

**COHERENT STATE REPRESENTATION
AND UNITARITY CONDITION
IN WHITE NOISE CALCULUS**

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ABSTRACT. White noise distribution theory over the complex Gaussian space is established on the basis of the recently developed white noise operator theory. Unitarity condition for a white noise operator is discussed by means of the operator symbol and complex Gaussian integration. Concerning the overcompleteness of the exponential vectors, a coherent state representation of a white noise function is uniquely specified from the diagonal coherent state representation of the associated multiplication operator.

Introduction

During the recent development of white noise calculus the complex white noise has played some interesting roles in a study of the infinite dimensional unitary group [9], [10], holomorphic white noise functions [11], an infinite dimensional analogue of Bargmann space [14], [23], and so forth. In the previous papers [17], [19] another point of interest is investigated concerning the coherent states (exponential vectors); namely, we established an infinite dimensional analogue of diagonal coherent state representation which is well known in quantum mechanics (e.g., [8], [12], [13], [22]) and, as natural consequences we come to the resolution of the identity, the inversion formulas for the S-transform and for the operator symbol.

The present paper contains two topics which supplement the study directed by the previous papers [17], [19]. First we shall discuss unitarity

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condition for white noise operators in terms of operator symbols. Apparently, our result possesses potential applications in the study of quantum stochastic evolution equations. We shall next revisit the overcompleteness of exponential vectors. For this famous property the coherent state representation of a white noise function is not unique, while the diagonal coherent state representation of a white noise operator is unique [19]. Being based on the quantum–classical correspondence in probability theory, we shall obtain a formula for the coherent state representation of a white noise function regarded as a multiplication operator. Further relevant investigation is now in progress, e.g., [20].

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1. Complex Gaussian space

We start with the real Gelfand triple:

$$(1) \quad E \equiv \mathcal{S}(\mathbf{R}) \subset H \equiv L^2(\mathbf{R}, dt) \subset E^* \equiv \mathcal{S}'(\mathbf{R}),$$

where $\mathcal{S}(\mathbf{R})$ is the space of rapidly decreasing functions and $\mathcal{S}'(\mathbf{R})$ the dual space, i.e., the space of tempered distributions. We denote by $\langle \cdot, \cdot \rangle$ the canonical bilinear form on $E^* \times E$ and by $|\cdot|_0$ the norm of H . For notational convenience, the \mathbf{C} -bilinear form on $E_{\mathbf{C}}^* \times E_{\mathbf{C}}$ is denoted by the same symbol so that $|\xi|_0^2 = \langle \bar{\xi}, \xi \rangle$ holds for $\xi \in H_{\mathbf{C}}$. (In general, the complexification of a real vector space \mathfrak{X} is denoted by $\mathfrak{X}_{\mathbf{C}}$.)

Let μ' be the Gaussian measure on E^* with variance 1/2, namely, a probability measure on E^* determined uniquely by the characteristic function:

$$e^{-|\xi|_0^2/4} = \int_{E^*} e^{i\langle x, \xi \rangle} \mu'(dx), \quad \xi \in E.$$

In view of the topological isomorphism $E_{\mathbf{C}}^* \cong E^* \times E^*$, we define a probability measure $\nu = \mu' \times \mu'$ on $E_{\mathbf{C}}^*$ by

$$\nu(dz) = \mu'(dx)\mu'(dy), \quad z = x + iy \in E_{\mathbf{C}}^*.$$

Following Hida [10, Chapter 6] the probability space $(E_{\mathbf{C}}^*, \nu)$ is called the *complex Gaussian space*. We record here two formulas:

$$(2) \quad \int_{E^*} e^{\langle x, \xi \rangle} \mu'(dx) = e^{\langle \xi, \xi \rangle / 4}, \quad \xi \in E_{\mathbf{C}},$$

$$(3) \quad \int_{E_{\mathbf{C}}^*} e^{\langle \bar{z}, \xi \rangle + \langle z, \eta \rangle} \nu(dz) = e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E_{\mathbf{C}},$$

where $\bar{z} = x - iy$ for $z = x + iy \in E^* + iE^*$. Remind that $\langle \cdot, \cdot \rangle$ is the canonical \mathbf{C} -bilinear form on $E_{\mathbf{C}}^* \times E_{\mathbf{C}}$.

2. CKS-space

We review the standard construction of a CKS-space [6]. For a Hilbert space H and a sequence $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ of positive numbers we put

$$\Gamma_{\alpha}(H) = \left\{ \phi = (f_n)_{n=0}^{\infty}; f_n \in H^{\widehat{\otimes} n}, \|\phi\|^2 \equiv \sum_{n=0}^{\infty} n! \alpha(n) |f_n|^2 < \infty \right\}.$$

In an obvious manner $\Gamma_{\alpha}(H)$ becomes a Hilbert space and is called a *weighted Fock space*. The Boson Fock space is by definition the special case of $\alpha(n) \equiv 1$ and is denoted by $\Gamma(H)$. In the sequel we assume that the weight sequence $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ satisfies the following conditions:

- (A1) $1 = \alpha(0) \leq \alpha(1) \leq \alpha(2) \leq \dots$;
- (A2) the generating function $G_{\alpha}(t) \equiv \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} t^n$ has an infinite radius of convergence;
- (A3) the function $\tilde{G}_{\alpha}(t) \equiv \sum_{n=0}^{\infty} t^n \frac{n^{2n}}{n! \alpha(n)} \left\{ \inf_{s>0} \frac{G_{\alpha}(s)}{s^n} \right\}$ has a positive radius of convergence;
- (A4) there exists a constant $C_1 > 0$ such that $\alpha(n)\alpha(m) \leq C_1^{n+m} \alpha(n+m)$ for all n, m ;
- (A5) there exists a constant $C_2 > 0$ such that $\alpha(n+m) \leq C_2^{n+m} \alpha(n)\alpha(m)$ for all n, m .

These conditions are sufficient for the characterization of S-transform [6], for the characterization of operator symbol and for another vital properties of white noise operators [4], [18]. However, they are not yet down to the minimum; the formulation proposed recently in [1], [7] singles out the essential requirements for $\{\alpha(n)\}$ in a different language but is easily transplanted to our argument.

For $p \geq 0$ let $E_{\pm p}$ be the Hilbert space obtained by completing $E = \mathcal{S}(\mathbf{R})$ with respect to the norm $|\xi|_{\pm p} = |A^{\pm p}\xi|_{L^2(\mathbf{R})}$, where $A = 1 + t^2 - d^2/dt^2$. We then have

$$E = \text{proj lim}_{p \rightarrow \infty} E_p, \quad E^* = \text{ind lim}_{p \rightarrow \infty} E_{-p}.$$

Given a sequence $\alpha = \{\alpha(n)\}_{n=0}^\infty$ satisfying the above conditions, we put

$$(4) \quad \Gamma_\alpha(E) = \text{proj lim}_{p \rightarrow \infty} \Gamma_\alpha(E_p).$$

Then $\Gamma_\alpha(E)$ becomes a nuclear space, the topology of which is given by the family of norms:

$$\|\phi\|_{p,+}^2 = \sum_{n=0}^\infty n! \alpha(n) |f_n|_p^2, \quad \phi = (f_n), \quad p \geq 0.$$

By a standard argument we see that

$$(5) \quad \Gamma_\alpha(E)^* \cong \text{ind lim}_{p \rightarrow \infty} \Gamma_{\alpha^{-1}}(E_{-p}),$$

where $\Gamma_\alpha(E)^*$ carries the strong dual topology and \cong stands for a topological linear isomorphism. We often write \mathcal{W} for $\Gamma_\alpha(E)$ for simplicity. Then, in view of (4) and (5) we come to a Gelfand triple:

$$(6) \quad \mathcal{W} \subset \Gamma(H_C) \subset \mathcal{W}^*,$$

which is called the *Cochran-Kuo-Sengupta space* or the *CKS-space* for short [6]. The canonical \mathbf{C} -bilinear form on $\mathcal{W}^* \times \mathcal{W}$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. Then

$$(7) \quad \langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^\infty n! \langle F_n, f_n \rangle, \quad \Phi = (F_n) \in \mathcal{W}^*, \quad \phi = (f_n) \in \mathcal{W},$$

and it holds that

$$|\langle\langle \Phi, \phi \rangle\rangle| \leq \|\Phi\|_{-p,-} \|\phi\|_{p,+},$$

where

$$\|\Phi\|_{-p,-}^2 = \sum_{n=0}^\infty \frac{n!}{\alpha(n)} |F_n|_{-p}^2, \quad \Phi = (F_n) \in \mathcal{W}^*.$$

3. CKS-space over complex Gaussian space

Let $\Gamma(H_C)$ be the Boson Fock space over H_C . Then, through the celebrated Wiener–Itô–Segal isomorphism we have

$$(8) \quad L^2(E^*, \mu') \cong \Gamma(H_C),$$

where the unitary isomorphism is uniquely determined by the correspondence:

$$(9) \quad \psi_\xi(x) \equiv e^{\sqrt{2}\langle x, \xi \rangle - (\xi, \xi)/2} \leftrightarrow \phi_\xi \equiv \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots \right), \quad \xi \in E_C.$$

The above ϕ_ξ is called an *exponential vector* or a *coherent state* (without normalization). In fact,

$$\langle\langle \phi_\xi, \phi_\eta \rangle\rangle = e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E_C,$$

where the left hand side stands for the canonical C -bilinear form on $\Gamma(H_C)$.

Since the exponential vectors $\{\phi_\xi; \xi \in E_C\}$ span a dense subspace of \mathcal{W} , white noise functions and operators are uniquely specified by their values on the exponential vectors. To be more precise, we recall notation. The *S-transform* of $\Phi \in \mathcal{W}$ is defined by

$$S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle, \quad \xi \in E_C,$$

and the *symbol* of $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \quad \xi, \eta \in E_C.$$

It is one of the most important features of white noise calculus that the S-transform and the operator symbol are characterized by their holomorphy and certain growth condition, for the S-transform see [6] and for the operator symbol see [4], [16].

By duplicating the Gelfand triple (6) we obtain

$$(10) \quad \mathcal{W} \otimes \mathcal{W} \subset \Gamma(H_C) \otimes \Gamma(H_C) \subset (\mathcal{W} \otimes \mathcal{W})^*.$$

On the other hand, identifying a function on E_C^* with one on $E^* \times E^*$ in such a way that

$$\phi \otimes \psi(x + iy) = \phi(x)\psi(y), \quad x, y \in E^*, \quad \phi, \psi \in L^2(E^*, \mu'),$$

we come to the isomorphisms:

$$(11) \quad \Gamma(H_C) \otimes \Gamma(H_C) \cong L^2(E^*, \mu') \otimes L^2(E^*, \mu') \cong L^2(E_C^*, \nu).$$

Combination of (10) and (11) gives rise to a Gelfand triple which we shall denote by

$$(12) \quad \mathcal{D} \subset L^2(E_{\mathbb{C}}^*, \nu) \subset \mathcal{D}^*.$$

This is referred to as a CKS-space over the complex Gaussian space $(E_{\mathbb{C}}^*, \nu)$. Since each equivalence class contains a unique continuous function on $E_{\mathbb{C}}^*$, we may regard \mathcal{D} as a space of continuous functions on $E_{\mathbb{C}}^*$. More detailed properties of functions in \mathcal{D} are examined in a similar manner as in [15], [21].

4. Diagonal coherent state representation

The definition of an exponential vector in (9) is applied also to defining an exponential vector $\phi_z \in \mathcal{W}^*$ for $z \in E_{\mathbb{C}}^*$. We then put

$$Q_z \phi = \langle\langle \phi_{\bar{z}}, \phi \rangle\rangle \phi_z, \quad \phi \in \mathcal{W}.$$

Note that $Q_z \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ and the map $z \mapsto Q_z$ is continuous.

It is proved [19] that for any $w \in \mathcal{D}^*$ there exists a unique operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ such that

$$(13) \quad \langle\langle \Xi \phi_{\xi}, \phi_{\eta} \rangle\rangle = \langle\langle w, q_{\xi, \eta} \rangle\rangle, \quad \xi, \eta \in E_{\mathbb{C}},$$

where

$$q_{\xi, \eta}(z) = \widehat{Q}_z(\xi, \eta) = \langle\langle Q_z \phi_{\xi}, \phi_{\eta} \rangle\rangle = e^{\langle \bar{z}, \xi \rangle + \langle z, \eta \rangle}, \quad z \in E_{\mathbb{C}}^*.$$

The operator Ξ defined as in (13) is naturally written in a formal integral:

$$(14) \quad \Xi = \int_{E_{\mathbb{C}}^*} w(z) Q_z \nu(dz)$$

and is called the *diagonal coherent state representation*. A significant consequence is the following

THEOREM 4.1. [19] *Every operator in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ admits a unique diagonal coherent state representation. In particular, the resolution of the identity holds:*

$$(15) \quad I = \int_{E_{\mathbb{C}}^*} Q_z \nu(dz).$$

For example, the annihilation and creation operators at a point $t \in \mathbf{R}$ are expressed respectively in the form:

$$(16) \quad a_t = \int_{E_{\mathbb{C}}^*} z(t) Q_z \nu(dz), \quad a_t^* = \int_{E_{\mathbb{C}}^*} \overline{z(t)} Q_z \nu(dz),$$

where $z(t) = \langle z, \delta_t \rangle$ is the complex white noise [10].

5. Unitarity condition

In order to discuss the unitarity of an operator Ξ on the Fock space $\Gamma(H_{\mathbf{C}})$ we need the hermitian inner product:

$$\langle\langle \phi, \psi \rangle\rangle = \langle\langle \bar{\phi}, \psi \rangle\rangle.$$

For an operator Ξ we denote by Ξ^\dagger its adjoint with respect to the above hermitian inner product. As is easily verified, it holds that

$$\Xi^\dagger \phi = \overline{\Xi^* \bar{\phi}}, \quad \phi \in \mathcal{W}.$$

By definition $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ is called an *isometry* on $\Gamma(H_{\mathbf{C}})$ if $\Xi^\dagger \Xi = I$; and is called a *unitary* if both Ξ and Ξ^\dagger are isometries, i.e., if $\Xi^* \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ and $\Xi^\dagger \Xi = \Xi \Xi^\dagger = I$.

Since the exponential vectors $\{\phi_\xi; \xi \in E_{\mathbf{C}}\}$ span a dense subspace of \mathcal{W} and hence of $\Gamma(H_{\mathbf{C}})$, the condition $\Xi^\dagger \Xi = I$ is equivalent to

$$\langle\langle \Xi \phi_\xi, \Xi \phi_\eta \rangle\rangle = \langle\langle \phi_\xi, \phi_\eta \rangle\rangle, \quad \xi, \eta \in E_{\mathbf{C}},$$

or in terms of the original \mathbf{C} -bilinear form:

$$(17) \quad \langle\langle \overline{\Xi \phi_\xi}, \Xi \phi_\eta \rangle\rangle = \langle\langle \phi_\xi, \phi_\eta \rangle\rangle = e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E_{\mathbf{C}}.$$

Similarly, under the assumption that $\Xi^* \in \mathcal{L}(\mathcal{W}, \mathcal{W})$, the condition $\Xi \Xi^\dagger = I$ is equivalent to

$$(18) \quad \langle\langle \overline{\Xi^* \phi_\xi}, \Xi^* \phi_\eta \rangle\rangle = \langle\langle \phi_\xi, \phi_\eta \rangle\rangle = e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E_{\mathbf{C}}.$$

We consider the isometricity condition (17). With the help of the resolution of the identity (see Theorem 4.1), the left hand side of (17) becomes

$$\begin{aligned} \langle\langle \overline{\Xi \phi_\xi}, \Xi \phi_\eta \rangle\rangle &= \int_{E_{\mathbf{C}}^*} \langle\langle \overline{\Xi \phi_\xi}, \phi_z \rangle\rangle \langle\langle \phi_z, \Xi \phi_\eta \rangle\rangle \nu(dz) \\ &= \int_{E_{\mathbf{C}}^*} \overline{\langle\langle \Xi \phi_\xi, \phi_z \rangle\rangle} \langle\langle \Xi \phi_\eta, \phi_z \rangle\rangle \nu(dz) \\ &= \int_{E_{\mathbf{C}}^*} \widehat{\Xi}(\bar{\xi}, z) \widehat{\Xi}(\eta, z) \nu(dz). \end{aligned}$$

In a similar manner, the left hand side of (18) becomes

$$\langle\langle \overline{\Xi^* \phi_\xi}, \Xi^* \phi_\eta \rangle\rangle = \int_{E_{\mathbf{C}}^*} \widehat{\Xi}(z, \bar{\xi}) \widehat{\Xi}(z, \eta) \nu(dz),$$

where we used the fact that the measure ν is invariant under the complex conjugation $z \mapsto \bar{z}$. We thus come to the following

THEOREM 5.1. *An operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ is an isometry on $\Gamma(H_{\mathbb{C}})$, i.e., $\Xi^\dagger \Xi = I$ if and only if*

$$(19) \quad \int_{E_{\mathbb{C}}^*} \overline{\widehat{\Xi}(\bar{\xi}, z)} \widehat{\Xi}(\eta, z) \nu(dz) = e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E_{\mathbb{C}}.$$

An operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ with $\Xi^ \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ is a unitary on $\Gamma(H_{\mathbb{C}})$, i.e., $\Xi^\dagger \Xi = \Xi \Xi^\dagger = I$ if and only if*

$$(20) \quad \int_{E_{\mathbb{C}}^*} \overline{\widehat{\Xi}(\bar{\xi}, z)} \widehat{\Xi}(\eta, z) \nu(dz) = \int_{E_{\mathbb{C}}^*} \overline{\widehat{\Xi}(z, \bar{\xi})} \widehat{\Xi}(z, \eta) \nu(dz) = e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E_{\mathbb{C}}.$$

Here is an example. In order to solve certain differential equations on Gaussian space a transformation group on white noise functions has been introduced in [2], see also [3] where a restricted case appeared. With $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$ and $B \in \mathcal{L}(E_{\mathbb{C}}, E_{\mathbb{C}})$ we associate an operator $G_{\kappa, B}$ defined by

$$(21) \quad G_{\kappa, B} \phi_{\xi} = e^{\langle \kappa, \xi^{\otimes m} \rangle} \phi_{B\xi}, \quad \xi \in E_{\mathbb{C}}.$$

In fact, that $G_{\kappa, B} \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ follows from the characterization theorem for operator symbols. It is also known that

$$G_{\kappa', B'} G_{\kappa, B} = G_{\kappa + (B^{\otimes m})^* \kappa', B' B}.$$

By definition (21) we have immediately

$$\widehat{G}_{\kappa, B}(\xi, \eta) = e^{\langle \kappa, \xi^{\otimes m} \rangle + \langle B\xi, \eta \rangle}$$

and, in view of (3),

$$\begin{aligned} & \int_{E_{\mathbb{C}}^*} \overline{\widehat{G}_{\kappa, B}(\bar{\xi}, z)} \widehat{G}_{\kappa, B}(\eta, z) \nu(dz) \\ &= e^{\langle \bar{\kappa}, \xi^{\otimes m} \rangle + \langle \kappa, \eta^{\otimes m} \rangle} \int_{E_{\mathbb{C}}^*} e^{\langle z, B\eta \rangle + \langle \bar{z}, \overline{B\xi} \rangle} \nu(dz) \\ &= \exp \left(\langle \bar{\kappa}, \xi^{\otimes m} \rangle + \langle \kappa, \eta^{\otimes m} \rangle + \langle B\eta, \overline{B\xi} \rangle \right). \end{aligned}$$

Hence by Theorem 5.1 the isometricity condition $G_{\kappa, B}^\dagger G_{\kappa, B} = I$ is equivalent to

$$\exp \left(\langle \bar{\kappa}, \xi^{\otimes m} \rangle + \langle \kappa, \eta^{\otimes m} \rangle + \langle B\eta, \overline{B\xi} \rangle \right) = e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E_{\mathbb{C}},$$

which is further equivalent to that $\kappa = 0$ and $B^\dagger B = I$. In this case $G_{0,B}$ coincides with the second quantization of B , that is, $G_{0,B} = \Gamma(B)$. Summing up,

PROPOSITION 5.2. *The operator $G_{\kappa,B}$ is an isometry on $\Gamma(H_C)$ if and only if $\kappa = 0$ and B is an isometry on H_C . Moreover, $G_{\kappa,B}$ is a unitary on $\Gamma(H_C)$ if and only if $\kappa = 0$ and B is a unitary on H_C .*

However, Theorem 5.1 is not applied to a simple example: $U_t = e^{iB(t)}$, where $\{B(t)\}$ is the standard Brownian motion. Note that $\{U_t\}$ is a unique solution to the stochastic differential equation:

$$dU = iUdB - \frac{1}{2}Udt, \quad U_0 = I,$$

or equivalently to the normal-ordered white noise equation:

$$\frac{dU}{dt} = \left(ia_t^* + ia_t - \frac{1}{2} \right) \diamond U, \quad U_0 = I.$$

(A general theory of normal-ordered white noise equations has been developed in [5].) The symbol of U_t is easily computed:

$$(22) \quad \widehat{U}_t(\xi, \eta) = \exp \left(-\frac{t}{2} + i\langle 1_{[0,t]}, \xi + \eta \rangle + \langle \xi, \eta \rangle \right),$$

and the verification of (20) is straightforward.

In fact, Theorem 5.1 is improved slightly with no difficulty as follows. For notational convenience an operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is called *regular* if the symbol $\widehat{\Xi}$, originally a \mathbf{C} -valued function on $E_C \times E_C$, admits an extension to a function on $E_C \times E_C^*$ as a function in $L^2(E_C^*, \nu)$ with respect to the second argument. For example, \widehat{U}_t in (22) is such a function. In that case, the integral in (19) exists for all $\xi, \eta \in E_C$ and the identity remains meaningful.

THEOREM 5.3. *Assume $\Xi \in \mathcal{L}(\mathcal{W}, \Gamma(H_C))$ is regular. Then Ξ is an isometry on $\Gamma(H_C)$, i.e., $\Xi^\dagger \Xi = I$ if and only if*

$$\int_{E_C^*} \overline{\widehat{\Xi}(\xi, z)} \widehat{\Xi}(\eta, z) \nu(dz) = e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E_C.$$

In addition, assume $\Xi^ \in \mathcal{L}(\mathcal{W}, \Gamma(H_C))$ and is regular. Then, Ξ is a unitary operator on $\Gamma(H_C)$, i.e., $\Xi^\dagger \Xi = \Xi \Xi^\dagger = I$ if and only if*

$$\int_{E_C^*} \overline{\widehat{\Xi}(\xi, z)} \widehat{\Xi}(\eta, z) \nu(dz) = \int_{E_C^*} \overline{\widehat{\Xi}(z, \xi)} \widehat{\Xi}(z, \eta) \nu(dz) = e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in E_C.$$

6. Coherent state representation of white noise functions

As is well known, the set of exponential vectors $\{\phi_\xi; \xi \in E_{\mathbb{C}}\}$ is linearly independent but is overcomplete as is illustrated by the identity:

$$(23) \quad \int_{E_{\mathbb{C}}^*} e^{(\bar{z}, \xi)} \phi_z \nu(dz) = \phi_\xi, \quad \xi \in E_{\mathbb{C}}.$$

This is verified immediately by the S-transform and is a consequence of the reproducing property of the exponential kernels. On the other hand, due to the uniqueness of the diagonal coherent representation (Theorem 4.1) one may specify a particular representation of a white noise function via coherent states with complex Gaussian integral.

Modifying the proof of [16, Chapter 3.5], one may easily check that the pointwise multiplication of two test white noise functions gives rise to a continuous bilinear form from $\mathcal{W} \times \mathcal{W}$ into \mathcal{W} . Hence, with each $\Phi \in \mathcal{W}^*$ we may associate a multiplication operator, denoted by the same symbol, by the relation:

$$\langle\langle \Phi \phi, \psi \rangle\rangle = \langle\langle \Phi, \phi \psi \rangle\rangle, \quad \phi, \psi \in \mathcal{W}.$$

Moreover, thus obtained injection: $\mathcal{W}^* \rightarrow \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is shown to be continuous.

As usual, let $\Gamma(\sqrt{2}) \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ denote the second quantization of the scalar operator $\sqrt{2}I$. Recall that $\Gamma(\sqrt{2})\phi_\xi = \phi_{\sqrt{2}\xi}$ for $\xi \in E_{\mathbb{C}}$ and $\Gamma(\sqrt{2}) \subset \Gamma(\sqrt{2})^* \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$.

THEOREM 6.1. *As a multiplication operator, the diagonal coherent state representation of $\Phi \in \mathcal{W}^*$ is given by*

$$(24) \quad \Phi = \int_{E_{\mathbb{C}}^*} w_\Phi(z) Q_z \nu(dz),$$

where

$$(25) \quad w_\Phi(z) = \Gamma(\sqrt{2})^* \Phi \left(\frac{z + \bar{z}}{\sqrt{2}} \right).$$

Proof. According to the argument in [19, Theorem 5.2], we need to find $\tilde{w} \in (\mathcal{W} \otimes \mathcal{W})^*$ such that

$$(26) \quad \langle\langle \tilde{w}, \phi_\xi \otimes \phi_\eta \rangle\rangle = \langle\langle \Phi \phi_{(\xi+i\eta)/\sqrt{2}}, \phi_{(\xi-i\eta)/\sqrt{2}} \rangle\rangle e^{-\langle \xi+i\eta, \xi-i\eta \rangle / 2}.$$

The right hand side is computed as

$$\begin{aligned} & \langle\langle \Phi, \phi_{(\xi+i\eta)/\sqrt{2}} \phi_{(\xi-i\eta)/\sqrt{2}} \rangle\rangle e^{-\langle \xi+i\eta, \xi-i\eta \rangle / 2} = \langle\langle \Phi, \phi_{\sqrt{2}\xi} \rangle\rangle \\ & = \langle\langle \Phi, \Gamma(\sqrt{2})\phi_\xi \rangle\rangle = \langle\langle \Gamma(\sqrt{2})^* \Phi, \phi_\xi \rangle\rangle = \langle\langle (\Gamma(\sqrt{2})^* \Phi) \otimes \phi_0, \phi_\xi \otimes \phi_\eta \rangle\rangle. \end{aligned}$$

We therefore have

$$\tilde{w} = (\Gamma(\sqrt{2})^* \Phi) \otimes \phi_0.$$

Let $w_\Phi \in \mathcal{D}^*$ be the element corresponding to \tilde{w} under the isomorphism $\mathcal{D}^* \cong (\mathcal{W} \otimes \mathcal{W})^*$ which extends the correspondence given in (9). Then, for $z = x + iy$ we have

$$w_\Phi(z) = \Gamma(\sqrt{2})^* \Phi(\sqrt{2}x) \phi_0(y) = \Gamma(\sqrt{2})^* \Phi(\sqrt{2}x),$$

from which (25) follows immediately. □

Consider the action of both sides of (24) on the vacuum ϕ_0 . Since $\Phi\phi_0 = \Phi$ and $Q_z\phi_0 = \phi_z$, it follows that

$$(27) \quad \Phi = \int_{E_{\mathbb{C}}^*} w_\Phi(z) \phi_z \nu(dz),$$

which is an identity in \mathcal{W}^* . Recall that there are continuously many different representations of a white noise function Φ due to the over-completeness of $\{\phi_z\}$. The coherent state representation of Φ specified as in (27) will be called *canonical* in the sense of classical-quantum correspondence.

For example, consider $\Phi = \phi_\xi$. By (25) we obtain

$$w_\Phi(z) = \Gamma(\sqrt{2})^* \phi_\xi \left(\frac{z + \bar{z}}{\sqrt{2}} \right) = \phi_{\sqrt{2}\xi} \left(\frac{z + \bar{z}}{\sqrt{2}} \right) = e^{\langle z + \bar{z}, \xi \rangle - \langle \xi, \xi \rangle}.$$

Hence, as a multiplication operator we have

$$\phi_\xi = \int_{E_{\mathbb{C}}^*} e^{\langle z + \bar{z}, \xi \rangle - \langle \xi, \xi \rangle} Q_z \nu(dz).$$

Therefore, the canonical coherent state representation of ϕ_ξ is given by

$$\phi_\xi = \int_{E_{\mathbb{C}}^*} e^{\langle z, \xi \rangle + \langle \bar{z}, \xi \rangle - \langle \xi, \xi \rangle} \phi_z \nu(dz),$$

which should be compared with (23).

Here is another example. The canonical coherent state representation of the classical white noise is given by

$$W_t = \int_{E_{\mathbb{C}}^*} (z(t) + \overline{z(t)}) \phi_z \nu(dz),$$

which is verified with the operator identity $W_t = a_t + a_t^*$ and (16). While,

$$W_t = \int_{E_{\mathbb{C}}^*} \overline{z(t)} \phi_z \nu(dz)$$

is also valid but is not canonical. In this connection we note the following more general identity:

$$(28) \quad \int_{E_{\mathbb{C}}^*} \langle z^{\otimes m}, \kappa \rangle \phi_z \nu(dz) = 0,$$

where $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$, $m \geq 1$. In fact, (28) follows from the diagonal coherent state representation of the integral kernel operator:

$$\Xi_{0,m}(\kappa) = \int_{E_{\mathbb{C}}^*} \langle z^{\otimes m}, \kappa \rangle Q_z \nu(dz).$$

A complete description of the null kernel of coherent state representation of a white noise function is investigated in [20].

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