\hat{v} + \frac{1}{2}x^2|\frac{\partial^2 \hat{v}}{\partial x^2}|\Delta\sigma^2 = 0,$$
$$\hat{v}(x,T) = g(x).$$

To solve this equation let us apply the small parameter approach denoting by $V^*(\sigma^2, \Delta\sigma^2)$ the capital of the minimal hedge in this stochastic volatility model and $V^*(\sigma^2, 0)$ the corresponding capital in the Black-Sholes model. We have

$$V_t^*(\sigma^2, \Delta\sigma^2) \simeq V_t^*(\sigma^2, 0) + \left(\frac{\partial V_t^*}{\partial \Delta\sigma^2}\right)|_{(\sigma^2,0)}\Delta\sigma^2$$
$$= V^{(0)} + V^{(1)}\Delta\sigma^2.$$

Thus $\hat{V}$ can be written in the form

$$\hat{V} = \hat{V}^{(0)} + \hat{V}^{(1)}\Delta\sigma^2$$

and therefore the Bellmann equation is reduced to

$$L\hat{V}^{(0)} + L\hat{V}^{(1)}\Delta\sigma^2 + \frac{1}{2}\left|\frac{\partial^2 \hat{V}^{(0)}}{\partial s^2} + \frac{\partial^2 \hat{V}^{(1)}}{\partial s^2}\Delta\sigma^2\right|\Delta\sigma^2 = 0$$

and

$$L\hat{V}^{(0)} + L\hat{V}^{(1)}\Delta\sigma^2 + \frac{1}{2}\left|\frac{\partial^2 \hat{V}^{(0)}}{\partial s^2}\right|\Delta\sigma^2 + \frac{1}{2}\left|\frac{\partial^2 \hat{V}^{(1)}}{\partial s^2}\right|(\Delta\sigma^2)^2 = 0,$$

where $L = \frac{\partial}{\partial t} + L^0 - r$. Taking into account that $\hat{V}^{(0)}$ is the capital of the minimal hedge in the Black-Sholes model we arrive to the following equation

$$L\hat{V}^{(1)}\Delta\sigma^2 + \frac{1}{2}\left|\frac{\partial^2 \hat{V}^{(0)}}{\partial s^2}\right|\Delta\sigma^2 = 0$$

(up to te rms of the $\Delta\sigma^2$ order).

Therefore the initial problem can be rewritten as a new boundary value problem

$$L\hat{V}^{(1)}(x,u) = h(x,u)$$
$$\hat{V}^{(1)}(x,T) = 0,$$

where $h(x,u) = -\frac{1}{2}\left|\frac{\partial^2 \hat{V}^{(0)}}{\partial s^2}\right|$.

Applying the Kolmogorov-Ito formula we have

$$e^{-rt}\hat{V}^{(1)}(S_t,t) - \int_0^t h(S_u,u)\,du = \hat{V}^{(1)}(S_0,0) + \int_0^t \frac{\partial \hat{V}^{(1)}}{\partial s}(S_u,u)\,de^{-ru}S_u + \int_0^t (L\hat{V}^{(1)}(S_u,u) - h(S_u,u))\,du,$$

and therefore

$$\begin{aligned}\hat{V}^{(1)}(x,0) &= E[e^{-rT}\hat{V}^{(1)}(S_T,T) - \int_0^T h(S_u,u)\,du - \int_0^T \frac{\partial \hat{V}^{(1)}}{\partial s}(S_u,u)\,de^{-ru}S_u] \\ &= -E\int_0^T h(S_u,u)\,du.\end{aligned}$$

As a result we get the following upper bound for the non-arbitrage price in the model under consideration:

$$C^*(\sigma,\Delta\sigma^2) = C(\sigma) + \left\{ \sup_{Q\in M(S/B)} E_Q \int_0^T \frac{1}{2}\left|\frac{\partial^2 \hat{V}^{(0)}}{\partial s^2}\,du\right| \right\}\Delta\sigma^2,$$

where $C(\sigma)$ is the fair price of in the Black-Sholes model with volatility σ.

Another method is based on the *mean-variance hedging approach* ([12], [29]), which gives another confirmation that in incomplete markets risks connected to non-redundant contingent claims can be minimized.

Now let us discuss a relationship between complete and incomplete markets ([25]).

Let us consider the difference

$$\Delta = \operatorname*{ess\,sup}_{\tilde{P}\in M(S/B)} \tilde{E}B_T^{-1}f_T - \operatorname*{ess\,inf}_{\tilde{P}\in M(S/B)} \tilde{E}B_T^{-1}f_T.$$

We shall interpret Δ as a natural measure of incompleteness of the market because $\Delta = 0$ for any complete market. This is so-called *spread*.

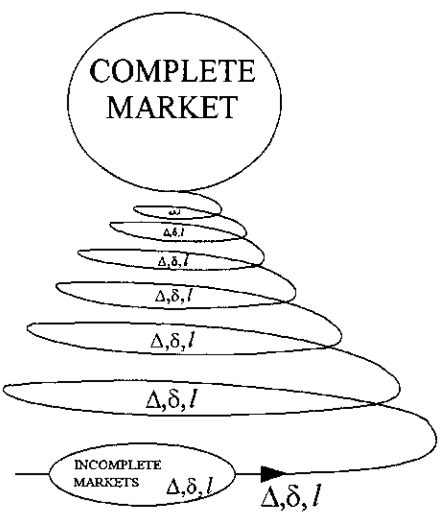
Let us note the other characteristics related to completeness and incompleteness: *leasing* and *transaction costs.* Usually, leasing of the stock S_t costs $l_t S_t$ and paymen t for transaction costs $\delta_t|\Delta\gamma_t|S_t$ where l_t and δ_t are *leasing* and *transaction costs coefficients.*

Involving new financial derivative securities makes the initial market "more complete" with smaller Δ,l and δ.

Merton was the first one ([27], [28]), who identified the movement of incomplete markets to a complete one as a "*financial innovation spiral*". He wrote ([27, p. 468]) about changes in financial system:

"From the perspective of our theory, these same facts about change are as consistent with a real-world dynamic path evolving toward an idealized target of an efficient financial market and intermediation system. On this promise, these changes can be interpreted as part of a "*financial innovation spiral*". That is, the profferation of new trading markets makes feasible the creation of new financial products; to hedge these products, producers trade in these new markets and volume expands; increased further implemetation of new products and trading strategies, which in turn leads to still more volume. Success of these trading markets encourages investment in creating additional markets and so on it goes ..., spiraling toward the theoretically limiting case of zero marginal transaction costs and dynamically complete markets".

That's why we call the next picture as "*Merton financial innovation spiral*" $(\Delta, \delta, l \to 0)$:



"Merton's spiral" $\Leftrightarrow$ Dynamics of incomplete markets towards a complete one when $\Delta, \delta, l \to 0$ and the volume of financial innovations is growing up.

4. Financial innovations and insurance risks

Any derivative security is characterized by insurance properties. Traditional insurance business pays the very important attention to risk of such a business. The difference between the traditional insurance and insurance by means of derivative securities is the following. In the first case an insurer sells his risk to an insurance company and in the second case the corresponding risk is distributed around the financial market.

The main object of the risk theory is the surplus process ([7], [14], [25]):

$$R_t = u + ct - \sum_{k=1}^{N_t} \xi_k,$$

where u is the initial capital, c is the rate of premiums, ξ_k are claims (independent identicaly distributed random variables), N_t is a Poisson process.

The key characteristic for any insurance company as its *solvency* can be characterized as a *positivity of the surplus process* R_t through the time. Traditionally, the probability of such an event (*ruin probability* $\phi(u)$) is used as the adequate deterministic characteristic for solvency:

$$\phi(u) = P\{\omega : R_t(\omega) < 0 \text{ for some } t > 0 | R_0 = u\}.$$

Under mentioned above classical assumptions this probability admits upper exponential bound (*Cramer-Lundberg estimates*):

$$\phi(u) \leq \exp\{-Ku\},$$

where K is the Lundberg constant.

The typical idea of the classical actuarial mathematics is the following: given risk level $\epsilon > 0$ is compared to this bound for $\phi(u)$ to find a *minimal solvency level* u_ϵ.

The traditional risk theory does not take into account any investment activity of the company. Therefore it is necessary to apply a financial market model to describe its investment strategies.

Starting from this point the natural expansion of the traditional risk theory is lying in the field of financial mathematics ([32], [9]).

Assume that the investment of the company is making to stock S, driven by Geometrical Brownian motion $dS_t = S_t(\mu_t \, dt + \sigma \, dw_t)$. Then the evolution of the corresponding risk process R_t can be represented in such a way:

$$dR_t = (\mu\, dt + \sigma\, dw_t)R_{t-} + c\, dt - \sum_{k=N_t}^{N_{t+dt}} \xi_k.$$

The exponential bound for $\phi(u)$ is the worst in this case. But one can derive an integro-dif ferential equation for $\phi(u)$ and find asymptotics for such a probability. Again one can repeat a previous procedure for finding u_ϵ by comparing the non-exponetial asymptotics (or appropriate approximations for solutions of the integro-differential equation) to a given risk level ϵ (for instance, [13], [22]).

As financial mathematics was enriched by financial innovations in the last 25 years the traditional actuarial mathematics should be enriched by involving of *insurance derivative securities*. Excellent examples in this direction are given by catastrophe insurance futures and options traded by Chicago Board of Trade (CBOT) since 1992 ([1], [4], [8], [11], [34], [35]).

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Steklov Mathematical Institute of
Russian Academy of Sciences
Gubkina str. 8, 117966 Moscow, Russia
E-mail: melnikov@mi.ras.ru