J. Korean Math. Soc. **38** (2001), No. 2, pp. 485–501

ANALYTIC FOURIER-FEYNMAN TRANSFORM AND FIRST VARIATION ON ABSTRACT WIENER SPACE

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ABSTRACT. In this paper we express analytic Feynman integral of the first variation of a functional F in terms of analytic Feynman integral of the product of F with a linear factor and obtain an *integration by parts formula* for the analytic Feynman integral of functionals on abstract Wiener space. We find the Fourier-Feynman transform for the product of functionals in the Fresnel class $\mathcal{F}(B)$ with n linear factors.

1. Introduction and preliminaries

The concept of an L_1 analytic Fourier-Feynman transform for functionals on classical Wiener space was introduced by Brue in [2]. In [4] Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform on classical Wiener space. In [13] Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \leq p \leq 2$ that extended the results in [2, 4] and gave various relationships between the L_1 and L_2 theories. In [10, 11, 12], Huffman, Park and Skoug defined a convolution product for functionals on classical Wiener space and they showed that the analytic Fourier-Feynman transform of convolution product is the product of transforms. In [3] Cameron obtained Wiener integral of first variation of functional F in terms of the Wiener integral of the product with a linear factor. In [6] Cameron and Storvick applied the result to Feynman integral and then gave formulas for Feynman integral of functionals on classical Wiener space that belong to the Banach algebra S' introduce by Cameron and Storvick in [5] . In [17]

Received December 31, 1999.

2000 Mathematics Subject Classification: 28C20, 44A35.

Key words and phrases: abstract Wiener space, Fourier-Feynman transform, first variation, integration by parts formula, Fresnel class.

Research supported by the Basic Science Research Institute Program, Korea Research Foundation under Grant KRF 2000-2-1137.

Park, Skoug and Storvick found the Fourier-Feynman transform of functional F from the Banach algebra $\mathcal{S}$ after it has been multiplied with n linear factors. Recently, Chang, Kim and Yoo established the relationships among Fourier-Feynman transform, first variation and convolution product on abstract Wiener space [8, 9]. In this paper we express analytic Feynman integral of the first variation of a functional F in terms of analytic Feynman integral of the product of F with a linear factor and obtain an integration by parts formula for the analytic Feynman integral of functionals on abstract Wiener space. We find the Fourier-Feynman transform for the product of functionals in the Fresnel class $\mathcal{F}(B)$ with n linear factors.

Let (H, B, ν) be an abstract Wiener space and let $\{e_j\}$ be a complete orthonormal system in H such that the e_j's are in B^*, the dual of B. For each $h \in H$ and $x \in B$, we define a stochastic inner product $(h, x)^\sim$ as follows;

$$(1.1) \qquad (h,x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle (x, e_j), & \text{if the limit exists} \\ 0, & \text{otherwise} \end{cases}$$

where $(\cdot, \cdot)$ is a natural dual pairing between B and B^*. It is well known [14, 15] that for each $h (\neq 0)$ in H, $(h, \cdot)^\sim$ is a Gaussian random variable on B with mean zero and variance $|h|^2$, that is,

$$(1.2) \qquad \int_B \exp\{i(h,x)^\sim\} d\nu(x) = \exp\{-\frac{1}{2}|h|^2\}.$$

Let $M(H)$ denote the space of complex-valued countably additive Borel measures on H. Under the total variation norm $\|\cdot\|$ and with convolution as multiplication, $M(H)$ is a commutative Banach algebra with identity [1].

A subset E of B is said to be scale-invariant measurable provided αE is measurable for each $\alpha > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $\nu(\alpha N) = 0$ for each $\alpha > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., then we write $F \approx G$. For more detail, see [7]. For a functional F on B, we denote by $[F]$ the equivalence class of functionals G which are equal to F s-a.e., that is,

$$[F] = \{G : G \approx F\}.$$

We now introduce the Fresnel class $\mathcal{F}(B)$ of functionals on B. The space $\mathcal{F}(B)$ is defined as the space of all equivalence classes of stochastic

Fourier transforms of elements of $M(H)$, that is,

(1.3)

$$\mathcal{F}(B) = \{[F] : F(x) = \int_H \exp\{i(h,x)^\sim\} d\sigma(h), x \in B, \sigma \in M(H)\}.$$

As is customary, we will identify a function with its s-equivalence class and think of $\mathcal{F}(B)$ as a collection of functionals on B rather than as a collection of equivalence classes.

It is well-known [14, 15] that $\mathcal{F}(B)$ is a Banach algebra with the norm $\|F\| = \|\sigma\|$ and the mapping $\sigma \longmapsto F$ is a Banach algebra isomorphism where $\sigma \in M(H)$ is related to F by

$$(1.4) \qquad F(x) = \int_H \exp\{i(h,x)^\sim\} d\sigma(h), \quad x \in B.$$

Let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ denote the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part, respectively.

Let F be a $\mathbb{C}$-valued scale-invariant measurable function on B such that

$$(1.5) \qquad J(\lambda) = \int_B F(\lambda^{-1/2}x) d\nu(x)$$

exists as a finite number for all real $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over B with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$(1.6) \qquad \int_B^{\text{anw}_\lambda} F(x) d\nu(x) = J^*(\lambda).$$

Let F be a functional on B such that $\int_B^{\text{anw}_\lambda} F(x) d\nu(x)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists for nonzero real q, then we call it the analytic Feynman integral of F over B with parameter q and we write

$$(1.7) \qquad \int_B^{\text{anf}_q} F(x) d\nu(x) = \lim_{\lambda \to -iq} \int_B^{\text{anw}_\lambda} F(x) d\nu(x)$$

where $\lambda \to -iq$ through $\mathbb{C}_+$.

Notation.

(i) For $\lambda \in \mathbb{C}_+$ and $y \in B$, let

$$(1.8) \qquad (T_\lambda(F))(y) = \int_B^{\text{anw}_\lambda} F(x+y) d\nu(x).$$

(ii) Given a number p with $1 \le p < \infty$, p and p' will always be related by $\frac{1}{p} + \frac{1}{p'} = 1$.

(iii) Let $1 < p < \infty$ and let G_n and G be scale-invariant measurable functionals such that, for each $\alpha > 0$,

$$\lim_{n\to\infty} \int_B |G_n(\alpha x) - G(\alpha x)|^{p'} d\nu(x) = 0. \tag{1.9}$$

Then we write

$$\underset{n\to\infty}{\text{l.i.m.}}\ (w_s^{p'})(G_n) \approx G \tag{1.10}$$

and call G the scale-invariant limit in the mean of order p'. A similar definition is understood when n is replaced by a continuously varying parameter.

DEFINITION 1.1. *Let $q \neq 0$ be a real number. For $1 < p < \infty$, we define the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ of F on B by the formula* ($\lambda \in \mathbb{C}_+$)

$$(T_q^{(p)}(F))(y) = \underset{\lambda\to -iq}{\text{l.i.m.}} (w_s^{p'})(T_\lambda(F))(y) \tag{1.11}$$

whenever this limit exists.

We define the L_1 analytic Fourier-Feynman transform $T_q^{(1)}(F)$ of F by ($\lambda \in \mathbb{C}_+$)

$$(T_q^{(1)}(F))(y) = \lim_{\lambda\to -iq} (T_\lambda(F))(y) \tag{1.12}$$

for s-a.e. $y \in B$ whenever this limit exists.

In particular, we set

$$(T_q^{(p)}(F))(0) = \int_B^{\text{anf}_q} F(x) d\nu(x), \quad 1 \le p < \infty. \tag{1.13}$$

We note that, for $1 \le p < \infty$, $T_q^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_q^{(p)}(F_1)$ exists and if $F_1 \approx F_2$, then $T_q^{(p)}(F_2)$ exists and $T_q^{(p)}(F_1) \approx T_q^{(p)}(F_2)$.

2. The Wiener integral of variations of functionals

In this section, we obtain a basic theorem which expresses the analytic Feynman integral of the first variation of a functional F in terms of the analytic Feynman integral of the product of F with a linear factor.

DEFINITION 2.1. *Let F be a Wiener measurable functional on B and let $w \in B$. Then*

$$\delta F(x|w) = \frac{\partial}{\partial t} F(x+tw)|_{t=0} \tag{2.14}$$

(if it exists) is called the first variation of $F(x)$ in the direction w.

The following theorem expresses the Wiener integral of the first variation of a functional F in terms of the Wiener integral of the product of F with a linear factor.

THEOREM 2.2. *Let (H, B, ν) be an abstract Wiener space and let $w \in H$. Let $F(x)$ be a Wiener integrable functional on B and let $F(x)$ have the first variation $\delta F(x|w)$ for $x \in B$. Suppose that there exists a Wiener integrable functional $G(x)$ such that for some positive η,*

$$\sup_{|t|\le\eta} |\delta F(x+tw|w)| \le G(x), \tag{2.15}$$

then both members of following equation exist and they are equal:

$$\int_B \delta F(x|w)d\nu(x) = \int_B F(x)[(w,x)^\sim]d\nu(x). \tag{2.16}$$

Proof. We note that

$$\begin{aligned} \delta F(x+tw|w) &= \frac{\partial}{\partial\lambda} F(x+tw+\lambda w)|_{\lambda=0} \\ &= \frac{\partial}{\partial\mu} F(x+\mu w)|_{\mu=t} \\ &= \frac{\partial}{\partial t} F(x+tw) \end{aligned} \tag{2.17}$$

and since the first member of this equation exists, so does the last. By the mean value theorem, we obtain $F(x+tw) = F(x) + t\delta F(x+\theta tw|w)$ for some θ in $0 < \theta < 1$ depending on t. Hence it follows from the integrability of (2.15) and of $F(x)$ that

$$\sup_{|t|\le\eta} |F(x+tw)| \tag{2.18}$$

is integrable on B. Now for $|t| \leq \eta$, we have the Cameron-Martin translation theorem in [16]

$$(2.19)\quad \int_B F(x)d\nu(x) = \exp\{-\frac{1}{2}t^2|w|^2\} \cdot \int_B F(x+tw)\exp\{-t(w,x)^\sim\}d\nu(x).$$

Differentiating formally with respect to t and the setting $t=0$, we obtain

$$(2.20)\quad \int_B \delta F(x|w)d\nu(x) = \int_B F(x)[(w,x)^\sim]d\nu(x).$$

To justify this differentiation under the integral sign, we must show that

$$(2.21)\quad \sup_{|t|\leq\eta_1} |\delta F(x+tw|w) - F(x+tw)\exp\{-t(w,x)^\sim\}(w,x)^\sim|$$

is Wiener integrable on B for some $\eta_1 > 0$. But it follows from the integrability of (2.18) that for some $\eta_2 > 0$

$$(2.22)\quad \sup_{|t|\leq\eta_2} |\delta F(x+tw|w)|\exp\{\eta_1|(w,x)^\sim|\}$$

is Wiener integrable on B. Similarly it follows from the integrability of (2.18) on B that for some $\eta_3 > 0$

$$(2.23)\quad \sup_{|t|\leq\eta_3} |F(x+tw)|\exp\{\eta_3|(w,x)^\sim|\}|(w,x)^\sim|$$

is Wiener integrable on B. Taking $\eta_1 = \min\{\eta_2,\eta_3\}$, we obtain the Wiener integrability of (2.21) on B. Thus the theorem is established. □

COROLLARY 2.3. *Let (H,B,ν) be an abstract Wiener space and let $w \in H$. For every $\rho > 0$ let $F(\rho x)$ be Wiener integrable on B. If $F(\rho x)$ have the first variation $\delta F(\rho x|\rho w)$ for all x in B. Suppose that there exists a Wiener integrable functional $G(x)$ such that for some positive function $\eta(\rho)$*

$$(2.24)\quad \sup_{|t|\leq\eta(\rho)} |\delta F(\rho x+\rho tw|\rho w)| \leq G(x),$$

then

$$(2.25)\quad \int_B \delta F(\rho x|\rho w)d\nu(x) = \int_B F(\rho x)[(w,x)^\sim]d\nu(x).$$

Proof. We apply Theorem 2.2 to the functional after a change of scale. To do this we set

$$H(x) = F(\rho x)$$

and note that

$$H(x+tw) = F(\rho x + t\rho w)$$

and

$$\frac{\partial}{\partial t} H(x+tw)|_{t=0} = \frac{\partial}{\partial t} F(\rho x + t\rho w)|_{t=0}$$

or

$$\delta H(x|w) = \delta F(\rho x|\rho w)$$

and the existence of either member implies that of the other. □

Our next basic theorem expresses the analytic Feynman integral of the first variation of a functional F in terms of analytic Feynman integral of the product of F with a linear factor.

THEOREM 2.4. *Let (H, B, ν) be an abstract Wiener space and let $w \in H$. For every $\rho > 0$ let $F(\rho x)$ be Wiener integrable on B. Let $F(\rho x)$ have the first variation $\delta F(\rho x|\rho w)$ for all x in B. Suppose that there exists Wiener integrable $G(x)$ such that for some positive function $\eta(\rho)$,*

$$\sup_{|t|\le\eta(\rho)} |\delta F(\rho x + \rho t w|\rho w)| \le G(x), \tag{2.26}$$

then if either member of the following equation exists, both analytic Feynman integrals below exist, and for each $q(\neq 0) \in \mathbb{R}$

$$\int_B^{\text{anf}_q} \delta F(x|w)\,d\nu(x) = -iq \int_B^{\text{anf}_q} F(x)[(w,x)^\sim]\,d\nu(x). \tag{2.27}$$

Proof. Let ρ be positive and set $z = \dfrac{w}{\rho}$. Then using (2.25), we have

$$\begin{aligned} \int_B \delta F(\rho x|w)\,d\nu(x) &= \int_B \delta F(\rho x|\rho z)\,d\nu(x) \\ &= \int_B F(\rho x)[(z,x)^\sim]\,d\nu(x) \\ &= \rho^{-2} \int_B F(\rho x)[(w,\rho x)^\sim]\,d\nu(x). \end{aligned} \tag{2.28}$$

If we let $\rho = \lambda^{-\frac{1}{2}}$, (2.28) becomes

$$\int_B \delta F(\lambda^{-\frac{1}{2}}x|w)\,d\nu(x) = \lambda \int_B F(\lambda^{-\frac{1}{2}}x)[(w,\lambda^{-\frac{1}{2}}x)^\sim]\,d\nu(x). \tag{2.29}$$

Thus by the definition of the analytic Wiener integral, if either side of the following equation exists, then both exist and we have

$$(2.30)\qquad \int_B^{\text{anw}_\lambda} \delta F(x|w)d\nu(x) = \lambda \int_B^{\text{anw}_\lambda} F(x)[(w,x)^\sim]d\nu(x).$$

Letting $\lambda \to -iq$ through $\mathbb{C}_+$, we have

$$(2.31)\qquad \int_B^{\text{anf}_q} \delta F(x|w)d\nu(x) = -iq \int_B^{\text{anf}_q} F(x)[(w,x)^\sim]d\nu(x). \qquad \square$$

3. Integration by parts formula

In this section we obtain an integration by parts formula for analytic Feynman integrals and for Fourier-Feynman transform. We first state several facts.

(i) Let F and G be in $\mathcal{F}(B)$ with associated measures f and g respectively. Then, as was shown in [14, 15], their product $K = FG$ is in $\mathcal{F}(B)$ with associated measure k satisfying $\|k\| \le \|f\|\|g\|$ where $\|\cdot\|$ is the total variation over H.

In [8, 9], Chang, Kim and Yoo obtained following facts for the Fourier-Feynman transform and the first variation on B.

(ii) Let F be in $\mathcal{F}(B)$ with associated measure f. Then, for all p with $1 \le p < \infty$, the Fourier-Feynman transform $T_q^{(p)}(F)$ exists for all $q \in \mathbb{R} - \{0\}$ and is given by the formula

$$(3.32)\qquad \begin{aligned}(T_q^{(p)}(F))(y) &= \int_H \exp\{i(h,y)^\sim - \frac{i}{2q}|h|^2\}df(h)\\ &= \int_H \exp\{i(h,y)^\sim\}d\mu(h)\end{aligned}$$

for s-a.e. y in B where μ is a complex Borel measure on H defined by

$$\mu(E) \equiv \int_E \exp\{-\frac{i}{2q}|h|^2\}df(h)$$

for every Borel set E in H, and so $\|\mu\| \le \|f\|$.

(iii) Let $F \in \mathcal{F}(B)$ so that

$$(3.33)\qquad F(x) = \int_H \exp\{i(h,x)^\sim\}df(h)$$

where f satisfies the condition $\int_H |h||df(h)| < \infty$. Then for each $w \in H$ and for s-a.e. $y \in B$, the first variation of F, $\delta F(y|w)$ is in $\mathcal{F}(B)$ and is given by the formula

$$\delta F(y|w) = \int_H i\langle h,w\rangle \exp\{i(h,y)^\sim\}df(h) = \int_H \exp\{i(h,y)^\sim\}df_w(h) \tag{3.34}$$

where $f_w(E) \equiv \int_E i\langle h,w\rangle df(h)$, $E \in \mathcal{B}(H)$, and so

$$\|f_w\| \le |w| \int_H |h||df(h)| < \infty.$$

(iv) Let F and G be elements of $\mathcal{F}(B)$ with associated measures f and g respectively, where f and g satisfy

$$\int_H |h|[|df(h)| + |dg(h)|] < \infty.$$

For each $w \in H$,

$$F(x)\delta G(x|w) + \delta F(x|w)G(x)$$

is an element of $\mathcal{F}(B)$.

(v) Let F be given as in (iv) and let $1 \le p < \infty$ and $q \in \mathbb{R} - \{0\}$. Then for each $w \in H$ and for s-a.e. $y \in B$,

$$T_q^{(p)}(\delta F(\cdot|w))(y) = \delta T_q^{(p)}(F)(y|w) = \int_H i\langle h,w\rangle \exp\Big\{i(h,y)^\sim - \frac{i}{2q}|h|^2\Big\}df(h). \tag{3.35}$$

In the following theorem, we obtain an integration by parts formula for analytic Feynman integral over B.

THEOREM 3.1. *Let F, G, f, g and w be given as (iv) above. Then for all $q \in \mathbb{R} - \{0\}$,*

$$\int_B^{\mathrm{anf}_q} [F(x)\delta G(x|w) + \delta F(x|w)G(x)]d\nu(x) = -iq \int_B^{\mathrm{anf}_q} F(x)G(x)[(w,x)^\sim]d\nu(x). \tag{3.36}$$

Proof. Let $K(x) = F(x)G(x)$. Then for all $\rho > 0$ and $t \in \mathbb{R}$.

$$\begin{aligned}(3.37)\qquad &|\delta K(\rho x + \rho t w|\rho w)| \\ &= |F(\rho x + \rho t w)\delta G(\rho x + \rho t w|\rho w) \\ &\quad + \delta F(\rho x + \rho t w|\rho w)G(\rho x + \rho t w)| \\ &\leq \rho||f|||w| \int_H |h||dg(h)| + \rho||g|||w| \int_H |h||df(h)|\end{aligned}$$

and the last member of the above expression is Wiener integrable in x for all $\rho > 0$. Also $K(x)$ is Wiener integrable and so by Theorem 2.4, stated in Section 2, equation (3.36) holds for all $q \in \mathbb{R} - \{0\}$. □

The following integration by parts formula for Fourier-Feynman transform follows from (i)∼(v) and Theorem 3.1.

THEOREM 3.2. *Let F, G, f, g and w be given as in Theorem 3.1. Then for $1 \leq p < \infty$ and $q \in \mathbb{R} - \{0\}$*

$$\begin{aligned}(3.38)&\int_B^{\text{anf}_q} [T_q^{(p)}(F)(x)\delta T_q^{(p)}(G)(x|w) + \delta T_q^{(p)}(F)(x|w)T_q^{(p)}(G)(x)]d\nu(x) \\ &= -iq \int_B^{\text{anf}_q} T_q^{(p)}(F)(x)T_q^{(p)}(G)(x)[(w,x)^\sim]d\nu(x).\end{aligned}$$

4. Transforms of functionals in $\mathcal{F}(B)$ multiplied with n linear factors

In this section we establish the Fourier-Feynman transform of functionals of the form

$$(4.39)\qquad F_n(x) = F(x)\prod_{j=1}^{n}(w_j, x)^\sim$$

with $F \in \mathcal{F}(B)$ and each $w_j \in H$.

We will show that the condition

$$(4.40)\qquad \int_H |h|^n |df(h)| < \infty$$

will ensure the existence of $T_q^{(p)}(F_n)(y)$ for s-a.e. $y \in B$. In addition, since
(4.40) implies that

$$(4.41)\qquad \int_H |h|^k |df(h)| < \infty$$

for $k = 1, \cdots, n-1$, condition (4.40) will also ensure the existence of $T_q^{(p)}(F_k)$ for $k = 1, \cdots, n-1$.

The next theorem gives a recurrence relation in which we express the transform of F_k in terms of the transforms and variation of F_{k-1}.

THEOREM 4.1. *Assume that* $T_q^{(p)}(\delta F_{k-1}(\cdot|w_k))(y) = \delta T_q^{(p)}(F_{k-1})(y|w_k)$ *exists for s-a.e.* $y \in B$. *Then* $T_q^{(p)}(F_k)(y)$ *exists for s-a.e.* $y \in B$ *and is given by the recurrence relation*

$$(4.42)\quad T_q^{(p)}(F_k)(y) = \left(\frac{i}{q}\right)T_q^{(p)}(\delta F_{k-1}(\cdot|w_k))(y) + (w_k, y)^\sim T_q^{(p)}(F_{k-1})(y).$$

Proof. Since $T_q^{(p)}(\delta F_{k-1}(\cdot|w_k))(y)$ exists, we know that $\delta F_{k-1}(\rho x + y|w_k)$ is Wiener integrable for each $\rho > 0$ and hence by Theorem 2.4,

$$\begin{aligned}
(4.43)\ & T_q^{(p)}(\delta F_{k-1}(\cdot|w_k))(y) \\
&= \int_B^{\text{anf}_q} \delta F_{k-1}(x+y|w_k)\,d\nu(x) \\
&= -iq \int_B^{\text{anf}_q} F_{k-1}(x+y)(w_k, x+y)^\sim d\nu(x) \\
&\quad + iq \int_B^{\text{anf}_q} F_{k-1}(x+y)(w_k, y)^\sim d\nu(x) \\
&= -iq \int_B^{\text{anf}_q} F_k(x+y)\,d\nu(x) + iq(w_k, y)^\sim \int_B^{\text{anf}_q} F_{k-1}(x+y)\,d\nu(x) \\
&= -iq T_q^{(p)}(F_k)(y) + iq(w_k, y)^\sim T_q^{(p)}(F_{k-1})(y).
\end{aligned}$$

Now solving (4.43) for $T_q^{(p)}(F_k)(y)$ yields (4.42) as desired. □

Our next result, which follows from Theorem 4.1 gives a recurrence relation for $T_q^{(p)}(\delta F_k(\cdot|w_{k+1}))(y) = \delta T_q^{(p)}(F_k)(y|w_{k+1})$.

THEOREM 4.2. *Assume that*

$$(4.44)\qquad \delta^2 T_q^{(p)}(F_{k-1})(\cdot|w_k)(y|w_{k+1}) = \delta T_q^{(p)}(\delta F_{k-1}(\cdot|w_k))(y|w_{k+1})$$

exists for s-a.e. $y \in B$. *Then* $T_q^{(p)}(\delta F_k(\cdot|w_{k+1}))(y)$ *exists for s-a.e.* $y \in B$ *and is given by the recurrence relation*

$$(4.45)\quad T_q^{(p)}(\delta F_k(\cdot|w_{k+1}))(y) = (\frac{i}{q})\delta T_q^{(p)}(\delta F_{k-1}(\cdot|w_k))(y|w_{k+1}) + \langle w_k, w_{k+1}\rangle T_q^{(p)}(F_{k-1})(y) + (w_k, y)^{\sim} T_q^{(p)}(\delta F_{k-1}(\cdot|w_{k+1}))(y).$$

Next we will use Theorem 4.1 and Theorem 4.2 to establish that equation (4.42) is valid for $k = 1, 2, \cdots, n$ where of course $F_0 = F$. First, for $F \in \mathcal{F}(B)$ assume that its associated measure f satisfies $\int_H |h||df(h)| < \infty$. Then by (ii) and (iii) in Section 3 above, we see that $\delta F(y|w_1)$ and $T_q^{(p)}(\delta F(\cdot|w_1))(y) = \delta T_q^{(p)}(F)(y|w_1)$ are in $\mathcal{F}(B)$. A direct calculation shows that

$$(4.46)\qquad \delta T_q^{(p)}(F)(y|w_1) = \int_H i\langle h, w_1\rangle \exp\Big\{i(h,y)^{\sim} - \frac{i}{2q}|h|^2\Big\}df(h)$$

holds for s-a.e. $y \in B$. Hence using Theorem 4.1 with $k = 1$, we see that

$$(4.47)\qquad T_q^{(p)}(F_1)(y) = (\frac{i}{q})\delta T_q^{(p)}(F)(y|w_1) + (w_1, y)^{\sim} T_q^{(p)}(F)(y)$$

for s-a.e. $y \in B$.

Next assume that f, the associated measure $F \in \mathcal{F}(B)$, satisfies

$$\int_H |h|^2 |df(h)| < \infty.$$

We see that

$$(4.48)\quad \delta^2 T_q^{(p)}(F)(\cdot|w_1)(y|w_2) = -\int_H \langle h, w_1\rangle\langle h, w_2\rangle \exp\Big\{i(h,y)^{\sim} - \frac{i}{2q}|h|^2\Big\}df(h)$$

for s-a.e. $y \in B$. In addition $\delta^2 T_q^{(p)}(F)$ is in $\mathcal{F}(B)$ and so by equation (4.45),

$$(4.49)\qquad \begin{aligned} &\delta T_q^{(p)}(F_1)(y|w_2) \\ &= T_q^{(p)}(\delta F_1(\cdot|w_2))(y) \\ &= (\frac{i}{q})\delta^2 T_q^{(p)}(F)(\cdot|w_1)(y|w_2) + \langle w_1, w_2\rangle T_q^{(p)}(F)(y) \\ &\quad + (w_1, y)^{\sim}\delta T_q^{(p)}(F)(y|w_2) \end{aligned}$$

for s-a.e. $y \in B$. Hence using Theorem 4.1 with $k = 2$, we see that

$$(4.50) \qquad T_q^{(p)}(F_2)(y) = (\frac{i}{q})\delta T_q^{(p)}(F_1)(y|w_2) + (w_2, y)^{\sim} T_q^{(p)}(F_1)(y).$$

for s-a.e. $y \in B$.

Continuing in this manner, we see that if f, the associated measure of $F \in \mathcal{F}(B)$, satisfies $\int_H |h|^n |df(h)| < \infty$, then

$$(4.51) \qquad \begin{aligned} &\delta^n T_q^{(p)}(F)(\cdot|w_1)(\cdot|w_2)\cdots(\cdot|w_{n-1})(y|w_n) \\ &= \int_H \Big(\prod_{j=1}^n i\langle h, w_j\rangle\Big) \exp\Big\{i(h,y)^{\sim} - \frac{i}{2q}|h|^2\Big\} df(h) \end{aligned}$$

for s-a.e. $y \in B$. In addition, $\delta^n T_q^{(p)}(F)$ is in $\mathcal{F}(B)$ with associated measure μ satisfying

$$\|\mu\| \le \Big(\prod_{j=1}^n |w_j|\Big) \int_H |h|^n |df(h)| < \infty.$$

Hence $\delta T_q^{(p)}(F_{n-1}(y|w_n))$ exists for s-a.e. $y \in B$ and is given by

$$(4.52) \quad \begin{aligned} &\delta T_q^{(p)}(F_{n-1})(y|w_n) \\ &= T_q^{(p)}(\delta F_{n-1}(\cdot|w_n))(y) \\ &= (\frac{i}{q})^0 \Big[\langle w_{n-1}, w_n\rangle T_q^{(p)}(F_{n-2})(y) + (w_{n-1}, y)^{\sim} \delta T_q^{(p)}(F_{n-2})(y|w_n)\Big] \\ &\quad + (\frac{i}{q})^1 \Big[\langle w_{n-2}, w_{n-1}\rangle \delta T_q^{(p)}(F_{n-3})(y|w_n) + \langle w_{n-2}, w_n\rangle \\ &\quad \cdot \delta T_q^{(p)}(F_{n-3})(y|w_{n-1}) + (w_{n-2}, y)^{\sim} \delta^2 T_q^{(p)}(F_{n-3})(\cdot|w_{n-1})(y|w_n)\Big] \\ &\quad + (\frac{i}{q})^2 \Big[\langle w_{n-3}, w_{n-2}\rangle \delta^2 T_q^{(p)}(F_{n-4})(\cdot|w_{n-1})(y|w_n) \\ &\quad + \langle w_{n-3}, w_{n-1}\rangle \delta^2 T_q^{(p)}(F_{n-4})(\cdot|w_{n-2})(y|w_n) \\ &\quad + \langle w_{n-3}, w_n\rangle \delta^2 T_q^{(p)}(F_{n-4})(\cdot|w_{n-2})(y|w_{n-1}) \\ &\quad + (w_{n-3}, y)^{\sim} \delta^3 T_q^{(p)}(F_{n-4})(\cdot|w_{n-2})(\cdot|w_{n-1})(y|w_n)\Big] \\ &\quad + \cdots + (\frac{i}{q})^{n-2} \Big[\langle w_1, w_2\rangle \delta^{n-2} T_q^{(p)}(F)(\cdot|w_3)(\cdot|w_4)\cdots(\cdot|w_{n-1})(y|w_n) \\ &\quad + \langle w_1, w_3\rangle \delta^{n-2} T_q^{(p)}(F)(\cdot|w_2)(\cdot|w_4)\cdots(\cdot|w_{n-1})(y|w_n) \\ &\quad + \cdots + \langle w_1, w_n\rangle \delta^{n-2} T_q^{(p)}(F)(\cdot|w_2)(\cdot|w_3)\cdots(\cdot|w_{n-2})(y|w_{n-1}) \\ &\quad + (w_1, y)^{\sim} \delta^{n-1} T_q^{(p)}(\cdot|w_2)(\cdot|w_3)\cdots(\cdot|w_{n-1})(y|w_n)\Big] \end{aligned}$$

$$+ (\frac{i}{q})^{n-1}\delta^n T_q^{(p)}(F)(\cdot|w_1)\cdots(\cdot|w_{n-1})(y|w_n).$$

Thus by Theorem 4.1 with $k = n$, we obtain that

$$T_q^{(p)}(F_n)(y) = (\frac{i}{q})\delta T_q^{(p)}(F_{n-1})(y|w_n) + (w_n, y)^{\sim} T_q^{(p)}(F_{n-1})(y) \tag{4.53}$$

for s-a.e. $y \in B$.

THEOREM 4.3. *Let $F_n(x) = F(x)\prod_{j=1}^n (w_j, x)^{\sim}$ with $F \in \mathcal{F}(B)$ whose associated measure f satisfies $\int_H |h|^n |df(h)| < \infty$. Then for $k = 1, 2, \ldots, n$,*

$$\begin{aligned} T_q^{(p)}(F_k)(y) &= (\frac{i}{q})\sum_{j=0}^{k-1}\Big[\delta T_q^{(p)}(F_j)(y|w_{j+1})\Big(\prod_{\ell=j+2}^{k}(w_\ell, y)^{\sim}\Big)\Big] \\ &\quad + T_q^{(p)}(F)(y)\Big(\prod_{j=1}^{k}(w_j, y)^{\sim}\Big) \end{aligned} \tag{4.54}$$

for s-a.e. $y \in B$.

Next, for special cases $n = 1, 2$ and 3, we express $T_q^{(p)}(F_1), T_q^{(p)}(F_2)$ and $T_q^{(p)}(F_3)$ in terms of $T_q^{(p)}(F), \delta T_q^{(p)}(F), \delta^2 T_q^{(p)}(F)$ and $\delta^3 T_q^{(p)}(F)$.

$$T_q^{(p)}(F_1)(y) = (\frac{i}{q})\delta T_q^{(p)}(F)(y|w_1) + (w_1, y)^{\sim} T_q^{(p)}(F)(y). \tag{4.55}$$

$$\begin{aligned} &T_q^{(p)}(F_2)(y) \\ = \;& (\frac{i}{q})^2\delta^2 T_q^{(p)}(F)(\cdot|w_1)(y|w_2) + (\frac{i}{q})\Big[(w_1, y)^{\sim}\delta T_q^{(p)}(F)(y|w_2) \\ &+ (w_2, y)^{\sim}\delta T_q^{(p)}(F)(y|w_1) + \langle w_1, w_2\rangle T_q^{(p)}(F)(y)\Big] \\ &+ (w_1, y)^{\sim}(w_2, y)^{\sim} T_q^{(p)}(F)(y). \end{aligned} \tag{4.56}$$

$$\begin{aligned} &T_q^{(p)}(F_3)(y) \\ = \;& (\frac{i}{q})^3\delta^3 T_q^{(p)}(F)(\cdot|w_1)(\cdot|w_2)(\cdot|w_3) \\ &+ (\frac{i}{q})^2\Big[(w_1, y)^{\sim}\delta^2 T_q^{(p)}(F)(\cdot|w_2)(y|w_3) \\ &+ (w_2, y)^{\sim}\delta^2 T_q^{(p)}(F)(\cdot|w_1)(y|w_3) \\ &+ (w_3, y)^{\sim}\delta^2 T_q^{(p)}(F)(\cdot|w_1)(y|w_2) \end{aligned} \tag{4.57}$$

$$+ \langle w_1, w_2\rangle \delta T_q^{(p)}(F)(y|w_3) + \langle w_1, w_3\rangle \delta T_q^{(p)}(y|w_2)$$
$$+ \langle w_2, w_3\rangle \delta T_q^{(p)}(F)(y|w_1)\Big]$$
$$+ (\frac{i}{q})\Big\{T_q^{(p)}(F)(y)\Big[(w_1, y)^{\sim}\langle w_2, w_3\rangle + (w_2, y)^{\sim}\langle w_1, w_3\rangle$$
$$+ (w_3, y)^{\sim}\langle w_1, w_2\rangle\Big] + (w_2, y)^{\sim}(w_3, y)^{\sim}\delta T_q^{(p)}(F)(y|w_1)$$
$$+ (w_1, y)^{\sim}(w_3, y)^{\sim}\delta T_q^{(p)}(F)(y|w_2) + (w_1, y)^{\sim}(w_2, y)^{\sim}$$
$$\cdot\, \delta T_q^{(p)}(F)(y|w_3)\Big\} + (w_1, y)^{\sim}(w_2, y)^{\sim}(w_3, y)^{\sim} T_q^{(p)}(F)(y).$$

Finally, setting $y \equiv 0$, we obtain the following Feynman integration formulas.

$$\text{(4.58)}\quad T_q^{(p)}(F_1)(0) = \int_B^{\mathrm{anf}_q} F(x)(w_1, x)^{\sim} d\nu(x)$$
$$= (\frac{i}{q}) \int_H i\langle h, w_1\rangle \exp\Big\{-\frac{i}{2q}|h|^2\Big\} df(h).$$

$$\text{(4.59)}\quad T_q^{(p)}(F_2)(0) = \int_B^{\mathrm{anf}_q} F(x)(w_1, x)^{\sim}(w_2, x)^{\sim} d\nu(x)$$
$$= -(\frac{i}{q})^2 \int_H \langle h, w_1\rangle\langle h, w_2\rangle \exp\Big\{-\frac{i}{2q}|h|^2\Big\} df(h)$$
$$+ (\frac{i}{q})\langle w_1, w_2\rangle \int_H \exp\Big\{-\frac{i}{2q}|h|^2\Big\} df(h).$$

(4.60)

$$T_q^{(p)}(F_3)(0) = \int_B^{\mathrm{anf}_q} F(x)(w_1, x)^{\sim}(w_2, x)^{\sim}(w_3, x)^{\sim} d\nu(x)$$
$$= -(\frac{i}{q})^3 \int_H i\langle h, w_1\rangle\langle h, w_2\rangle\langle h, w_3\rangle \exp\Big\{-\frac{i}{2q}|h|^2\Big\} df(h)$$
$$+ (\frac{i}{q})^2 \int_H i \exp\Big\{-\frac{i}{2q}|h|^2\Big\}\Big[\langle w_2, w_3\rangle\langle h, w_1\rangle$$
$$+ \langle w_1, w_3\rangle\langle h, w_2\rangle + \langle w_1, w_2\rangle\langle h, w_3\rangle\Big] df(h).$$

By the way, if $n = 4$, we get the following analytic Feynman integration formula:

$$
\begin{aligned}
(4.61)\quad & T_q^{(p)}(F_4)(0) \\
&= \int_B^{\mathrm{anf}_q} F(x)\Big(\prod_{j=1}^{4}(w_j, x)^{\sim}\Big)d\nu(x) \\
&= (\frac{i}{q})^4 \int_H \Big(\prod_{j=1}^{4} i(h, w_j)\Big)\exp\Big\{-\frac{i}{2q}|h|^2\Big\}df(h) \\
&\quad + (\frac{i}{q})^3 \int_H \exp\Big\{-\frac{i}{2q}|h|^2\Big\}\Big[-\langle w_1, w_2\rangle\langle h, w_3\rangle\langle h, w_4\rangle \\
&\quad - \langle w_1, w_3\rangle\langle h, w_2\rangle\langle h, w_4\rangle - \langle w_1, w_4\rangle\langle h, w_2\rangle\langle h, w_3\rangle \\
&\quad - \langle w_2, w_4\rangle\langle h, w_1\rangle\langle h, w_3\rangle - \langle w_2, w_3\rangle\langle h, w_1\rangle\langle h, w_4\rangle \\
&\quad - \langle w_3, w_4\rangle\langle h, w_1\rangle\langle h, w_2\rangle\Big]df(h) + (\frac{i}{q})^2\Big[\langle w_1, w_2\rangle\langle w_3, w_4\rangle \\
&\quad + \langle w_1, w_3\rangle\langle w_2, w_4\rangle\langle w_1, w_4\rangle\langle w_2, w_3\rangle\Big] \int_H \exp\Big\{-\frac{i}{2q}|h|^2\Big\}df(h).
\end{aligned}
$$

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