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CONSTRUCTION OF SOME PROCESSES ON THE WIENER SPACE ASSOCIATED TO SECOND ORDER OPERATORS

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ABSTRACT. We show that it is possible to associate diffusion processes to second order perturbations of the Ornstein-Uhlenbeck operator L on the Wiener space of the form

$$\mathcal{L} = L + \frac{1}{2}\sum_k \mathcal{L}^2_{\xi_k}$$

where the ξ_k are "tangent processes" (i.e., semimartingales with antisymmetric diffusion coefficients).

1. Introduction and background

In the last years the development of the study of geometry of path spaces over a Riemannian manifold ([4, 5, 3]) has shown the need to consider variations of Wiener functionals along more general paths than the Cameron-Martin ones traditionally used in Malliavin Calculus. Indeed, a Cameron-Martin variation on the path space over a Riemannian manifold corresponds to a variation on the Wiener space along a semimartingale whose diffusion coefficient, given by the curvature tensor of the manifold, is antisymmetric (and therefore by Levy's theorem, still keeps the Wiener measure invariant under its action). More precisely we have the following:

DEFINITION 1.1 ([3]). A tangent process on the Wiener space X is a $\mathbb{R}^d$-valued semimartingale process defined on X with Itô differential

$$d_\tau \xi_\alpha = a^\beta_\alpha \, dx_\beta(\tau) + b_\alpha \, d\tau$$

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where $\alpha, \beta = 1, \ldots, d$, $a_\alpha^\beta = -a_\beta^\alpha$, $a_\alpha^\beta(0) = 0$, and such that ξ, besides its representation as an Itô integral, can be represented in terms of a Stratonovich integral.

Let us recall the notion of Ornstein-Uhlenbeck operator on the Wiener space $X = \mathcal{C}([0,1]; \mathbb{R}^d)$.

For a cylindrical functional $F(x) = f(x(\tau_1), \ldots, x(\tau_m))$, $x \in X$, $0 \le \tau_1 < \cdots < \tau_m \le 1$, and f a smooth bounded function on $\mathbb{R}^m$, the derivative operators $D_{\tau,\alpha}F$, $\alpha = 1, \ldots, d$, $\tau \in [0,1]$, are defined by (cf. [9])

$$D_{\tau,\alpha}F(x) = \sum_{i=1}^{m} \big(\partial_i f\big)^\alpha \big(x(\tau_1), \ldots, x(\tau_m)\big)\, \mathbf{1}_{\tau<\tau_i}$$

and, for $h \in H^1 = \{h \in X : \exists \dot{h} \in L^2[0,1]\}$,

$$D_h F = \sum_\alpha \int_0^1 D_{\tau,\alpha}\, F\, \dot{h}^\alpha(\tau) d\tau. \tag{1.1}$$

We denote by W_1^p the domain of D in L_μ^p, where μ is the Wiener measure. This space is the closure of the class of cylindrical functionals with respect to the Sobolev norm

$$\|F\|_{1,p}^p = E_\mu(|F|^p + \|DF\|^p),$$
$$\|DF\| = \left(\sum_\alpha \int_0^1 |D_{\tau,\alpha}F|^2\, d\tau\right)^{1/2}$$

The dual of the derivative operator in L_μ^2, namely the operator defined for H^1-valued processes $u(\cdot)$ by

$$E_\mu(D_u F) = E_\mu(F\delta(u)), \quad \forall F \in W_1^2, \tag{1.2}$$

is called the divergence operator.

It was discovered by Gaveau and Trauber [6] the fundamental relation between this operator and stochastic calculus: the divergence coincides with the Skorohod integral, which is an extension for anticipative processes of the Itô stochastic integral.

If u is a process in $L^2([0,1] \times X)$ such that $u_\tau \in W_1^2$ for a.a. τ, and

$$E \int_0^1 \int_0^1 \|D_\sigma u_\tau\|^2\, d\sigma\, d\tau < +\infty,$$

the Skorohod integral can be characterized [9] as the limit of the following sums (when the mesh of the corresponding partition tends to zero):

$$\delta(u) = \lim\Bigg[\sum_k \frac{1}{\tau_{k+1}-\tau_k}\int_{\tau_k}^{\tau_{k+1}} u_\sigma \, d\sigma\big(x(\tau_{k+1}) - x(\tau_k)\big) - \sum_k \frac{1}{\tau_{k+1}-\tau_k}\int_{\tau_k}^{\tau_{k+1}}\int_{\tau_k}^{\tau_{k+1}} D_\tau.u_\sigma \, d\tau\, d\sigma\Bigg]. \tag{1.3}$$

Under (much) stronger assumptions on u the first term of these sums converges to the so-called Stratonovich-Skorohod integral, denoted by $\int_0^1 u \circ dx$, and one has the identity

$$\delta(u) = \int_0^1 u \circ dx - \frac{1}{2}\int_0^1 (D_\tau^+ + D_\tau^-).u_\tau \, d\tau, \tag{1.4}$$

where

$$D_\tau^\pm .u_\tau = \lim_{\sigma\to\tau^\pm} \sum_{\alpha=1}^d D_\tau^\alpha u_\sigma^\alpha.$$

The question of describing conditions under which the Stratonovich-Skorohod integral exists is actually a delicate one. The assumptions are usually made via the Skorohod integral and it is required that both $\delta(u)$ and $\frac{1}{2}\int_0^1 (D_\tau^+ + D_\tau^-).u_\tau \, d\tau$ converge.

We also recall the following commutation formula:

$$D_h(\delta u) = \delta(D_h u) + \int_0^1 \dot{u}\dot{h} \, d\tau. \tag{1.5}$$

The Ornstein-Uhlenbeck operator on the Wiener space, that we shall denote by L [8], is defined by

$$LF = -\delta DF. \tag{1.6}$$

This operator is a fundamental object in Quantum Field Theory in $1+1$ dimensions and corresponds to the free case.

The Poincaré-type inequalities due to Krée and Meyer state that

$$c_1 \|LF\|_{L^p_\mu} \le \|D^2 F\|_{L^p} \le c_2 \|LF\|_{L^p},$$

with c_1, c_2 constants, $1 < p < +\infty$, and where D^2 denotes the second derivative of the functional F. Therefore the domain of L is the Sobolev space $W_2^2(X)$. The operator L is the generator of a process, the Ornstein-Uhlenbeck process (see [8, 9], for example). This is a stationary Gaussian continuous Markov process with values on X which has μ as invariant measure.

From its definition and the characterization of the divergence it is easily seen how L is a second order operator. We also observe that, in the assumption that both terms are well defined, we can write

$$LF = -\int_0^1 D_\tau F \circ dx(\tau) + \frac{1}{2}\int_0^1 (D_\tau^+ + D_\tau^-) \,.\, D_\tau F \, d\tau, \tag{1.7}$$

the second term being responsible for the second order derivatives.

2. Second order operators associated to tangent processes

We shall consider operators on the Wiener space of the Hormander-type form $\mathcal{L}_\xi^2$ where ξ is a tangent process (see Definition 1.1) and $\mathcal{L}_\xi F = D_\xi F$. We need a representation for such a derivative, in analogy with (1.1) for Cameron-Martin vectors. Such a formula uses anticipative calculus (since $\tau \mapsto D_\tau$ is not adapted) and was shown in [3] (cf. (2.3.9)):

THEOREM 2.1. *Let ξ be a tangent process, $d_\tau \xi^\alpha = a_\alpha^\beta dx_\beta(\tau) + b_\alpha d\tau$, such that $E\int_0^1 \|Da(\tau)\|^p < +\infty$ for every $1 < p < +\infty$ and $E\int_0^1 \|b(\tau)\|^2 d\tau < +\infty$. Then W_2^p is contained in the domain of D_ξ for all $p > 1$ and we have:*

$$D_\xi F = \int_0^1 D_{\tau,\alpha} F(a_\alpha^\beta \, dx_\beta + b_\alpha \, d\tau)$$

for $F \in W_2^p(X)$ and where the stochastic integral is taken in the sense of Skorohod. If $d_\tau \xi_\alpha = a_\alpha^\beta \circ dx_\beta(\tau) + b_\alpha d\tau$, we have

$$D_\xi F = \int_0^1 D_{\tau,\alpha} F(a_\alpha^\beta \circ dx_\beta + b_\alpha \, d\tau).$$

According to the expression of the Skorohod integral, which coincides with a divergence (see (1.4)), it is not clear a priori why the operator $\mathcal{L}_\xi^2$ should be of second order. Let us consider the case $b = 0$. We have (without specifying regularity conditions):

(2.1)

$$\begin{aligned}
\mathcal{L}_\xi F &= \sum_\alpha \int_0^1 D_{\tau,\alpha} F a_\alpha^\beta(\tau) \, dx_\beta(\tau) \\
&= \sum_\alpha \int_0^1 \left(D_{\tau,\alpha} F a_\alpha^\beta(\tau)\right) \circ dx_\beta(\tau) - \frac{1}{2}\sum_\alpha \int_0^1 D_\tau . (D_{\tau,\alpha} F a_\alpha(\tau)) \, d\tau \\
&= \sum_\alpha \int_0^1 \left(D_{\tau,\alpha} F a_\alpha^\beta(\tau)\right) \circ dx_\beta(\tau) - \frac{1}{2}\sum_{\alpha,\beta} \int_0^1 D_{\tau,\alpha} F D_{\tau,\beta} a_\alpha^\beta(\tau) \, d\tau.
\end{aligned}$$

We have assumed continuity in time of the derivatives and used the fact that, since a is an antisymmetric matrix, and $D_{\tau,\beta}D_{\tau,\alpha} = D_{\tau,\alpha}D_{\tau,\beta}$,

$$\sum_{\alpha,\beta} \int_0^1 D_{\tau,\beta}D_{\tau,\alpha}F a_\alpha^\beta(\tau)\, d\tau = 0.$$

The operator $\mathcal{L}_\xi F$ is therefore a derivation and $\mathcal{L}_\xi^2 F$ only involves second order derivatives.

Let us write a more explicit expression for this operator in the case where $d_\tau \xi^\alpha = a_\alpha^\beta \circ dx_\beta(\tau)$. We have:

$$\mathcal{L}_\xi^2 F = \sum_\alpha \int_0^1 D_{\tau,\alpha}(\mathcal{L}_\xi F)\, a_\alpha^\beta(\tau) \circ dx_\beta(\tau)$$

and

$$\begin{aligned} D_{\tau,\alpha}(\mathcal{L}_\xi F) = \sum_\gamma \int_0^1 & D_{\tau,\alpha}\big(D_{\sigma,\gamma}F a_\gamma^{\beta'}(\sigma)\big) \circ dx_{\beta'}(\sigma) \\ & + \sum_\gamma D_{\tau,\gamma}F a_\gamma^\alpha(\tau). \end{aligned}$$

Therefore,

(2.2)

$$\begin{aligned} \mathcal{L}_\xi^2 F &= \sum_\alpha \int_0^1 D_{\tau,\alpha}(\mathcal{L}_\xi F)\, a_\alpha^\beta(\tau) \circ dx_\beta(\tau) \\ &\quad - \frac{1}{2} \sum_{\alpha,\beta} \int_0^1 D_{\tau,\alpha}(\mathcal{L}_\xi F)\, D_{\tau,\beta}\, a_\alpha^\beta(\tau)\, d\tau \\ &= \sum_{\alpha,\gamma} \int_0^1 \left(\int_0^1 D_{\tau,\alpha}D_{\sigma,\gamma}F a_\gamma^{\beta'}(\sigma) \circ dx_{\beta'}(\sigma) \right) a_\alpha^\beta(\tau) \circ dx_\beta(\tau) \\ &\quad + \sum_{\alpha,\gamma} \int_0^1 \left(\int_0^1 D_{\sigma,\gamma}F D_{\tau,\alpha} a_\gamma^{\beta'}(\sigma) \circ dx_{\beta'}(\sigma) \right) a_\alpha^\beta(\tau) \circ dx_\beta(\tau) \\ &\quad + \sum_{\alpha,\gamma} \int_0^1 D_{\tau,\gamma}F a_\gamma^\alpha(\tau)\, a_\alpha^\beta(\tau) \circ dx_\beta(\tau) \\ &\quad - \frac{1}{2} \sum_{\alpha,\beta,\gamma} \int_0^1 \left(\int_0^1 D_{\tau,\alpha}D_{\sigma,\gamma}F a_\gamma^{\beta'}(\sigma) \circ dx_{\beta'}(\sigma) \right) D_{\tau,\beta} a_\alpha^\beta(\tau)\, d\tau \end{aligned}$$

$$-\frac{1}{2}\sum_{\alpha,\beta,\gamma}\int_0^1\left(\int_0^1 D_{\sigma,\gamma}F D_{\tau,\alpha}a_\gamma^{\beta'}(\sigma)\circ dx_{\beta'}(\sigma)\right) D_{\tau,\beta}a_\alpha^\beta(\tau)\,d\tau$$
$$-\frac{1}{2}\sum_{\alpha,\beta,\gamma}\int_0^1 D_{\tau,\gamma}F a_\gamma^\alpha(\tau)\,D_{\tau,\beta}a_\alpha^\beta(\tau)\,d\tau.$$

Finally, for every $F,G\in\bigcap_p W_2^p(X)$, we have

$$\begin{aligned}E_\mu(\mathcal{L}_\xi F.G) &= E_\mu(\mathcal{L}_\xi(FG)) - E_\mu(F\mathcal{L}_\xi G)\\ &= -E_\mu(F\mathcal{L}_\xi G),\end{aligned}$$

since $d_\tau\xi^\alpha = a_\alpha^\beta\,dx_\beta(\tau)$ defines a measure preserving isomorphism on the Wiener space. In particular, if $F,G\in\bigcap_p W_4^p(X)$,

$$E_\mu(\mathcal{L}_\xi^2 F.G) = -E_\mu(\mathcal{L}_\xi F.\mathcal{L}_\xi G), \tag{2.3}$$

a property that will be used in the next paragraph.

3. Construction of the associated processes

As in [1], where we showed the existence of flows associated to tangent processes on the Wiener space, we look at these processes as functionals with values on the Wiener space, $\xi: X\to X$. The gradient of ξ is a linear operator in $\mathcal{L}(H;X)$ defined by

$$\nabla\xi(x)(h) = D_h\xi(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[\xi(x+\epsilon h)-\xi(x)],$$

where the limit is taken in the supremum (Banach) norm and a.e. with respect to the Wiener measure. Considering $\mathcal{L}(H;X)$ as a Banach space, we can define the second gradient and proceed in the same way for higher derivatives. Then we consider the following norms:

$$\|\nabla^j\xi\| = \sup_{h_i\in H;\|h_i\|_H\le 1}\|\nabla^j\xi(h_1,\dots,h_j)\|_X$$

and the corresponding Sobolev spaces $W_2^p(X;X)$.

We present here a generalization of a result in [2] for operators of the form $L+(1/2)\sum_k \mathcal{L}_{\xi_k}^2$, where ξ_k are tangent processes (and not Cameron-Martin vector fields). We restrict ourselves to processes which are martingales since the general case is a modification of the result via Girsanov transformation, as in [2].

THEOREM 3.1. *Let* $\{\xi_k\}$, $d_\tau(\xi_k)_\alpha = (a_k)_\alpha^\beta \, dx_\beta(\tau)$, *be a family of tangent processes on the Wiener space such that*

$$\sum_k \|\xi_k\|_{W_3^p(X,X)} < +\infty, \quad \forall p$$

and

$$\Big[I + \sum_{k,\gamma,\alpha} \int_0^1 \Big(\int_0^1 (a_k)_\alpha^{\beta'} \circ dx_{\beta'}(\sigma) \Big) (a_k)_\alpha^\beta \circ dx_\beta(\tau) + \sum_{k,\gamma,\alpha} \int_0^1 \Big(\int_0^1 (a_k)_\alpha^{\beta'}(\sigma) \circ dx_{\beta'}(\sigma) \Big) D_{\tau,\beta}\,(a_k)_\alpha^\beta \, d\tau \Big]^{1/2} - I \in L^4(X).$$

Let $k_0 \in L^2(X)$, *with* $k_0 \geq 0$ *and* $E_\mu k_0 = 1$, *be such that* $E_\mu(\|x\|_X^2 k_0) < +\infty$.

Then there exists a probability space $(\Omega, \mathcal{F}, P)$, *a process* $x_\omega(\cdot) \in \mathcal{C}(\mathbb{R}^+; X)$, $\omega \in \Omega$, *and a function* $k_t \in L^2(X)$ *such that, if* ν *denotes the law of* x_ω *on the space* $\mathcal{C}(\mathbb{R}^+; X)$, *we have:*

1. $\nu(x : x(0) \in T) = \int_T k_0 \, d\mu$;
2. $\int f(x(t)) \, d\nu(x) = \int f \, k_t \, d\mu$, $\forall t > 0$, *for every* $f \in L^2(X)$;
3. $\|k_t\|_{L^2(X)} \leq \|k_0\|_{L^2(X)}$, $\forall t \geq 0$;
4. $E_\mu(f \, k_t) = E_\mu(f \, x_0) + E_\mu \int_0^t \mathcal{L} f \, k_s \, ds$ *for every* $f \in \bigcap_p W_2^p(X)$, *where*

$$\mathcal{L} = L + \frac{1}{2} \sum_k \mathcal{L}_{\xi_k}^2.$$

Idea of the proof. The proof follows the lines of the proof of Theorem 1.3.1 in [2] with the modifications in the approximation procedure corresponding to the fact that we deal with Wiener space valued functionals, as in [1]. The finite dimensional approximations of the tangent processes ξ are defined in the following way: for $x \in X$, we denote by $\pi_n x$ the poligonal line linking the points $x(k2^{-n})$ for $k = 0, 1, \ldots, 2^n$ and by $\mathcal{A}_n$ the finite σ-algebra whose atoms are the dyadic intervals of length 2^{-n}. Then the orthogonal projection $L^2([0,1]) \to L^2(\mathcal{A}_n)$ induces the conditional expectations $E^{\mathcal{A}_n}$ from X to a finite dimensional Gaussian space V_n. We put

$$\xi^{(n)}(x) = \pi_n\big(E^{\sigma_n}(\xi)(x)\big),$$

where σ_n denotes the σ-algebra based on cylinder sets supported on V_n. It was shown in [1] that, for every p,

$$E_\mu\|\xi^{(n)}\|^p \leq E_\mu\|\xi\|^p,$$
$$E_\mu\|\xi^{(n)} - \xi\|^p \to 0 \quad \text{and}$$
$$E_\mu\|\nabla^j\xi^{(n)}\|^p \leq E_\mu\|\nabla^j\xi\|^p.$$

It is possible to define, globally in time, solutions of the stochastic differential equations associated to the finite-dimensional operators

$$\mathcal{L}^n = L^n + \frac{1}{2}\sum_k \mathcal{L}^2_{\xi_k^{(n)}},$$

where L^n is the Ornstein-Uhlenbeck operator on V_n (similarly to [2]).

In passing to the limit, one of the main ingredients is that, by property (2.3), we have the following a priori estimate for the density of the process:

$$\|k_t\|_{L^2(X)} \leq \|k_0\|_{L^2(X)}.$$

Indeed, since $\frac{\partial}{\partial t}k_t = \mathcal{L}^* k_t = \mathcal{L}k_t$,

$$\begin{aligned}\frac{d}{dt}E_\mu(k_t^2) &= 2E_\mu(k_t\mathcal{L}k_t) = 2E_\mu(k_t L k_t) + \sum_k E_\mu(k_t\mathcal{L}^2_{\xi_k}k_t)\\ &= -2\|k_t\|_{W_1^2} - \sum_k E_\mu(\mathcal{L}_{\xi_k}k_t)^2 \leq 0.\end{aligned}$$

□

REMARK 3.1. Examples of tangent processes satisfying the assumptions of theorem 3.1 can be found in [1]. In the simplest case, one can take the random variables a_k of the form $a_k(x)(\tau) = a_k(x(\tau))$, where a_k are smooth functions.

REMARK 3.2. It is possible that a Dirichlet form type approach (see for example [7]) would also provide a process associated to $\mathcal{L}$. The Dirichlet form to be considered would be

$$\mathcal{E}(u,v) = -E(\mathcal{L}u.v).$$

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