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WHITE NOISE APPROACH TO FEYNMAN INTEGRALS

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ABSTRACT. The trajectory of a classical dynamics is detrmined by the least action principle. As soon as we come to quantum dynamics, we have to consider all possible trajectories which are proposed to be a sum of the classical trajectory and Brownian fluctuation. Thus, the action involves the square of the derivative $\dot{B}(t)$ (white noise) of a Brownian motion $B(t)$. The square is a typical example of a generalized white noise functional. The Feynman propagator should therefore be an average of a certain generalized white noise functional. This idea can be applied to a large class of dynamics with various kinds of Lagrangians.

1. Introduction

As is well known, in the classical Hamiltonian mechanics, the trajectory of a dynamical system is given by the Lagrangian in such a way that the action attains the extremal values at the trajectory which is to be actually realized.

We are interested in a transition from classical mechanics to quantum dynamics, having been motovated by Dirac's approach. According to the Feynman's original idea to establish the third formulation of nonrelativistic quantum mechanics, a probability amplitude should be associated with the entire motion of a particle which is a function of time, rather than simply with a position of the particle at particular instant.

From the probabilistic viewpoint, we are led to consider an ensemble of sample functions of a certain stochastic process, which we shall specify later. The Feynman's theory (1948) has taken this viewpoint and

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proposed a method of the path integral to form the quantum mechanical propagator, if we understand correctly. His profound idea is to take the average of a certain functional, involving the action integral, over an ensemble of possible trajectories.

Many attempts have been made to give mathematical interpretations to this formulation. Under such circumstances, we have proposed (1983) a reasonable, visualized and of course mathematically rigorous formulation of of the Feynman path integral. The present note aims at a quick review of our results and some discussions for further interpretations of this theory.

Actually, in the present report a white noise approach to the Feynman path integral is discussed. For this purpose some background of stochastic process is given in Section 2. From the viewpoint of dynamical theory, the Brownian bridge plays an important role, so that it is briefly discussed, although it is well known. The significance of a Brownian bridge is that, like in the classical mechanics, it has time reversibility property.

We then come to our formulation of the path integral by using a Brownian bridge in Section 3. For the setup it is necessary to prepare the theory of generalized white noise functionals. With this notion it is possible to give a visualized expression of the so-called Feynman functional without any tricks, like imaginary variance or taking approximation. This is just the advantage of our method.

It is also well known that many results have been obtained in this direction by various authors, in particular by LS!(L. Streit and his group have developed extensively (some of the literatures are listed in the References; there are many others). The present note will not come to mention those results, however just a short remark related to this topic will be given in the concluding remark in the last section.

2. Background

White noise is a standard Gaussian measure space (E^*, μ), where E^* is the dual space of a nuclear space E which is a subspace of $L^2(R)$ and μ is a Gaussian measure on E^* such that its characteristic functional is given by

$$C(\xi) = \exp[-\frac{1}{2}\|\xi\|^2].$$

Each member x in E^* with the measure μ is viewed as a sample function of $\dot{B}(t)$ which is the time derivative of a Brownian motion $B(t)$.

Thus, the $\dot{B}(t)$ itself is also called white noise. Now set

$$B(t,x)(=B(t)) = \langle x, \chi_{[0,t]}\rangle.$$

Then, it can be shown that $B(t)$ is a version of Brownian motion.

The complex Hilbert space $(L^2) = L^2(E^*, \mu)$ is a Fock space in the sense that it admits a direct sum decomposition into homogeneous chaos $H_n, n = 0, 1, 2, \ldots$:

$$(L^2) = \oplus H_n.$$

Brownian motion $B(t)$ lives in the subspace H_1. It should be noted that if a Brownian motion $B(t)$ shares the time propagation with the white noise, then its expression is unique up to sign. Namely, it is the one defined just above.

In order to come to our main topic we need to introduce a class of white noise functionals much wider than (L^2). This will be done in the Section 4.

3. Brownian bridge

A Brownian bridge is, intuitively speaking, a Gaussian process that starts from the origin 0 and returns to 0 again at a fixed time, say at instant T. Inbetween it behaves like an ordinary Brownian motion (see, e.g. [7] §2.5.). It can be realized, for example, as follows.

Take a time interval $[0, T]$. Set

$$X_T(t) = B(t) - \frac{t}{T}B(T).$$

Then, $X_T(t), t \in [0, T]$ is a Brownian bridge for the time interval $[0, T]$. It is a Gaussian Markov process. The canonical representation (for definition, see [5], or briefly [7]) is given by

$$X_T(t) = (T-t)\int_0^t \frac{1}{T-u}\dot{B}_1(u)du.$$

It has mean 0 and covariance function

$$\Gamma(t,s) = T^{-1}(t \wedge s)((T-t) \wedge (T-s)),$$

where $\wedge$ means the minimum. Note that the $\dot{B}_1(t)$ is the innovation of the $X_T(t)$.

If $X_T(t)$ is normalized (i.e. divided by the standard deviation to have unit variance), then the resultant process $Y_T(t)$enjoys the so-called *projective invariance* property. It is stated as follows.

THEOREM 1. *If g is a projective transformation acting on the time interval $[0, T]$, then $\{Y_T(t)\}$ and $\{Y_T(gt)\}$ are the same Gaussian process.*

This theorem is known; for proof see e.g. [7] Chapt.5.

This property gives us a suggestion to our further directions.

The backward canonical representation, which can be defined as a counterpart of the canonical representation, is given by

THEOREM 2. The expression

$$X^T(t) = t\int_t^T \frac{1}{u}\dot{B}_2(u)du$$

is the backward canonical representation of a Brownian bridge.

The proof comes from the computation of the covariance function and the fact that the $\dot{B}_2(t)$ is obtained from the values $X^T(u), u > t$.

The projective invariance gives a bridge that connects the two representations; one is the (forward) canonical representation and the other is the backward canonical one. More analytically, they are connected through a member of the infinite dimensional rotation group acting on E.

Note that for two representations, the directions of the time evolution are different, one is ordinary direction and the other is the reverse. We therefore claim that a Brownian bridge is time reversible in the sense of stochastic process. Further discussion on this notion will be given in the separate paper.

4. The Feynman functional

We are now ready to propose an expression of path integral. Assume that a Lagrangian $L(x, \dot{x})$ is given for a time interval $[0, T]$. Then, the classical path, denoted by $y(t)$, is uniquely determined. As soon as we come to quantum mechanics, we have to consider all possible trajectories $x(t)$. We propose that a quantum mechanical trajectory $x(t)$ is a sum of the classical path and a fluctuation which is given by a Brownian bridge $X_T(t)$:

$$x(t) = y(t) + (\frac{\hbar}{m})^{\frac{1}{2}} X_T(t), \quad t \in [0, T].$$

Now we have to give a plausible interpretation why such a proposal seems to be fitting in line with Feynman's idea (we hope).

The fluctuation term $X_T(t)$ should be a Markov process (see [2] Chapt.5). The time reversible property is acceptable, if we understand correctly, because of the property of mechanics. A certain kind of optimality from the viewpoint of probability theory implies that the distribution should be Gaussian. Thus, we can accept the formula for $x(t)$ (see, e.g. [7], [9])

The classical action is

$$S[x] = \int_0^T L(x, \dot{x})dt.$$

Typical Lagrangian is of the form

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x).$$

Hence, the propagator $G(y_1.y_2, T)$ is given by

$$G(y_1, y_2, T) = E\{N \exp\left[\frac{im}{2\hbar}\int_0^T \dot{x}(t)^2 dt + \frac{1}{2}\int_0^T \dot{B}(t)^2 dt\right] \cdot \exp\left[-\frac{i}{\hbar}\int_0^T V(x(t))dt\right]\}.$$

The factor involving $\dot{B}(t)^2$ serves to flatten the Gaussian measure.

REMARK. In our earlier paper [9], we take just a Brownian motion as the fluctuation to form a possible trajectory. We therefore put the delta function (indeed, the Donsker's delta function) to have pinning effect at instant T.

As is seen in the expression of the propagator, we have a good visualized and even illustrative formula, however we have to pay a price; namely the term $\dot{B}(t)^2$ is involved even twice. We have therefore to appeal to the theory of generalized white noise functionals. A short note on this fact is stated below.

Starting from the Fock space, we can define a Gel'fand triple by using the second quantization technique with the operator

$$A = -D^2 + u^2 + 1, \quad D : \text{differential operator},$$

such that

$$(S) \subset (L^2) \subset (S)^*.$$

The space (S) is the space of test functionals and $(S)^*$ is that of generalized (white noise) functionals.

Good examples are now in order.

EXAMPLE 1. A polynomial in $x(t)$ of degree 2 may be written as

$$: x(t)^2 := x(t)^2 - \frac{1}{dt},$$

where x is a sample function of white noise; hence the above functional may be written as : $\dot{B}(t)^2$:. These formulas look like formal expressions, however we can give good interpretation. In this case we need the additive renormalization as much as $\frac{1}{dt}$.

EXAMPLE 2. Exponential function.

$$N \exp[\int x(t)^2 dt],$$

where N is the factor put for the multiplicative renormalization.

EXAMPLE 3. Donsker's delta function. It is given by

$$\delta_0(B(t) - a).$$

The functionals in Example 1 and Example 2 are used in the formulation of the propagator $G(y_1, y_2, T)$ and the Donsker's deltafunction has been used in [9].

These generalized functionals make the formulation rigorous and visualized.

5. Concluding remark

As was mentioned before, our setup for Feynman's path integral has effectively been applied to systems with various Lagrangians and generalizations of the theory have been reported; we mainly refer to the results by L.Streit and others, where further developments are included. Also, as is suggested by [1], and by the Tomonaga-Schwinger equation, we are naturally led to the case of multi-dimensional parameter. This is one of the reason why we expect the study of random fields.

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References

[1] P. A. M. Dirac, *The Lagrangian in Quantum Mechanics*, Phys. Zeitsch. der Sowjetunion, **3** (1933).

[2] ———, *The Principles of Quantum Mechanics*, 4th ed. Oxford Univ. Press, 1958.

[3] R. Feynman, *Space-time approach to non-relativistic quantum mechanics*, Rev. of Modern Phys. **20** (1948), 367–387.

[4] R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill Book Co. 1965.

[5] T. Hida, *Stationary Stochastic Processes*, Princeton Univ. Press, 1970.

[6] ———, *Analysis of Brownian Functionals*, Carleton University Math. Lecture Notes, (1975), no. 13.

[7] ———, *Brownian Motion*, Springer-Verlag, Applications of Math. **11** (1980).

[8] T. Hida, H.-H. Kuo, J. Potthoff, and I. Streit, *White Noise. An infinite dimensional calculus*, Kluwer Academic Pub. 1993.

[9] L. Streit and T. Hida, *Generalized Brownian functionals and the Feynman integral*, Stoch. Proc. Appl. **16** (1983), 55–69.

[10] L. Streit, *The Feynman integral*, Recent results. Univ. da Madeira, 1992.

[11] ———, *White noise analysis and applications in Quantum Physiscs*, Univ. da Madeira, 1993.

[12] L. Streit et al, *Feynman integrals for a class of exponentially growing potentials*, J. Math. Phys. **39** (1998), 4476–4491.

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