J. Korean Math. Soc. **38** (2001), No. 5, pp. 955–970

ISOSPECTRAL MANIFOLDS WITH DIFFERENT LOCAL GEOMETRY

CAROLYN S. GORDON

ABSTRACT. Two compact Riemannian manifolds are said to be *isospectral* if the associated Laplace-Beltrami operators have the same eigenvalue spectrum. We describe a method, based on the used of Riemannian submersions, for constructing isospectral manifolds with different local geometry and survey examples constructed through this method.

Inverse spectral geometry is the study of the extent to which the geometry and topology of a Riemannian manifold can be determined from spectral data. For example, Mark Kac's [23] appealingly phrased question "Can one hear the shape of a drum?" asks whether the spectrum of characteristic frequencies of vibration of an idealized drumhead determines the shape of a drum. Viewing the drumhead as a bounded plane domain, the frequency spectrum corresponds to the eigenvalue spectrum of the Laplacian acting on smooth functions satisfying Dirichlet boundary conditions, i.e., functions which are zero on the boundary. Thus Kac's question asks whether the eigenvalue spectrum of the Laplacian determines the domain up to congruence. The analog of this question for arbitrary compact Riemannian manifolds asks whether the eigenvalue of the associated Laplacian acting on functions determines the manifold up to isometry.

We will say two Riemannian manifolds are *isospectral* if the associated Laplacians acting on functions have the same eigenvalue spectrum. If the manifolds have boundary, one must impose boundary conditions on the eigenfunctions. We will use the term isospectral in this case to mean that the Laplacians are both Dirichlet and Neumann isospectral.

Received October 16, 2000.

2000 Mathematics Subject Classification: 58J50, 53C20.

Key words and phrases: isospectral manifolds, Laplacian.

The author's research is supported in part by a grant from the National Science Foundation.

At roughly the time that Kac posed his question, J. Milnor [27] found the first example of isospectral, non-isometric Riemannian manifolds, a pair of 16-dimensional flat tori. In the early 1980's, new examples began to appear: Riemann surfaces and hyperbolic manifolds [35], spherical space forms [21], [22] and continuous families of isospectral nilmanifolds [15]. The sporadic nature of the examples changed in 1985 when T. Sunada [31] introduced an elegant and simple method for constructing isospectral manifolds with a common finite cover. The many examples that followed included, among others, a profundity of isospectral Riemann surfaces ([3], [4], [5]) and examples of isospectral plane domains [13]. Various generalizations of Sunada's Theorem [1], [2], [7], [28] also allowed most of the earlier examples to be viewed in the same context. These representation theoretic techniques always produce isospectral manifolds with a common cover and thus the same local geometry. The examples reveal various *global* geometric and topological properties which are not spectrally determined such as the diameter and the fundamental group.

Another tool for constructing isospectral manifolds is the use of Riemannian submersions. (See the classical result 1.2 below.) The articles [9], [16], [18], [19] used a combination of representation theoretic techniques and Riemannian submersions to construct new examples of isospectral manifolds, again with a common cover and thus the same local geometry.

The main goal of this article is to describe a more recent method for constructing isospectral manifolds involving only the use of Riemannian submersions. The method was first introduced in [10] in the special context of Riemannian nilmanifolds and then expanded in a series of papers ([11], [12], [14], [17], [29], [30]). The method in general produces isospectral manifolds with different local geometry and thus enables us to identify local geometric invariants which are not spectrally determined. The method has resulted in many examples of continuous isospectral deformations, including deformations of left-invariant Riemannian metrics on simply-connected Lie groups and deformations of Riemannian metrics on balls and spheres. We will describe the technique, consider a few examples in detail, survey other examples and compare the geometry of the isospectral manifolds.

We remark that Z. Szabo [32], [33], [34] independently constructed pairs of isospectral manifolds with different local geometry, in fact, the first such examples; see the discussion in 2.14 and 2.19 below.

The author would like to thank Dorothee Schueth and Emily Proctor for helpful comments.

Technique for constructing isospectral manifolds

Definition 1.1. Let M and N be Riemannian manifolds, and let $\pi : M \to N$ be a smooth submersion. For $p \in M$, the tangent space $V_p := \ker(\pi_{*p})$ to the fiber $\pi^{-1}(\pi(p))$ through p is called the *vertical space* at p and its orthogonal complement $H_p := T_p(M) \ominus V_p$ relative to the Riemannian metric is the *horizontal space* at p. The submersion π is said to be a *Riemannian submersion* if for each $p \in M$, the linear isomorphism $\pi_* : H_p \to T_{\pi(p)}(N)$ is an inner product space isometry relative to the Riemannian inner products on $H_p \subset T_p(M)$ and on $T_{\pi(p)}(N)$. We say the fibers are *totally geodesic* if any M-geodesic which starts tangent to a fiber stays in the fiber.

We will describe a technique for constructing isospectral manifolds based on the following classical fact.

Proposition 1.2 [36]. *Let $\pi : M \to N$ be a Riemannian submersion with totally geodesic fibers. Then the Laplacians Δ_M and Δ_N satisfy*

$$\pi^*\Delta_N(f) = \Delta_M \pi^*(f)$$

for all functions f on N. In particular,

$$spec(\Delta_N) = spec(\Delta_{M|\pi^* C^\infty(N)}).$$

The rough idea behind the technique below (Theorem 1.3) is to construct a pair of Riemannian manifolds M_1 and M_2 admitting many pairs of Riemannian submersions $M_1 \to N_1$ and $M_2 \to N_2$ with the base manifolds N_1 and N_2 being isospectral. By Proposition 1.2, each such pair of submersions allows us to conclude that part of the spectra of M_1 and M_2 coincide. One needs sufficiently many such submersions to conclude that M_1 and M_2 are isospectral.

Theorem 1.3. *Let T be a torus of dimension greater than one, viewed as an abelian Lie group. Suppose that M_1 and M_2 are compact Riemannian manifolds on which T acts freely by isometries and that the fibers (the orbits of T), with the induced Riemannian metrics, are totally geodesic flat tori. For each subtorus K of T of codimension at most one, suppose that the quotient manifolds $K\backslash M_1$ and $K\backslash M_2$, with the induced metrics, are isospectral. Then M_1 and M_2 are isospectral.*

Proof. We give the key ideas of the elementary proof. The torus T acts on $\mathrm{L}^2_{\mathbf{C}}(M_i)$, $i = 1, 2$, and by a Fourier decomposition for this action, we have

$$\mathrm{L}^2_{\mathbf{C}}(M_i) = \Sigma_{\alpha\in\hat{T}}\mathcal{H}_i^{\alpha}$$

where $\hat{T}$ consists of all characters on T, i.e., all homomorphisms from the group T to the unit complex numbers, and

$$\mathcal{H}_i^{\alpha} = \{f \in \mathrm{L}^2_{\mathbf{C}}(M_i) : zf = \alpha(z)f \text{ for all } z \in T\}.$$

Since the torus action on M_i is by isometries, the Laplacian leaves each of these subspaces invariant.

For $\alpha = 1$ the trivial character, the space $\mathcal{H}_i^1$ consists of those functions constant on the fibers of the submersion $M_i \to T\backslash M_i$. By hypothesis, $T\backslash M_1$ and $T\backslash M_2$ are isospectral, so by Proposition 1.2, the restrictions of the Laplacians of M_1 and M_2 to $\mathcal{H}_1^1$ and $\mathcal{H}_2^1$, respectively, are isospectral.

For non-trivial $\alpha \in \hat{T}$, the kernel of α is a subtorus K_α of T of codimension one. Let $\mathcal{H}_i^{[\alpha]} = \Sigma_{\{\beta\in\hat{T}:\ker(\beta)=\ker(\alpha)\}}\mathcal{H}_i^{\beta}$. Then the space of all functions on M_i constant on the fibers of the submersion $M_i \to K_\alpha\backslash M_i$ coincides with $\mathcal{H}_i^{[\alpha]} \oplus \mathcal{H}_i^1$. We can again use the hypothesis of the theorem together with Proposition 1.2 to conclude that the restrictions of the Laplacians of M_i to the subspaces $\mathcal{H}_i^{[\alpha]} \oplus \mathcal{H}_i^1$, $i = 1, 2$ are isospectral. Since we already know that the restrictions to the subspaces $\mathcal{H}_i^1$ are isospectral, we conclude that the restrictions to $\mathcal{H}_i^{[\alpha]}$ are isospectral, and the theorem follows. □

For many (but not all) of the isospectral manifolds constructed via Theorem 1.3, the manifolds are diffeomorphic to a product $T \times N$, although the metrics are not product metrics. D. Schueth gave a simple reformulation of Theorem 1.3 in this case, which we now describe.

NOTATION 1.4. Let (N, g_0) be a Riemannian manifold and let $T = \mathcal{L}\backslash\mathbf{R}^k$ be a torus. Given an $\mathbf{R}^k$-valued one-form λ on N, define a Riemannian metric g^λ on $T \times N$ as follows: For $u = (\bar{z}, p) \in T \times N$, the tangent space $T_u(T \times N)$ is isomorphic to $\mathbf{R}^k \times T_p(N)$. For $x, y \in \mathbf{R}^k$, define $g_u^\lambda((x, 0), (y, 0)) = \langle x, y\rangle$ where $\langle\, ,\, \rangle$ is the standard inner product on $\mathbf{R}^k$. Define the g_u^λ-orthogonal complement H_u to $\mathbf{R}^k$ in $T_u(T \times N)$ to be

$$H_u = \{(\lambda(x), x) : x \in T_p(N)\}$$

and set $g_u^\lambda((\lambda(x), x), (\lambda(y), y)) = g_0(x, y)$ for $x, y \in T_p(N)$. Then g^λ satisfies: (i) the restriction of g^λ to each submanifold $T \times \{p\}$, $p \in N$, is isometric to T with its standard flat metric, and (ii) the canonical projection $\pi : T \times N \to N$ is a Riemannian submersion with respect to the metric g^λ on $T \times N$ and the metric g_0 on N. Moreover, it is easy to verify that the fibers $T \times \{p\}$, $p \in N$ of the Riemannian submersion $(T \times N, g^\lambda) \to (N, g_0)$ are totally geodesic. (For details, see [30].)

Conversely, every Riemannian metric on $T \times N$ satisfying (i) and (ii) is of the form g^λ for some $\mathbf{R}^k$-valued one-form λ on N.

LEMMA 1.5. *Let (N, g_0) be a Riemannian manifold and let λ and λ' be $\mathbf{R}^k$-valued one-forms on N. In the notation of 1.4, if there exists an isometry α of N such that $\lambda' = \alpha^*(\lambda)$, then the metrics g^λ and $g^{\lambda'}$ are isometric. The isometry is given by $(\bar{z}, p) \to (\bar{z}, \alpha^{-1}(p))$.*

COROLLARY 1.6 TO THEOREM 1.3 ([30]). *Let (N, g_0) be a compact Riemannian manifold and let λ and λ' be $\mathbf{R}^k$-valued 1-forms on N. For each unit vector $z \in \mathbf{R}^k$, define a real-valued one-form λ_z on N by $\lambda_z(x) = \langle \lambda(x), z \rangle$ for all $x \in TN$, where $\langle\, , \rangle$ is the canonical inner product on $\mathbf{R}^k$. Define λ'_z similarly. If for each unit vector $z \in \mathbf{R}^k$, there exists an isometry α_z on N such that $\lambda'_z = \alpha_z^* \lambda_z$, then the metrics g^λ and $g^{\lambda'}$ on $T \times N$ are isospectral.*

Proof. The key points of the proof are the following:

(i) The metric induced on $N = T\backslash M$ by both g^λ and $g^{\lambda'}$ is g_0.

(ii) Let K be a subtorus of T of codimension one. K is defined by a codimension one subspace $\mathfrak{k}$ of $\mathbf{R}^k$ spanned by lattice vectors in $\mathcal{L}$. Let z be a unit vector in $\mathbf{R}^k$ orthogonal to $\mathfrak{k}$. Identify the span $\mathbf{R}z$ with $\mathbf{R}$, and let $\mathcal{L}_z$ be the image of $\mathcal{L}$ under the orthogonal projection $\mathbf{R}^k \to \mathbf{R}z$. Then $K\backslash T$ with the flat metric induced from T is isometric to the circle $T^1 = \mathcal{L}_z\backslash\mathbf{R}$, and the Riemannian metric induced by g^λ on $K\backslash M = T^1 \times N$ is isometric to g^{λ_z}, defined as in 1.4. A similar statement holds for $g^{\lambda'}$.

By (ii) and Lemma 1.5, the Riemannian metrics induced on $K\backslash M$ by g^λ and $g^{\lambda'}$ are isometric whenever K is a subtorus of T of codimension one. By (i), the same is true when K has codimension zero, i.e., when $K = T$. Thus by Theorem 1.3, the metrics g^λ and $g^{\lambda'}$ are isospectral.□

Schueth [30], Theorem 1.6, also gave a very useful reformulation of the general case of Theorem 1.3 in terms of $\mathbf{R}^k$-valued T-invariant one-forms on the total space M.

Examples of isospectral manifolds

Most of the known examples of isospectral manifolds obtained from Theorem 1.3 were constructed via the use of linear maps $j : \mathbf{R}^k \to \mathfrak{so}(m)$ as we now explain.

DEFINITION 2.1. Let $j, j' : \mathbf{R}^k \to \mathfrak{so}(m)$ be linear maps, where $\mathfrak{so}(m)$ is the space of skew-symmetric $m \times m$ matrices.

(i) We will say that j and j' are *isospectral* if for each $z \in \mathbf{R}^k$, the eigenvalue spectra (with multiplicities) of $j(z)$ and $j'(z)$ coincide, i.e., there exists an orthogonal linear operator $A_z \in O(m)$ for which

$$A_z j(z) A_z^{-1} = j'(z).$$

(ii) We will say that j is *strongly equivalent* to j' if there exists an orthogonal transformation A of $\mathbf{R}^m$ such that

$$Aj(z)A^{-1} = j'(z)$$

for all $z \in \mathbf{R}^k$.

(iii) We will say that j and j' are *weakly equivalent* if there exists orthogonal transformations A of $\mathbf{R}^m$ and C of $\mathbf{R}^k$ such that

$$Aj(z)A^{-1} = j'(C(z))$$

for all $z \in \mathbf{R}^k$.

REMARK. Our notation here differs from the notation in the various references: Our notion of *weak equivalence* is referred to merely as *equivalence* in the references. The notion of *strong* equivalence does not appear in the references but is introduced here to simplify the presentation below.

2.2. PHILOSOPHY OF THE EXAMPLES. We will give various constructions of Riemannian manifolds associated with each linear map j. The constructions are designed so that:

- Isospectral j-maps will produce isospectral manifolds;
- Strongly equivalent j-maps will produce isometric manifolds;
- In general, the manifolds will not be isometric, or even locally isometric, unless the j-maps are at least weakly equivalent.

For some of the constructions, the proof of the third statement requires an extra genericity assumption on the j-maps, although we expect that the statement is actually true in all the constructions without the genericity conditions. We will not go into details concerning non-isometry proofs here but refer the interested reader to the references in which the various constructions were first developed. In every case, the constructions produce examples - frequently continuous families - of isospectral manifolds which are not isometric. In situations when it makes sense to compare the local geometry, e.g. when the manifolds are locally homogeneous, the local geometry generally differs as well.

2.3. EXAMPLES OF ISOSPECTRAL j-MAPS. (i) Let $k = 3$ and $m = 6$ and view $\mathbf{R}^6$ as $\mathbf{R}^3 \times \mathbf{R}^3$. For $z \in \mathbf{R}^3$ and $(x, y) \in \mathbf{R}^6$, define $j(z)(x, y) = (z \times x, z \times y)$ and $j'(z)(x, y) = (z \times x, -z \times y)$, where $u \times v$ denotes the vector cross product of $u, v \in \mathbf{R}^3$. Clearly the linear operators $j(z)$ and $j'(z)$ are isospectral for each $z \in \mathbf{R}^3$, so j is isospectral to j'. However, j and j' are not equivalent, either weakly or strongly.

(ii) Following A. Kaplan [24], [25], we will say that a linear map $j : \mathbf{R}^k \to \mathfrak{so}(m)$ is of *Heisenberg type* if $j(z)^2 = -\|z\|^2 Id$ for all $z \in \mathbf{R}^k$. The classification of linear maps j of Heisenberg type corresponds to the classification of Clifford modules ([24]). If $j, j' : \mathbf{R}^k \to \mathfrak{so}(m)$ are both of Heisenberg type, then they are easily seen to be isospectral in the sense of Definition 2.1.

For a specific example, let $k = 3$ and $m = 4l$ with $l \geq 2$. Identify $\mathbf{R}^3$ with the purely imaginary quaternions and $\mathbf{R}^m$ with the direct sum of l copies of the quaternions. Choose non-negative integers a and b with $l = a + b$. Define the map $j_{a,b} : \mathbf{R}^3 \to \mathrm{so}(m)$ by

$$j_{a,b}(p)(q_1, \ldots, q_a, q'_1, \ldots, q'_b) = (pq_1, \ldots, pq_a, q'_1 p, \ldots, q'_b p)$$

where pq_i and $q'_j p$ denote quaternionic multiplication. The map $j_{(a,b)}$ is of Heisenberg type. If (a', b') is another pair of non-negative integers such that $a + b = a' + b'$, then $j_{(a,b)}$ and $j_{(a',b')}$ are weakly equivalent in the sense of Definition 2.1 if and only if (a',b') is a permutation of (a,b) and strongly equivalent only if $a=a'$ and $b=b'$.

For a specific example of a continuous family of isospectral maps $j_t:\mathbf{R}^2\to\mathfrak{so}(6)$ which are not weakly equivalent, see [17]. In general, at least when $k=2$, examples of continuous deformations are plentiful although not easy to construct explicitly. In fact, we have:

PROPOSITION 2.4 ([17], Theorem 2.2). *Let m be any positive integer other than $1,2,3,4$, or 6. Let W be the real vector space consisting of all linear maps $j:\mathbf{R}^2\to\mathfrak{so}(m)$. Then there is a Zariski open subset $\mathcal{O}$ of W (i.e., $\mathcal{O}$ is the complement of the zero locus of some non-zero polynomial function on W) such that each $j\in\mathcal{O}$ belongs to a d-parameter family of isospectral elements of W which are not weakly equivalent. Here $d\geq m(m-1)/2-[m/2]([m/2]+2)>1$. In particular, d is of order at least $O(m^2)$.*

The key property of j-maps which will be used to show the various manifolds we construct are isospectral is the following:

LEMMA 2.5. *If $j,j':\mathbf{R}^k\to\mathfrak{so}(m)$ are isospectral, then the restrictions of j and j' to any one-dimensional subspace of $\mathbf{R}^k$ are strongly equivalent.*

The lemma is immediate from Definition 2.1.

We can now describe a specific construction of isospectral manifolds.

NOTATION 2.6 ([30]). Recall that a left-invariant Riemannian metric on a Lie group G (i.e., a metric for which the left-translations are isometries) corresponds to an inner product on its Lie algebra. A bi-invariant Riemannian metric (i.e., one for which both the left and right translations are isometries) is given by an $Ad(G)$-invariant inner product on the Lie algebra. In the notation of 1.4, let N be either the Lie group $SO(m)$ or its simply-connected covering $Spin(m)$, and let g_0 be a bi-invariant Riemannian metric on N. Let $T=\mathcal{L}\backslash\mathbf{R}^k$ be a k-dimensional torus. Then $T\times N$ is a Lie group, which we denote by G. Recall that an $\mathbf{R}^k$-valued left-invariant 1-form on a Lie group is defined by a linear map from the associated Lie algebra into $\mathbf{R}^k$. The Lie algebra of both $SO(m)$ and of $Spin(m)$ is given by $\mathfrak{so}(m)$. Define $\lambda_j:\mathfrak{so}(m)\to\mathbf{R}^k$ to be the transpose of j. We can then define a metric g^{λ_j} on G as in 1.4. Since both the metric g_0 on N and the 1-form λ_j are left-invariant, the metric g^{λ_j} on G is also left-invariant.

THEOREM 2.7 ([30]). *Let G be the Lie group $T \times SO(m)$ or $T \times Spin(m)$. In the notation of* 1.4 *and* 2.6*, if $j, j' : \mathbf{R}^k \to \mathfrak{so}(m)$ are isospectral linear maps, then the left-invariant Riemannian metrics g^{λ_j} and $g^{\lambda'_j}$ on G are isospectral.*

LEMMA 2.8. *Let N denote either $SO(m)$ or $Spin(m)$ with a bi-invariant Riemannian metric. Suppose j is strongly equivalent to j', say $j'(z) = Aj(z)A^{-1}$ for all $z \in \mathbf{R}^k$ where $A \in O(m)$. Let α be the isometry of N given by conjugation I_A by A (in case $N = SO(m)$) or by the lift to $Spin(m)$ of I_A (in case $N = Spin(m)$). Then $\lambda = \alpha^*\lambda'$ as 1-forms on N.*

Proof of Lemma 2.8. In either case, α is an automorphism of the Lie group N whose differential $\alpha_* : \mathfrak{so}(m) \to \mathfrak{so}(m)$ is given by conjugation by A, an orthogonal automorphism of $\mathfrak{so}(m)$. Thus $j' = \alpha_* \circ j$ and $\lambda' = \lambda \circ (\alpha_*)^{-1} = (\alpha^{-1})^*\lambda$. □

Proof of Theorem 2.7. For z a unit vector in $\mathbf{R}^k$, it is straightforward to check that $(\lambda_j)_z = \lambda_{(j_{|\mathbf{R}z})}$ in the notation of 1.6. Thus the Theorem follows from Corollary 1.6 and Lemmas 2.5 and 2.8. □

COROLLARY 2.9 ([30]). *Let T be a two-torus and let $m \geq 5$. Then there exist continuous families of isospectral, non-isometric left-invariant Riemannian metrics on $T \times SO(m)$.*

In fact, Schueth constructed isospectral deformations using Theorem 2.7 in such a way that the norm of the Ricci curvature (a constant function on the Lie group since the metric is homogeneous) depends non-trivially on the deformation parameter.

REMARK 2.10. By letting $\mathfrak{su}(m)$ play the role of $\mathfrak{so}(m)$ and $SU(m)$ the role of $O(m)$ in 2.1-2.8, Schueth also constructed non-trivial isospectral families of left-invariant metrics on $T \times SU(m)$, $m \geq 3$.

NOTATION 2.11. Given a torus $T = \mathcal{L}\backslash\mathbf{R}^k$ and a linear map $j : \mathbf{R}^k \to \mathfrak{so}(m)$, we now define Riemannian metrics on $B \times T$ and on $S \times T$ where B is the unit ball and S the unit sphere centered at zero in $\mathbf{R}^m$. First define a bilinear anti-symmetric map $[\,,\,] : \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^k$ by

$$\langle [x,y], z\rangle = \langle j(z)x, y\rangle$$

for all $x, y \in \mathbf{R}^m$ and $z \in \mathbf{R}^k$, where we are using the notation $\langle\,,\,\rangle$ for the standard inner products both on $\mathbf{R}^k$ and on $\mathbf{R}^m$. Identifying the tangent space $T_p(\mathbf{R}^m)$ at a point p in $\mathbf{R}^m$ with $\mathbf{R}^m$ itself, define an $\mathbf{R}^k$-valued 1-form on $\mathbf{R}^m$ by $\lambda_p(x) = [p, x]$ for all $x \in T_p(\mathbf{R}^m)$ and all $p \in \mathbf{R}^m$. Continuing to denote by λ the restriction of λ to the ball B and the pull-back of λ to the sphere S via the inclusion map (i.e., the restriction of λ to $T(S)$), we can then define metrics g^λ on $T \times B$ and on $T \times S$ as in 1.4.

LEMMA 2.12. *We use the notation of* 2.1 *and* 2.11. *Suppose that j is strongly equivalent to j', say $j'(z) = Aj(z)A^{-1}$ for all $z \in \mathbf{R}^k$. Viewing $A \in O(m)$ as an isometry of $\mathbf{R}^m$, which restricts to an isometry of B and of S, then $\lambda_j = A^*\lambda_{j'}$ as $\mathbf{R}^k$-valued forms on $\mathbf{R}^m$, B and S.*

Proof. For $z \in \mathbf{R}^k$, $p \in \mathbf{R}^m$ and $x \in T_p(\mathbf{R}^m) = \mathbf{R}^m$, we have

$$\begin{aligned}\langle (A^*\lambda_{j'})_p(x), z\rangle &= \langle (\lambda_{j'})_{Ap}(Ax), z\rangle = \langle j'(z)(Ap), Ax\rangle \\ &= \langle A^{-1}j'(z)Ap, x\rangle = \langle j(z)(p), x\rangle = \langle (\lambda_j)_p(x), z\rangle. \quad \square\end{aligned}$$

THEOREM 2.13. *In the notation of* 2.11, *if j is isospectral to j', then the metrics g^{λ_j} and $g^{\lambda'_j}$ on $T \times B$ (or on $T \times S$) are isospectral.*

Proof. For z a unit vector in $\mathbf{R}^k$, $(\lambda_j)_z = \lambda_{(j_{|\mathbf{R}z})}$ under the identification of $\mathbf{R}z$ with $\mathbf{R}$. Thus the Theorem follows from Corollary 1.6, and Lemmas 2.5 and 2.12. $\square$

2.14. ISOSPECTRAL METRICS ON $T \times B$ AND $T \times S$ (HISTORY). The first examples of isospectral metrics with different local geometry were found by Z. Szabo (later published in [32]). They were metrics of the form g^{λ_j} and $g^{\lambda'_j}$ on $T \times B$ where j and j' were of Heisenberg type (see 2.3). His proof of isospectrality was entirely different and involved an explicit computation of the eigenvalues. The general case of Theorem

2.13 was proved in the case of $T \times B$ in [17] and in the case of $T \times S$ in [12] using Theorem 1.3. These papers then use Theorem 2.13 along with Proposition 2.4 to obtain continuous isospectral deformations on $T \times B$ and $T \times S$, where T is a two-torus and S and B are the sphere and ball in $\mathbf{R}^m$, $m \geq 5$. Simultaneously but independently of the latter paper, Szabo [32] constructed pairs of isospectral metrics on $T \times S$, again using j-maps of Heisenberg type and proving isospectrality by explicit computation. One pair of isospectral metrics constructed on $T^3 \times S^7$ by Szabo is especially interesting in that one metric is homogeneous and the other not.

The presentation of this construction in 2.11-2.13 above follows [30]. In the original (equivalent) construction in [12] and [17], the manifolds $T \times S$ and $T \times B$ were viewed as submanifolds of certain 2-step nilpotent Lie groups and the metrics g^{λ_j} were obtained by restriction of suitable left-invariant metrics on the Lie groups.

Note that in 2.11-2.13, the one-form λ_j depends linearly on the point p. The lowest dimensional examples constructed by Theorem 2.13 and Proposition 2.4 are 6-dimensional. Schueth [30] constructed a pair of $\mathbf{R}^2$-valued one-forms on the 2-sphere depending quadratically on the point p and satisfying the hypothesis of Corollary 1.6. In this way she obtained a pair of isospectral metrics with different local geometry on $T^2 \times S^2$. This is the lowest dimensional example known of isospectral metrics with different local geometry. The lowest dimension of a manifold of the form $T \times S$ known to admit a *continuous isospectral deformation* remains six.

2.15. Isospectral deformations of metrics on simply-connected manifolds constructed via Theorem 1.3.

(i) (Schueth [29]) The first examples of isospectral, non-isometric Riemannian metrics on simply-connected manifolds were continuous families of isospectral metrics on $G \times S$ where S is the sphere in $\mathbf{R}^m$, $m \geq 5$ and G is a compact Lie group of rank at least two. These metrics were obtained from the metrics on $T \times S$ in Theorem 2.13 (with T a 2-torus) by embedding T in G.

(ii) In [30] Schueth constructed isospectral deformations of left-invariant Riemannian metrics on the Lie groups $SO(n)$ and $Spin(n)$, $n \geq 9$ and on $SU(n)$, $n \geq 6$. We describe the metrics on $SO(n)$; the other cases are similar. A 2-torus T can be embedded in $SO(4)$ as the subgroup $SO(2) \times SO(2)$. Thus $T \times SO(m)$ can be embedded as a Lie subgroup in $SO(m+4)$ with Lie algebra $\mathfrak{g} := \mathfrak{so}(2) \times \mathfrak{so}(2) \times \mathfrak{so}(m) \subset \mathfrak{so}(m+4)$.

We set $n = m + 4$. Given a linear map $j : \mathbf{R}^2 \to \mathfrak{so}(m)$, define an inner product g_j on $\mathfrak{so}(n)$ as follows: Let h be a bi-invariant inner product on $\mathfrak{so}(n)$. Define the restriction of g_j to the subspace $\mathfrak{g}$ to be the inner product defined in 2.6 (under the identification of $\mathfrak{g}$ with $\mathbf{R}^2 \times \mathfrak{so}(m)$. Declare the h-orthogonal complement $\mathfrak{u}$ of $\mathfrak{g}$ to be g_j orthogonal to $\mathfrak{g}$, and set $g_{j|\mathfrak{u}\times\mathfrak{u}} = h_{|\mathfrak{u}\times\mathfrak{u}}$.

2.16. HOW DOES THE GEOMETRY OF THE ISOSPECTRAL MANIFOLDS DIFFER? The various examples of isospectral manifolds with different local geometry reveal many geometric properties which are not spectrally determined.

(i) As already noted, an example of Szabo shows that you cannot tell from the spectrum whether a metric is homogeneous.

(ii) In many of the examples of isospectral deformations of locally homogeneous manifolds, the norm of the Ricci curvature varies during the deformation. This phenomenon was first observed in [17].

(iii) While the total scalar curvature is known to be a spectral invariant (see the discussion of the heat invariants below), the maximum (and minimum) of the scalar curvature function can vary during the isospectral deformations. This phenomenon was first observed in [12].

(iv) [14] There exists a pair of Riemannian metrics on $T^3 \times S^5$, one of which has constant scalar curvature and the other variable scalar curvature.

2.17. THE HEAT INVARIANTS AND THE p-SPECTRUM. Let (M, g) be a closed Riemannian manifold and let $\lambda_1, \lambda_2, \ldots$ be the spectrum of the Laplacian of (M, g). Minakshisundaram and Pleijel showed that the function $\zeta(t) = \Sigma_j e^{-\lambda_j t}$ (the heat trace) has an asymptotic expansion as $t \to 0^+$ of the form

$$Z(t) = (4\pi t)^{-n/2} \Sigma_{k=0}^{\infty} a_k t^k$$

where n is the dimension of M. The coefficients a_k are integrals over M of universal homogeneous polynomials in the curvature and its covariant derivatives. These coefficients, the so-called *heat invariants*, are spectrally determined. The first few are given by:

$$a_0 = \mathrm{vol}(M);\ a_1 = \frac{1}{6}\int_M \tau;$$

$$a_2 = \frac{1}{360}\int_M (5\tau^2 - 2\|\mathrm{Ric}\|^2 + 2\|R\|^2)$$

where τ is the scalar curvature, Ric is the Ricci tensor and R is the curvature tensor. See [8] for more details. In particular, from a_1 we see that the spectrum determines the total scalar curvature, as noted in 2.16.

Observe that a_2 is a linear combination of three terms $\int_M \tau^2$, $\int_M \|\mathrm{Ric}\|^2$ and $\int_M \|R\|^2)$. While a_2 is a spectral invariant, the three individual terms actually vary during some of the isospectral deformations and thus are not spectral invariants. This phenomenon was first observed by Schueth for the examples described in 2.15(i) above.

Associated to the Riemannian metric, one can also define a Laplace operator, the Hodge Laplacian, acting on p-forms for each $p = 1, \dots, \dim(M)$. For closed manifolds, the analogously defined heat trace has a similar asymptotic expansion. The heat coefficients $a_i(p)$ are linear coefficients of the same curvature expressions as the heat coefficients $a_i = a_i(0)$ for the Laplacian on functions. In particular, $a_2(p)$ is a linear combination of $\int_M \tau^2$, $\int_M \|\mathrm{Ric}\|^2$ and $\int_M \|R\|^2)$. However the coefficient of each of the three terms in the linear combination depends on p. In the examples computed by Schueth in which the three individual terms vary during the isospectral deformation, the resulting heat coefficient $a_2(1)$ for the Laplacian acting on 1-forms thus changes during the deformation. This fact shows that the Laplacians acting on 1-forms are not isospectral. Our expectation is that, in general, isospectral manifolds constructed by submersion methods as described here are not isospectral on p-forms for $p \geq 1$.

Most examples of isospectral manifolds constructed by earlier methods were strongly isospectral; in particular, the Laplacians acting on p-forms were isospectral for every p. Exceptions include certain Heisenberg manifolds [9], Lens spaces [22], continuous families of isospectral metrics on certain nilmanifolds [19] and recent examples of isospectral flat manifolds [26]. The article [26] by R. Miatello and J. P. Rossetti also contains the first examples of manifolds for which the Laplacians acting on p-forms for some values of p are isospectral while the Laplacians acting on functions are not.

2.18. Manifolds of negative sectional or Ricci curvature. In [14], the technique of Theorem 1.3 for constructing isospectral manifolds was extended to a larger class of Riemannian manifolds admitting free torus actions by weakening the hypothesis that the toral fibers be totally geodesic. Examples obtained in that paper include:

(i) Continuous families of isospectral, locally nonisometric negatively curved metrics on a manifold with boundary. These examples contrast with a result of Guillemin-Kazhdan [20] in dimension 2, generalized to arbitrary dimension by Croke-Sharafutdinov [6], stating that a closed Riemannian manifold of negative curvature cannot be isospectrally deformed.

(ii) A pair of isospectral metrics on a manifold with boundary, one of which has negative curvature and the other mixed curvature.

(iii) A pair of isospectral metrics on a manifold with boundary, one of which has constant Ricci curvature and the other variable.

(iv) A pair of isospectral metrics on a manifold with boundary, one of which has parallel curvature tensor and the other not.

2.19. Isospectral metrics on spheres and balls. In [11] a further generalization of the technique of Theorem 1.3 was obtained: the hypothesis that the torus action on the manifolds be free was weakened. Using this generalization, the author constructed continuous isospectral deformations of Riemannian metrics on balls and spheres in $\mathbf{R}^m$ with $m \geq 9$. The metrics on the spheres can be chosen arbitrarily close to the round metric; in particular, they can be chosen to be positively curved. The metrics on the ball can be chosen arbitrarily close to the flat metric.

Szabo [33], [34] independently and very slightly earlier constructed by different methods pairs of isospectral metrics on balls and spheres, including a pair of metrics on the 11-sphere one of which is homogeneous and the other not. The metrics constructed by Szabo do not appear to admit an explanation using the technique of Theorem 1.3 or any known generalization.

References

[1] P. Bérard, *Transplantation et isospectralité I*, Math. Ann. **292** (1992), 547–559.

[2] ———, *Transplantation et isospectralité II*, J. London Math. Soc. **48** (1993), 565–576.

[3] R. Brooks, R. Gornet, and W. Gustafson, *Mutually isospectral Riemann surfaces*, Adv. Math. **138** (1998), 306–322.

[4] R. Brooks and R. Tse, *Isospectral surfaces of small genus*, Nagoya Math. J. **107** (1987), 13–24.

[5] P. Buser, *Isospectral Riemann surfaces*, Ann. Inst. Fourier (Grenoble) **36** (1986), 167–192.

[6] C. Croke and V. Sharafutdinov, *Spectral rigidity of a compact negatively curved manifold*, Topology **37** (1998), 1265–1273.

[7] D. DeTurck and C. Gordon (appendix by K. B. Lee), *Isospectral Deformations II: trace formulas, metrics, and potentials*, Comm. Pure Appl. Math. **42** (1989), 1067–1095.

[8] P. B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, Publish or Perish, Inc., Wilmington, DE, 1984.

[9] C. S. Gordon, *Riemannian manifolds isospectral on functions but not on 1-forms*, J. Differential Geom. **24** (1986), 79–96.

[10] ———, *Isospectral closed Riemannian manifolds which are not locally isometric, II*, Contemporary Mathematics: Geometry of the Spectrum (R. Brooks, C. Gordon and P. Perry, eds.), vol. 173, AMS, 1994, pp. 121–131.

[11] ———, *Isospectral deformations of metrics on spheres*, Invent. Math. (to appear).

[12] C. Gordon, R. Gornet, D. Schueth, D. Webb, and E. N. Wilson, *Isospectral deformations of closed Riemannian manifolds with different scalar curvature*, Ann. Inst. Fourier (Grenoble) **48** (1998), 593-607.

[13] C. Gordon, D. Webb, and S. Wolpert, *Isospectral plane domains and surfaces via Riemannian orbifolds*, Invent. Math. **110** (1992), 1–22.

[14] C. S. Gordon and Z. I. Szabo, *Isospectral deformations of negatively curved Riemannian manifolds with boundary which are not locally isometric*, Duke Math. J. (to appear).

[15] C. Gordon and E. N. Wilson, *Isospectral deformations of compact solvmanifolds*, J. Differential Geom. **19** (1984), 241–256.

[16] ———, *The spectrum of the Laplacian on Riemannian Heisenberg manifolds*, Michigan Math. J. **33** (1986), 253–271.

[17] ———, *Continuous families of isospectral Riemannian manifolds which are not locally isometric*, J. Differential Geom **47** (1997), 504-529.

[18] Ruth Gornet, *A new construction of isospectral Riemannian nilmanifolds with examples*, Michigan Math. J. **43** (1996), no. 1.

[19] ———, *Continuous families of Riemannian manifolds isospectral on functions but not on 1-forms*, J. Geom. Anal. **10** (2000), 281–298.

[20] V. Guillemin and D. Kazhdan, *Some inverse spectral results for negatively curved 2-manifolds*, Topology **19** (1980), 301–312.

[21] A. Ikeda, *Isospectral problem for spherical space forms*, Spectra of Riemannian Manifolds (M. Berger, S. Murakami, and T. Ochiai, eds.), Kaigai Publications, 1983, pp. 57–63.

[22] ———, *Riemannian manifolds p-isospectral but not $(p+1)$-isospectral*, Geometry of Manifolds (Matsumoto), Perspect. Math., vol. 8, Academic Press, Boston, MA, 1989, pp. 383–417.

[23] M. Kac, *Can one hear the shape of a drum?*, Amer. Math. Monthly **73 part II** (1966), 1–23.

[24] A. Kaplan, *Riemannian nilmanifolds attached to Clifford modules*, Geom. Dedicata **11** (1981), 127–136.

[25] ———, *On the geometry of groups of Heisenberg type*, Bull. London Math. Soc. **15** (1983), 35–42.

[26] R. Miatello and J. P. Rossetti, *Flat manifolds isospectral on p-forms*, J. Geom. Anal. (to appear).

[27] J. Milnor, *Eigenvalues of the Laplace operator on certain manifolds*, Proc. Natl. Acad. Sci. USA **51** (1964), 542.

[28] Hubert Pesce, *Représentations relativement équivalentes et variétés Riemanniennes isospectrales*, C. R. Acad. Sci. Paris **3118, Série I** (1994), 657–659.

[29] D. Schueth, *Continuous families of isospectral metrics on simply connected manifolds*, Ann. of Math. **149** (1999), 287–308.

[30] ———, *Isospectral manifolds with different local geometries*, J. Reine Angew. Math. (to appear).

[31] T. Sunada, *Riemannian coverings and isospectral manifolds*, Ann. of Math. **121** (1985), 169–186.

[32] Z. I. Szabo, *Locally non-isometric yet super isospectral spaces*, Geom. Funct. Anal. **9** (1999), 185–214.

[33] ———, *Isospectral balls and spheres*, Ann. of Math. (to appear).

[34] ———, *Cornocopia of isospectral pairs of metrics constructed on balls and spheres with different local geometries*, Ann. of Math. (to appear).

[35] M. F. Vignéras, *Variétés Riemanniennes isospectrales et non isométriques*, Ann. of Math. **112** (1980), 21–32.

[36] B. Watson, *Manifold maps commuting with the Laplacian*, J. Differential Geom. **8** (1973), 85–94.

Department of Mathematics
Dartmouth College
Hanover, New Hampshire 03755, USA
E-mail: carolyn.s.gordon@dartmouth.edu