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A FUBINI THEOREM FOR ANALYTIC FEYNMAN INTEGRALS WITH APPLICATIONS

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Abstract. In this paper we establish a Fubini theorem for various analytic Wiener and Feynman integrals. We then proceed to obtain several integration formulas as corollaries.

1. Introduction and preliminaries

Let $C_0[0,T]$ denote one-parameter Wiener space; that is the space of $\mathbf{R}$-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. In section 2, we establish a Fubini theorem for the analytic Wiener integral and the analytic Feynman integral for various functionals $F : C_0[0,T] \to \mathbf{C}$. In section 3, we use these Fubini theorems to establish several Feynman integration formulas.

The usual Fubini theorem, see for example [14, p. 307], does not apply to analytic Wiener and Feynman integrals since they are not defined in terms of a countably additive nonnegative measure. Rather, they are defined in terms of a process of analytic continuation and a limiting procedure applied to a Wiener integral which is however based on such a measure.

Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. $(C_0[0,T],\mathcal{M},m)$ is a complete measure space and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)m(dx).$$

A subset E of $C_0[0,T]$ is said to be scale-invariant measurable [5,11] provided $\rho E \in \mathcal{M}$ for all $\rho > 0$, and a scale-invariant measurable set N

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is said to be scale-invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere ($s-a.e.$). If two functionals F and G are equal $s-a.e.$, we write $F \approx G$. For a rather detailed discussion of scale-invariant measurability and its relation with other topics see [11]. In [15], Segal gives an interesting discussion of the relation between scale change in Wiener space and certain questions in quantum field theory.

In [2,10], all of the functionals F on Wiener space and all the $\mathbf{C}$-valued functions f on $\mathbf{R}^n$ were assumed to be Borel measurable. But, as was pointed out in [11,p.170], the concept of scale-invariant measurability in Wiener space and Lebesgue measurability in $\mathbf{R}^n$ is precisely correct for the analytic Fourier-Feynman transform theory and the analytic Feynman integration theory.

Throughout this paper we will assume that each functional F we consider satisfies the conditions:

$$F : C_0[0,T] \to \mathbf{C} \text{ is scale-invariant measurable.} \tag{1.1}$$

$$\int_{C_0[0,T]} |F(\rho x)| m(dx) < \infty \text{ for each } \rho > 0. \tag{1.2}$$

REMARK 1. Using Theorem 9 of [11], it follows that condition (1.2) is equivalent to the condition

$$\int_{C_0^2[0,T]} |F(ay+bz)| d(m \times m)(y,z) < \infty \text{ for all } a, b > 0. \tag{1.3}$$

REMARK 2. For $F : C_0[0,T] \to \mathbf{C}$ satisfying conditions (1.1) and (1.2) above, the usual Fubini theorem [14,p.307] implies that

$$\begin{aligned} &\int_{C_0[0,T]} \Big(\int_{C_0[0,T]} F(ay+bz) m(dy) \Big) m(dz) \\ &= \int_{C_0^2[0,T]} F(ay+bz) d(m \times m)(y,z) \\ &= \int_{C_0[0,T]} \Big(\int_{C_0[0,T]} F(ay+bz) m(dz) \Big) m(dy) \end{aligned} \tag{1.4}$$

for all $a, b > 0$. In addition, by [11,Theorem 9], it follows that

$$(1.5)\quad \int_{C_0^2[0,T]} F(ay+bz)d(m\times m)(y,z) = \int_{C_0[0,T]} F(\sqrt{a^2+b^2}x)m(dx).$$

Let $\mathbf{C}_+ = \{\lambda \in \mathbf{C} : Re\,\lambda > 0\}$ and $\tilde{\mathbf{C}}_+ = \{\lambda \in \mathbf{C} : \lambda \neq 0 \text{ and } Re\,\lambda \geq 0\}$. Let $F : C_0[0,T] \to \mathbf{C}$ be defined $s-a.e.$ and satisfy conditions (1.1) and (1.2) above, and for $\lambda > 0$, let

$$J(\lambda) = \int_{C_0[0,T]} F(\lambda^{-1/2}x)m(dx).$$

If there exists a function $J^*(\lambda)$ analytic in $\mathbf{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0,T]$ with parameter λ, and for λ in $\mathbf{C}_+$ we write

$$(1.6)\qquad \int_{C_0[0,T]}^{anw_\lambda} F(x)m(dx) \equiv J^*(\lambda).$$

Let q be a real parameter ($q \neq 0$) and let F be a functional whose analytic Wiener integral exists for all $\lambda \in \mathbf{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with parameter q and we write

$$(1.7)\qquad \int_{C_0[0,T]}^{anf_q} F(x)m(dx) = \lim_{\lambda\to -iq} \int_{C_0[0,T]}^{anw_\lambda} F(x)m(dx)$$

where $\lambda \to -iq$ through values in $\mathbf{C}_+$.

2. A Fubini theorem

In our first theorem we obtain a Fubini theorem for analytic Wiener integrals.

THEOREM 1. *Let $F : C_0[0,T] \to \mathbf{C}$ satisfy conditions (1.1) and (1.2) above. Then*

$$(2.1)\qquad \begin{aligned} &\int_{C_0[0,T]}^{anw_\beta} \left(\int_{C_0[0,T]}^{anw_\lambda} F(y+z)m(dy) \right) m(dz) \\ &\doteq \int_{C_0[0,T]}^{anw_\lambda} \left(\int_{C_0[0,T]}^{anw_\beta} F(y+z)m(dz) \right) m(dy) \end{aligned}$$

where $\doteq$ *means that if either side exists for all* $(\lambda,\beta)\in \mathbf{C}_+\times\mathbf{C}_+$, *then both sides exist for all* $(\lambda,\beta)\in\mathbf{C}_+\times\mathbf{C}_+$ *and equality holds.*

Proof. We begin the proof by observing that the iterated analytic Wiener integrals in (2.1) are defined by analytic continuation of the Wiener integral of the functional $F(\frac{y}{\sqrt{\lambda}}+\frac{z}{\sqrt{\beta}})$.

This integrand has symmetric properties in the sense that if we let

$$K(\lambda,y,\beta,z)\equiv F(\frac{y}{\sqrt{\lambda}}+\frac{z}{\sqrt{\beta}}) \tag{2.2}$$

for

$$(\lambda,y,\beta,z)\in(0,+\infty)\times C_0[0,T]\times(0,+\infty)\times C_0[0,T],$$

then

$$K(\lambda,y,\beta,z)\equiv K(\beta,z,\lambda,y).$$

Consequently the Wiener integrals

$$\int_{C_0[0,T]}F(\frac{y}{\sqrt{\lambda}}+\frac{z}{\sqrt{\beta}})m(dy)$$

and

$$\int_{C_0[0,T]}F(\frac{y}{\sqrt{\lambda}}+\frac{z}{\sqrt{\beta}})m(dz)$$

are actually the same (with λ and β interchanged and y and z interchanged). Therefore we point out that

$$\int_{C_0[0,T]}^{anw_\lambda}F(y+\frac{z}{\sqrt{\beta}})m(dy)$$

exists for all $\lambda\in\mathbf{C}_+,\ \beta>0$ and $s-a.e.\ z\in C_0[0,T]$ if and only if

$$\int_{C_0[0,T]}^{anw_\beta}F(\frac{y}{\sqrt{\lambda}}+z)m(dz)$$

exists for all $\beta\in\mathbf{C}_+,\ \lambda>0$ and $s-a.e.\ y\in C_0[0,T]$.

Because the functional F is a scale-invariant measurable functional on Wiener space we may apply Theorem 9 of [11] and the usual Fubini theorem (see Royden, [14], p. 307) to conclude that for all $(\lambda,\beta) \in (0,+\infty)\times(0,+\infty)$ we have

$$
\begin{aligned}
&\int_{C_0[0,T]} \Big(\int_{C_0[0,T]} F\Big(\frac{y}{\sqrt{\lambda}}+\frac{z}{\sqrt{\beta}}\Big)m(dy)\Big)m(dz)\\
&= \int_{C_0[0,T]} \Big(\int_{C_0[0,T]} F\Big(\frac{y}{\sqrt{\lambda}}+\frac{z}{\sqrt{\beta}}\Big)m(dz)\Big)m(dy)\\
&= \int_{C_0^2[0,T]} F\Big(\frac{y}{\sqrt{\lambda}}+\frac{z}{\sqrt{\beta}}\Big)d(m\times m)(y,z)\\
&= \int_{C_0[0,T]} F\Big(\sqrt{\frac{\beta+\lambda}{\lambda\beta}}x\Big)m(dx)\\
&= \int_{C_0[0,T]} F\Big(\frac{x}{\sqrt{\frac{\lambda\beta}{\lambda+\beta}}}\Big)m(dx).
\end{aligned}
\tag{2.3}
$$

This last expression is defined for $\lambda > 0$ and $\beta > 0$. For each $\beta > 0$ it can be analytically continued in λ for $\lambda \in \mathbf{C}_+$. Also for $\lambda > 0$ it can be analytically continued in β for $\beta \in \mathbf{C}_+$. Therefore since $\lambda \in \mathbf{C}_+$, $\beta \in \mathbf{C}_+$ implies that $\frac{\lambda\beta}{\lambda+\beta} \in \mathbf{C}_+$, an application of Lemma 1 of [1] enables us to conclude that the last expression in (2.3) can be analytically continued into $\mathbf{C}_+$ to equal the analytic Wiener integral

$$
\int_{C_0[0,T]}^{anw_{\frac{\lambda\beta}{\lambda+\beta}}} F(x)m(dx)
\tag{2.4}
$$

and Theorem 1 is proved. □

NOTATION. To simplify some expressions it is helpful to let

$$
\int_{C_0[0,T]}^{an_\lambda} F(x)m(dx) \equiv \begin{cases} \displaystyle\int_{C_0[0,T]}^{anw_\lambda} F(x)m(dx) & ,\ \lambda \in \mathbf{C}_+ \\[2ex] \displaystyle\int_{C_0[0,T]}^{anf_q} F(x)m(dx) & ,\ \lambda = -iq \in \tilde{\mathbf{C}}_+ - \mathbf{C}_+. \end{cases}
\tag{2.5}
$$

REMARK 3. Note that in the definition of the analytic Feynman integral (1.7), we assumed that λ could approach $-iq$ in an arbitrary fashion through values in $\mathbf{C}_+$; i.e., the limit exists and is the same no matter how $\lambda \to -iq$ through values in $\mathbf{C}_+$.

The following lemma is a consequence of Remark 3.

LEMMA 1. *Let $F : C_0[0,T] \to \mathbf{C}$ be as in Theorem 1 above. Furthermore, assume that*

$$G(\lambda) \equiv \int_{C_0[0,T]}^{an_\lambda} F(x)m(dx) \tag{2.6}$$

exists for all $\lambda \in \tilde{\mathbf{C}}_+$. Then $G(\lambda)$ is a continuous function of λ on $\tilde{\mathbf{C}}_+$, and hence is a uniformly continuous function of λ on all compact subsets of $\tilde{\mathbf{C}}_+$.

THEOREM 2. *Let $F : C_0[0,T] \to \mathbf{C}$ be as in Theorem 1 above. Furthermore, assume that the analytic Feynman integral $\int_{C_0[0,T]}^{anf_q} F(x)m(dx)$ exists for all real $q \neq 0$. Let q_1 and q_2 be elements of $\mathbf{R} - \{0\}$ with $q_1 + q_2 \neq 0$. Then*

$$\begin{aligned} &\int_{C_0[0,T]}^{anf_{q_2}} \left(\int_{C_0[0,T]}^{anf_{q_1}} F(y+z)m(dy) \right) m(dz) \\ &\doteq \int_{C_0[0,T]}^{anf_{q_1}} \left(\int_{C_0[0,T]}^{anf_{q_2}} F(y+z)m(dz) \right) m(dy) \end{aligned} \tag{2.7}$$

where $\doteq$ means that if either side exists, both sides exist and equality holds.

Proof. *Let E be any subset of $\tilde{\mathbf{C}}_+ \times \tilde{\mathbf{C}}_+$ containing the point $(-iq_1, -iq_2)$ and is such that $(\lambda, \beta) \in E$ implies that $\lambda + \beta \neq 0$. Then (see Remark 3 and Lemma 1) the function*

$$H(\lambda, \beta) \equiv \int_{C_0[0,T]}^{an_\beta} \left(\int_{C_0[0,T]}^{an_\lambda} F(y+z)m(dy) \right) m(dz)$$

is continuous on E and is uniformly continuous on E provided E is compact. Now assume that the left hand side of equation (2.7) exists. Then by (1.7), the continuity of H, and Theorem 1, we obtain that

$$\int_{C_0[0,T]}^{anf_{q_2}} \left(\int_{C_0[0,T]}^{anf_{q_1}} F(y+z)m(dy) \right) m(dz)$$

$$
\begin{aligned}
&= \lim_{\beta \to -iq_2} \int_{C_0[0,T]}^{anw_\beta} \left(\lim_{\lambda \to -iq_1} \int_{C_0[0,T]}^{anw_\lambda} F(y+z)m(dy) \right) m(dz) \\
&= \lim_{\beta \to -iq_2} \lim_{\lambda \to -iq_1} \int_{C_0[0,T]}^{anw_\beta} \left(\int_{C_0[0,T]}^{anw_\lambda} F(y+z)m(dy) \right) m(dz) \\
&= \lim_{\beta \to -iq_2} \lim_{\lambda \to -iq_1} \int_{C_0[0,T]}^{anw_\lambda} \left(\int_{C_0[0,T]}^{anw_\beta} F(y+z)m(dz) \right) m(dy) \\
&= \lim_{\lambda \to -iq_1} \int_{C_0[0,T]}^{anw_\lambda} \left(\lim_{\beta \to -iq_2} \int_{C_0[0,T]}^{anw_\beta} F(y+z)m(dz) \right) m(dy) \\
&= \int_{C_0[0,T]}^{anf_{q_1}} \left(\int_{C_0[0,T]}^{anf_{q_2}} F(y+z)m(dz) \right) m(dy)
\end{aligned}
$$

as desired. □

THEOREM 3. *Let $F : C_0[0,T] \to \mathbf{C}$ be as in Theorem 1 above. Furthermore, assume that the analytic Feynman integral $\int_{C_0[0,T]}^{anf_q} F(x)m(dx)$ exists for all real $q \neq 0$. Then (using the notation given in equation (2.5)) for all $(\lambda, \beta) \in \tilde{\mathbf{C}}_+ \times \tilde{\mathbf{C}}_+$ with $\lambda + \beta \neq 0$,*

$$
\begin{aligned}
&\int_{C_0[0,T]}^{an_\beta} \left(\int_{C_0[0,T]}^{an_\lambda} F(y+z)m(dy) \right) m(dz) \\
&= \int_{C_0[0,T]}^{an_{\frac{\lambda\beta}{\lambda+\beta}}} F(x)m(dx) \qquad (2.8) \\
&= \int_{C_0[0,T]}^{an_\lambda} \left(\int_{C_0[0,T]}^{an_\beta} F(y+z)m(dz) \right) m(dy).
\end{aligned}
$$

Proof. We first note that if λ and β are in $\mathbf{C}_+$, then $\gamma = \frac{\lambda\beta}{\lambda+\beta}$ and $\frac{\lambda+\beta}{\lambda\beta}$ are in $\mathbf{C}_+$. Thus, our assumption that the analytic Feynman integral $\int_{C_0[0,T]}^{anf_q} F(x)m(dx)$ exists for all real $q \neq 0$, implies that the integral $\int_{C_0[0,T]}^{an_{\frac{\lambda\beta}{\lambda+\beta}}} F(x)m(dx)$ exists for all $(\lambda, \beta) \in \tilde{\mathbf{C}}_+ \times \tilde{\mathbf{C}}_+$ with $\lambda + \beta \neq 0$. But by Theorem 9 of [11] we see that for all $(\lambda, \beta) \in (0, +\infty) \times (0, +\infty)$,

$$
\int_{C_0[0,T]} \left(\int_{C_0[0,T]} F\left(\frac{y}{\sqrt{\lambda}} + \frac{z}{\sqrt{\beta}}\right) m(dy) \right) m(dz)
$$

$$= \int_{C_0[0,T]} F\left(\frac{x}{\sqrt{\frac{\lambda\beta}{\lambda+\beta}}}\right) m(dx)$$

$$= \int_{C_0[0,T]} \left(\int_{C_0[0,T]} F(\frac{y}{\sqrt{\lambda}} + \frac{z}{\sqrt{\beta}}) m(dz)\right) m(dy).$$

Thus equation (2.8) is valid for all $(\lambda,\beta) \in (0,+\infty)\times(0,+\infty)$. Analytic continuation yields equation (2.8) for all $(\lambda,\beta) \in \mathbf{C}_+ \times \mathbf{C}_+$.

Finally, continuity (established in Lemma 1) yields equation (2.8) whenever λ and/or β are elements of $\mathbf{C}\tilde{}_+ - \mathbf{C}_+$. Recall that if both λ and β are in $\mathbf{C}\tilde{}_+ - \mathbf{C}_+$, we are assuming that $\lambda + \beta \neq 0$. □

3. Applications

We first note that the hypotheses (and hence the conclusions) of Theorems 1-3 in section 2 above are indeed satisfied by several large classes of functionals. These classes of functionals include:

(a) The Banach algebra S defined by Cameron and Storvick in [3]: also see [6,13].

(b) Various spaces of functionals of the form

$$F(x) = f(\int_0^T \alpha_1(t)dx(t), \ldots, \int_0^T \alpha_m(t)dx(t))$$

for appropriate f as discussed in [7,12].

(c) Various spaces of functionals of the form

$$F(x) = \exp\{\int_0^T f(t,x(t))dx\}$$

for appropriate $f : [0,T]\times\mathbf{R}\to\mathbf{C}$ as discussed in [8].

(d) Various spaces of functionals of the form

$$F(x) = \exp\{\int_0^T \int_0^T f(s,t,x(s),x(t))dsdt\}$$

for appropriate $f : [0,T]^2\times\mathbf{R}^2\to\mathbf{C}$ as discussed in [9].

Throughout this section it is assumed that the functional $F{:}C_0[0,T]\to\mathbf{C}$ satisfies the hypotheses of Theorem 3 above. We will state our results in terms of analytic Feynman integrals; similar results of course hold for analytic Wiener integrals.

To obtain equation (3.1) below, we simply let $\lambda = \beta = -iq$ in Theorem 3.

Corollary 1 (of Theorem 3). For all real $q \neq 0$,

$$\int_{C_0[0,T]}^{anf_q} \left(\int_{C_0[0,T]}^{anf_q} F(y+z)m(dy) \right) m(dz) = \int_{C_0[0,T]}^{anf_{q/2}} F(x)m(dx). \tag{3.1}$$

In fact, for any positive integer $n \geq 2$,

$$\int_{C_0[0,T]}^{anf_q} \int_{C_0[0,T]}^{anf_q} \cdots \int_{C_0[0,T]}^{anf_q} F(y_1 + \cdots + y_n)m(dy_1)\ldots m(dy_{n-1})m(dy_n) = \int_{C_0[0,T]}^{anf_{q/n}} F(x)m(dx). \tag{3.2}$$

Corollary 2 (of Theorem 3). *Let* q_1, q_2 *and* q_3 *be elements of* $\mathbf{R} - \{0,\}$ *with* $q_1+q_2 \neq 0$, $q_1+q_3 \neq 0$, $q_2+q_3 \neq 0$ *and* $q_1q_2+q_1q_3+q_2q_3 \neq 0$. *Then*

$$\begin{aligned} &\int_{C_0[0,T]}^{anf_{q_3}} \Big(\int_{C_0[0,T]}^{anf_{q_2}} \big(\int_{C_0[0,T]}^{anf_{q_1}} F(y_1+y_2+y_3)m(dy_1)\big)m(dy_2)\Big)m(dy_3) \\ &= \int_{C_0[0,T]}^{anf_{q_3}} \big(\int_{C_0[0,T]}^{anf_{(q_1q_2)/(q_1+q_2)}} F(z+y_3)m(dz)\big)m(dy_3) \\ &= \int_{C_0[0,T]}^{anf_{(q_1q_2q_3)/(q_1q_2+q_1q_3+q_2q_3)}} F(x)m(dx). \end{aligned} \tag{3.3}$$

Remark 4. (i) Note that each of the iterated integrals in equation (3.3) above can also be expressed in five other similar ways; for example, all of the expressions in (3.3), also equal the expression

$$\int_{C_0[0,T]}^{anf_{(q_2q_3)/(q_2+q_3)}} \left(\int_{C_0[0,T]}^{anf_{q_1}} F(z+y_1)m(dy_1) \right) m(dz).$$

(ii) Clearly there is an n-dimensional version of the above corollary.

LEMMA 2. *For all real $q \neq 0$ and all $a > 0$,*

$$\int_{C_0[0,T]}^{anf_{aq}} F(x)m(dx) = \int_{C_0[0,T]}^{anf_q} F(\frac{x}{\sqrt{a}})m(dx). \tag{3.4}$$

Proof. We first note that for all $\lambda > 0$,

$$\begin{aligned}\int_{C_0[0,T]}^{anw_{a\lambda}} F(x)m(dx) &= \int_{C_0[0,T]} F(\frac{\lambda^{-1/2}x}{\sqrt{a}})m(dx)\\ &= \int_{C_0[0,T]}^{anw_\lambda} F(\frac{x}{\sqrt{a}})m(dx).\end{aligned}$$

Equation (3.4) now follows by analytic continuation in λ. □

THEOREM 4. *For $a, b \in \mathbf{R}$ and $q_1, q_2 \in \mathbf{R} - \{0\}$ with $q_1 b^2 + q_2 a^2 \neq 0$,*

$$\begin{aligned}&\int_{C_0[0,T]}^{anf_{q_2}} \left(\int_{C_0[0,T]}^{anf_{q_1}} F(ay+bz)m(dy)\right) m(dz)\\ &\qquad = \int_{C_0[0,T]}^{anf_{(q_1q_2)/(q_1b^2+q_2a^2)}} F(x)m(dx).\end{aligned} \tag{3.5}$$

Proof. If either $a = 0$ or $b = 0$, the proof is immediate. Since for Wiener integrals,

$$\int_{C_0[0,T]} F(-x)m(dx) = \int_{C_0[0,T]} F(x)m(dx),$$

we may assume that both a and b are positive. Then using Theorem 9 of [11] and Lemma 2 above, we see that for all $(\lambda, \beta) \in (0, +\infty) \times (0, +\infty)$,

$$\begin{aligned}&\int_{C_0[0,T]}^{anw_\beta} \left(\int_{C_0[0,T]}^{anw_\lambda} F(ay+bz)m(dy)\right) m(dz)\\ &\quad = \int_{C_0[0,T]} \left(\int_{C_0[0,T]} F(\frac{ay}{\sqrt{\lambda}} + \frac{bz}{\sqrt{\beta}})m(dy)\right) m(dz)\\ &\quad = \int_{C_0[0,T]} F\left(\sqrt{\frac{a^2}{\lambda} + \frac{b^2}{\beta}}x\right) m(dx)\\ &\quad = \int_{C_0[0,T]}^{anw_{(\lambda\beta)/(\lambda b^2+\beta a^2)}} F(x)m(dx).\end{aligned}$$

Then equation (3.5) follows by analytic continuation in λ and β. $\square$

We finish this paper by simply writing down some interesting special cases of equations (3.1)-(3.5) above:

$$\int_{C_0[0,T]}^{anf_{q/2}} F(x)m(dx) = \int_{C_0[0,T]}^{anf_q} F(\sqrt{2}x)m(dx), \tag{3.6}$$

$$\int_{C_0[0,T]}^{anf_{2q}} F(x)m(dx) = \int_{C_0[0,T]}^{anf_q} F(x/\sqrt{2})m(dx), \tag{3.7}$$

$$\int_{C_0[0,T]}^{anf_q} \left(\int_{C_0[0,T]}^{anf_q} F(\frac{y \pm z}{\sqrt{2}})m(dy) \right) m(dz) = \int_{C_0[0,T]}^{anf_q} F(x)m(dx), \tag{3.8}$$

$$\begin{aligned} &\int_{C_0[0,T]}^{anf_{q_2}} \left(\int_{C_0[0,T]}^{anf_{q_1}} F(y \pm \frac{z}{\sqrt{2}})m(dy) \right) m(dz) \\ &\quad = \int_{C_0[0,T]}^{anf_{(q_1q_2)/(q_1+2q_2)}} F(x/\sqrt{2})m(dx)\ , \end{aligned} \tag{3.9}$$

provided $q_1 + 2q_2 \neq 0$.

References

[1] R. H. Cameron and D. A. Storvick, *Analytic Continuation for Functions of Several Complex Variables*, Trans. Amer. Math. Soc. **125** (1966), 7–12.

[2] ______, *An L_2 Analytic Fourier-Feynman Transform*, Michigan Math. J. **23** (1976), 1–30.

[3] ______, *Some Banach Algebras of Analytic Feynman Integrable Functionals, Analytic Functions (Kozubnik, 1979), Lecture Notes in Math. Springer, Berlin* **798** (1980), 18–67.

[4] ______, *A Simple Definition of the Feynman Integral, with Applications*, Mem. Amer. Math. Soc. **46** (1983), 1–46.

[5] K. S. Chang, *Scale-invariant Measurability in Yeh-Wiener Space*, J. Korean Math. Soc. **19** (1982), 61–67.

[6] K. S. Chang, G. W. Johnson, and D. L. Skoug, *Functions in the Banach Algebra $S(v)$*, J. Korean Math. Soc. **24** (1987), 151–158.

[7] T. Huffman, C. Park, and D. Skoug, *Analytic Fourier-Feynman Transforms and Convolution*, Trans. Amer. Math. Soc. **347** (1995), 661–673.

[8] ———, *Convolution and Fourier-Feynman Transforms*, Rocky Mountain J. Math. **27** (1997), 827–841.

[9] ———, *Convolutions and Fourier-Feynman Transforms of Functionals InvolvingMultiple Integrals*, Michigan Math. J. **43** (1996), 247–261.

[10] G. W. Johnson and D. L. Skoug, *An L_p Analytic Fourier-Feynman Transform*, Michigan Math. J. **26** (1979), 103–127.

[11] ———, *Scale-invariant Measurability in Wiener Space*, Pacific J. Math. **83** (1979), 157–176.

[12] J. Kim, J. Ko, C. Park, and D. Skoug, *Relationships among Transforms, Convolutions, and First Variations*, Int. J. Math. Math. Sci. **22** (1999), 191–204.

[13] C. Park, D. Skoug, and D. Storvick, *Relationships among the First Variation, the Convolution Product and the Fourier-Feynman Transform*, Rocky Mountain J. Math. **28** (1998), 1447–1468.

[14] H. L. Royden, *Real Analysis*, Macmillan (1988).

[15] I. Segal, *Transformations in Wiener Space and Squares of Quantum Fields*, Adv. Math. **4** (1970), 91–108.

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