J. Korean Math. Soc. **38** (2001), No. 5, pp. 937–954

Γ-DEVIATION AND LOCALIZATION

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ABSTRACT. This paper is a natural continuation of [2], [3], [4] and [5]. Localization techniques for modular lattices are developed. These techniques are applied to study liftings of linear order types from quotient lattices and to find Γ–dense sets in certain lattices without Γ–deviation in the sense of [4], where Γ is a set of indecomposable linear order types.

Introduction

In the classical theory of commutative rings, the theories of localization and Krull dimension play important roles. For modules over a commutative ring, localizations can be obtained by using tensor products. These types of localization were generalized to an Abelian category setting by Gabriel [7] and others. This more general localization was used to study torsion theories and general Krull dimension for modules (e.g., see [1], [5], [16]). The idea of Krull dimension was extended to partially ordered sets and studied by many authors (e.g., [9], [10], [11], [12]), often under the name of deviation or Γ–deviation. In our recent papers [4], [5] we investigated the general properties of the Γ–deviation of posets and modular lattices, and in other recent papers, Albu and Smith [2],[3] studied the localization of modular lattices in conjunction with Krull dimension.

In the present paper, we combine ideas from [2], [3], [4], and [5] to obtain a theory of localization for Γ–deviation by using quotient lattices. In Section 1, we present some properties of our localization of lattices and compare it to the localizations of [8] and [13]. In Section 2, we develop the concept of Γ–deviation by studying the interaction of chains

Received October 13, 2000. Revised January 26, 2001.

2000 Mathematics Subject Classification: 06A11, 06B23, 06C05, 16P60, 18E15, 18E35, 18E40.

Key words and phrases: poset, modular lattice, quotient lattice, Krull dimension, dual Krull dimension, Γ–deviation, Serre class, hereditary torsion theory, localization.

and congruence relations. For a lattice L, we determine some situations in which it is possible to lift a chain of linear order type γ in a quotient of L to a chain of type γ in L. Finally, in Section 3, we apply some of the ideas of the previous sections to obtain new results about the existence of Γ–dense sets in lattices without Γ–deviation; these results provide partial answers to questions raised in [4] and [9].

0. Terminology and notation

We will follow the notation and terminology used in [2], [3], [4]. Thus, $\mathcal{M}$ (resp. $\mathcal{C}$) will denote the class of all modular lattices with 0 and 1 (resp. complete modular lattices).

Throughout this paper a lattice will mean a member of $\mathcal{M}$, and $(L, \leq, \wedge, \vee, 0, 1)$, or more simply, just L, will always denote such a lattice. The opposite lattice of L will be denoted by L^*. If x, y are elements in L with $x \leq y$, then y/x will denote the interval $[x, y]$; i.e.,

$$y/x = \{\, a \in L \,|\, x \leq a \leq y \,\}.$$

For all undefined notation and terminology on lattices, the reader is referred to [4], [6], [15], and/or [16]. In particular, we draw freely on our earlier paper [4].

1. Krull dimension via quotient lattices

The aim of this section is to present several definitions of the Krull dimension of a modular lattice via quotient lattices and to reveal the relationships between them.

Throughout this section we shall use freely some notation, terminology, and facts from [2] and [3]. So, as in [2], by an *abstract class of lattices* we mean a nonempty subclass $\mathcal{X}$ of the class $\mathcal{M}$ of all modular lattices with 0 and 1, that is closed under isomorphisms (i.e., if $P, Q \in \mathcal{M}$, $P \simeq Q$ and $P \in \mathcal{X}$, then $Q \in \mathcal{X}$).

If $L \in \mathcal{M}$, then a nonempty subclass $\mathcal{X}$ of $\mathcal{M}$ is called a *Serre class for* L if $\mathcal{X}$ is an abstract class of lattices, and for all $a \leq b \leq c$ in L, $c/a \in \mathcal{X}$ if and only if $b/a \in \mathcal{X}$ and $c/b \in \mathcal{X}$. A *Serre class* of lattices is an abstract class of lattices that is a Serre class for all lattices $L \in \mathcal{M}$.

If $L \in \mathcal{C}$, we say that $\mathcal{X}$ is a *localizing class for* L if $\mathcal{X}$ is a Serre class for L, and for any $x \in L$ and for any family $(x_i)_{i \in I}$ of elements of $1/x$ such that $x_i/x \in \mathcal{X}$ for all $i \in I$, we have $(\bigvee_{i \in I} x_i)/x \in \mathcal{X}$.

By a *localizing class of lattices* we mean a Serre class of lattices that is a localizing class for all $L \in \mathcal{C}$.

Let $\mathcal{X}$ be an arbitrary nonempty subclass of $\mathcal{M}$ and let $L \in \mathcal{M}$. As in [2] we define a relation $\sim_{\mathcal{X}}$ on L by:

$$a \sim_{\mathcal{X}} b \iff (a \vee b)/(a \wedge b) \in \mathcal{X}\,.$$

Note that if $a \leq b$ in L, then

$$a \sim_{\mathcal{X}} b \iff b/a \in \mathcal{X}\,.$$

It was proved in [2, Proposition 2.4] that if $\mathcal{X}$ is an abstract class of lattices, then $\sim_{\mathcal{X}}$ is congruence on L if and only if $\mathcal{X}$ is a Serre class for L, and in this case, the lattice $L/{\sim_{\mathcal{X}}}$ is called the *quotient lattice* (or *factor lattice*) of L by (or *modulo*) the Serre class $\mathcal{X}$.

As in [2], one defines for $\mathcal{X}$ and L a certain subset $\mathrm{Sat}_{\mathcal{X}}(L)$ of L as follows:

$$\mathrm{Sat}_{\mathcal{X}}(L) = \{\, x \in L \mid x \leq y \in L\,,\; y/x \in \mathcal{X} \implies x = y \,\}\,,$$

which is called the *$\mathcal{X}$–saturation* of L or the *$\mathcal{X}$–closure* of L. This is the precise analogue of the lattice $\mathrm{Sat}_{\tau}(M) = \{\, N \leq M_R \mid M/N \in \mathcal{F} \,\}$ of all τ–closed submodules of a given right R–module M, where $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on the category Mod–R of all unital right R–modules over an associative ring R with nonzero identity.

Of particular interest are those lattices $L \in \mathcal{M}$ and classes $\mathcal{X}$ such that $\mathrm{Sat}_{\mathcal{X}}(L)$ has a natural structure of a modular lattice. This is, by [2, Proposition 3.6], the case when $\mathcal{X}$ is a Serre class for L and additionally L possesses an *$\mathcal{X}$–closure operator*; i.e., a map

$$L \longrightarrow \mathrm{Sat}_{\mathcal{X}}(L), \;\; x \longmapsto \overline{x}$$

such that

(1) $x \leq \overline{x}$ and $\overline{x}/x \in \mathcal{X}$ for all $x \in L$.
(2) $x \leq y$ in $L \implies \overline{x} \leq \overline{y}$.

It is known that if $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on the category Mod–R and M_R is a right R–module, then the lattice $\mathrm{Sat}_{\tau}(M)$ of all τ–closed submodules of M is isomorphic to the lattice $\mathcal{L}(T(M))$ of all subobjects of the object $T(M)$ in the quotient category Mod–$R/\mathcal{T}$, where $T :$ Mod–$R \longrightarrow$ Mod–$R/\mathcal{T}$ is the canonical functor (see [1, Proposition 7.10]). The result below is precisely the latticial analogue of this fact.

LEMMA 1.1. ([2, Lemma 3.7]) *Let $\mathcal{X}$ be a Serre class for a modular lattice L such that L has an $\mathcal{X}$–closure operator. Then*

$$L/\sim_{\mathcal{X}} \simeq \mathrm{Sat}_{\mathcal{X}}(L).$$

Recall that if $\mathcal{X}$ is an arbitrary nonempty subclass of $\mathcal{M}$, a lattice L is called *$\mathcal{X}$–Noetherian* (resp. *$\mathcal{X}$–Artinian*) if for every ascending chain $x_1 \leq x_2 \leq \dots$ (resp. descending chain $x_1 \geq x_2 \geq \dots$) of elements in L there exists a positive integer s such that $x_{i+1}/x_i \in \mathcal{X}$ (resp. $x_i/x_{i+1} \in \mathcal{X}$) for all $i \geq s$.

The next result is an extension of the equivalence $(1) \Longleftrightarrow (3)$ in [2, Proposition 4.2].

LEMMA 1.2. *Let $\mathcal{X}$ be a Serre class for a modular lattice L. Then the following statements are equivalent:*

(1) *L is $\mathcal{X}$–Artinian (resp. $\mathcal{X}$–Noetherian).*

(2) *The quotient lattice $L/\sim_{\mathcal{X}}$ is Artinian (resp. Noetherian).*

Proof. We shall discuss only the $\mathcal{X}$–Artinian case. For any $x \in L$ we denote by $\widehat{x}$ the congruence class of x.

$(1) \Longrightarrow (2)$. Let $\widehat{x_1} > \widehat{x_2} > \widehat{x_3} > \cdots$ be a descending chain in $L/\sim_{\mathcal{X}}$, with $x_i \in L$ for all $i \geq 1$. We construct a descending chain $y_1 > y_2 > y_3 > \cdots$ in L such that $y_i \in \widehat{x_i}$ for all $i \geq 1$ as follows: Set $y_1 = x_1$. Inductively assume that we have constructed $y_1 > y_2 > \cdots > y_n$, with $y_k \in \widehat{x_k}$ for all $1 \leq k \leq n$. Now

$$\widehat{x_n} > \widehat{x_{n+1}} \Longrightarrow \widehat{x_{n+1}} = \widehat{x_n} \wedge \widehat{x_{n+1}} = \widehat{y_n} \wedge \widehat{x_{n+1}} = \widehat{y_n \wedge x_{n+1}}$$
$$\Longrightarrow y_n \wedge x_{n+1} \in \widehat{x_{n+1}},$$

since the canonical map $L \longrightarrow L/\sim_{\mathcal{X}}$ is a lattice morphism. Set $y_{n+1} = y_n \wedge x_{n+1}$. Then $y_n > y_{n+1}$. Since L is $\mathcal{X}$–Artinian by (1), there exists m such that $y_m/y_{m+1} \in \mathcal{X}$, and hence $y_m \sim_{\mathcal{X}} y_{m+1}$. But this contradicts the fact that y_m and y_{m+1} are in different congruence classes $\widehat{x}_m$ and $\widehat{x}_{m+1}$.

$(2) \Longrightarrow (1)$: Assume that the lattice $L/\sim_{\mathcal{X}}$ is Artinian, and let

$$x_1 \geq x_2 \geq \dots$$

be a descending chain in L. Then, we obtain the descending chain

$$\widehat{x_1} \geq \widehat{x_2} \geq \dots$$

in $L/\sim_{\mathcal{X}}$. Since $L/\sim_{\mathcal{X}}$ is Artinian, there exists a positive integer s such that $\widehat{x_n} = \widehat{x_{n+1}}$; i.e., $x_n/x_{n+1} \in \mathcal{X}$ for all $n \geq s$. This shows that L is $\mathcal{X}$–Artinian. $\square$

Now recall from [3] some notations and facts on Krull dimension and dual Krull dimension:

$\mathcal{K} = \{ L \in \mathcal{M} \mid L \text{ has Krull dimension} \}$,
$\mathcal{K}^0 = \{ L \in \mathcal{M} \mid L \text{ has dual Krull dimension} \}$,
$\mathcal{K}_\alpha = \{ L \in \mathcal{K} \mid k(L) < \alpha \}$,
$\mathcal{K}_\alpha^0 = \{ L \in \mathcal{K}^0 \mid k^0(L) < \alpha \}$,

where $\alpha \geq 0$ is an arbitrary ordinal and $k(L)$ (resp. $k^0(L)$) denotes the Krull dimension (resp. dual Krull dimension) of the lattice L.

It is well known that $\mathcal{K}$, $\mathcal{K}_\alpha$, $\mathcal{K}_\alpha^0$ are all Serre classes of lattices. Moreover, $\mathcal{K} = \mathcal{K}^0$ according to a well known result due to Lemonnier [10].

For any ordinal $\alpha \geq 0$ we shall briefly denote $\sim_{\mathcal{K}_\alpha}$ by $\sim_\alpha$, and $\sim_{\mathcal{K}_\alpha^0}$ by $\sim_{\alpha^0}$. So, if $a, b \in L$, then

$$a \sim_\alpha b \iff k((a \vee b)/(a \wedge b)) < \alpha,$$

$$a \sim_{\alpha^0} b \iff k^0((a \vee b)/(a \wedge b)) < \alpha.$$

Since $\mathcal{K}_\alpha$ and $\mathcal{K}_\alpha^0$ are Serre classes of lattices, we have that both $\sim_\alpha$ and $\sim_{\alpha^0}$ are congruences on any lattice $L \in \mathcal{M}$. In particular, $L/\sim_\alpha$ and $L/\sim_{\alpha^0}$ are modular lattices, which will be denoted throughout this paper by L_α and L_{α^0} respectively.

Remark that if $\mathcal{X}$ is a Serre class of lattices for a lattice L, then

$$L \in \mathcal{X} \iff L/\sim_{\mathcal{X}} = \mathbf{1},$$

where $\mathbf{1}$ denotes the *trivial* lattice, i.e., the lattice having only one element.

The first half of the next result is a particular case of [3, Proposition 3.3]:

Lemma 1.3. *Let $L \in \mathcal{M}$ and let $\alpha \geq 0$ be an ordinal. Then,*

$$k(L) = \alpha \iff L \notin \mathcal{K}_\alpha \text{ and } L \text{ is } \mathcal{K}_\alpha - \text{Artinian}.$$

$$k^0(L) = \alpha \iff L \notin \mathcal{K}_\alpha^0 \text{ and } L \text{ is } \mathcal{K}_\alpha^0 - \text{Noetherian}.$$

Now using Lemma 1.2, Lemma 1.3, and the remark made just before Lemma 1.3, we deduce at once:

PROPOSITION 1.4. *Let $L \in \mathcal{M}$ and let $\alpha \geq 0$ be an ordinal. Then,*

$$\begin{aligned} k(L) = \alpha &\iff k(L_\alpha) = 0 \\ &\iff \alpha \text{ is the least ordinal such that } L_{\alpha+1} = \mathbf{1}, \end{aligned}$$

and

$$\begin{aligned} k^0(L) = \alpha &\iff k^0(L_{\alpha^0}) = 0 \\ &\iff \alpha \text{ is the least ordinal such that } L_{(\alpha+1)^0} = \mathbf{1}. \end{aligned}$$

Our next aim is to compare the definition of Krull dimension of a modular lattice with the dimensions discussed in [8], [13], and [14].

As in [13, Section 10.2], for any lattice L and for any ordinal $\alpha \geq 0$ we shall define inductively a congruence $\approx^\alpha$ on L and the corresponding quotient lattice L^α of L as follows:

$$L^0 := L\,,\ \approx^0 := =\,,\ \approx^1 := \sim_1\,,\ L^1 := L/\approx^1\,.$$

Thus,

$$x \approx^1 y \iff (x \vee y)/(x \wedge y) \text{ is Artinian.}$$

Let $\alpha \geq 2$ be an ordinal and suppose that we have defined the quotient lattices L^γ of L for any ordinal $\gamma < \alpha$ and the congruences $\approx^\gamma$ on L such that $L^\gamma \simeq L/{\approx^\gamma}$ canonically. If α is a nonlimit ordinal, say $\alpha = \beta + 1$, then define $L^\alpha = L^{\beta+1} := (L^\beta)^1$, and denote by $\approx^\alpha$ the congruence on L such that $L^\alpha \simeq L/{\approx^\alpha}$.

In case α is a limit ordinal, then define

$$\approx^\alpha := \bigcup_{\beta<\alpha} \approx^\beta\,, \text{ and } L^\alpha := L/{\approx^\alpha}\,.$$

We say that the lattice L has *Krull*-dimension* if there exists an ordinal γ with $L^{\gamma+1} = \mathbf{1}$; in this case, define $k^*(L)$ as being the least ordinal α such that $L^{\alpha+1} = \mathbf{1}$. It was asserted in [13, p. 217] that "*modulo quibbles at limit ordinals*", $k^*(L)$ coincides with the usual Krull dimension $k(L)$ of L. This is in fact so, because of the next result.

PROPOSITION 1.5. *For any lattice $L \in \mathcal{M}$ and any ordinal $\alpha \geq 0$ the congruences $\approx^\alpha$ and $\sim_\alpha$ on L coincide.*

Proof. This can be proved by transfinite induction. The cases $\alpha =$ 0, 1, or a limit ordinal are immediate from the definitions.

For the nonlimit ordinal case, assume the result is true for α. Now $x \sim_{\alpha+1} y$ if and only if any descending chain between $x \vee y$ and $x \wedge y$

has only finitely many factors with Krull dimension $\not< \alpha$. This happens exactly when any descending chain

$$x \vee y \geq x_1 \geq x_2 \geq \ldots \geq x_{n-1} \geq x_n \geq \ldots \geq x \wedge y$$

has $x_i \sim_\alpha x_{i+1}$ for all but finitely many i. But by the induction hypothesis, $x_i \sim_\alpha x_{i+1}$ if and only if $x_i \approx^\alpha x_{i+1}$, and the latter condition holding for all but finitely many i in any chain means $x \approx^{\alpha+1} y$. □

For a lattice L, the dimensions $k(L)$ and $k^0(L)$ are special cases of a more general concept of Γ–deviation, which can be found in [4], [5], or [12]. In particular, $k(L)$ and $k^0(L)$ are obtained by using $\Gamma = \{\omega^*\}$ and $\Gamma = \{\omega\}$, respectively, ([4, Proposition 3.10]) in the following definition.

DEFINITION 1.6. Let $(P, \leq)$ be an arbitrary poset and Γ an arbitrary nonempty set of linear order types. The Γ–*deviation* of $(P, \leq)$, also called Γ–*Krull dimension* of $(P, \leq)$ and denoted in the sequel by $k_\Gamma(P)$, is an ordinal number defined recursively as follows: $k_\Gamma(P) = -1$, where -1 is assumed to be the predecessor of zero, if and only if P is a trivial poset (or antichain), that is if it has no two distinct comparable elements.

$k_\Gamma(P) = 0$ if and only if $k_\Gamma(P) \neq -1$ and P contains no chain of order type γ for any $\gamma \in \Gamma$.

Let $\alpha > 0$ be an ordinal number and assume that we have already defined which posets have Γ–deviation β for any ordinal $\beta < \alpha$. Then we say that $k_\Gamma(P) = \alpha$ if $k_\Gamma(P)$ has not been previously defined, and if, for any $\gamma \in \Gamma$ and any chain C of P of type γ, there exists $a < b$ in C such that b/a, considered as an interval in P, has $k_\Gamma(b/a) = \beta$ for some $\beta < \alpha$.

In case $\Gamma = \{\gamma\}$, then instead of $k_{\{\gamma\}}(P)$, we shall simply write $k_\gamma(P)$, and call it γ–deviation.

In [8] H. Krause implicitly defines a Krull dimension $h(L)$ for a lattice L in a manner similar to Prest [13]: for any ordinal $\alpha \geq 0$ one defines inductively a congruence $\asymp^\alpha$ on L and the corresponding quotient lattice H_α of L as follows:

$$H_{-1} := L\,,\ \asymp_{-1} := =\,,\ \asymp_0 := \sim_1\,,\ H_0 := L/\asymp_0\,.$$

Thus,

$$x \asymp_0 y \Longleftrightarrow (x \vee y)/(x \wedge y) \text{ is Artinian.}$$

Let $\alpha \geq 1$ be an ordinal and suppose that we have defined the lattices H_γ for any ordinal $\gamma < \alpha$ and the congruences $\asymp_\gamma$ on L such that $H_\gamma \simeq L/\asymp_\gamma$ canonically. If α is a nonlimit ordinal, say $\alpha = \beta + 1$, then define $H_\alpha = H_{\beta+1} := (H_\beta)_0$, and denote by $\asymp_\alpha$ the congruence on L such that $H_\alpha \simeq L/\asymp_\alpha$.

In case α is a limit ordinal, then define

$$\asymp_\alpha := \bigcup_{\beta<\alpha} \asymp_\beta \quad \text{and} \quad H_\alpha := L/\asymp_\alpha .$$

We say that the lattice L has *h–dimension* if there exists an ordinal γ with $H_\gamma = \mathbf{1}$; in this case, define $h(L)$ as being the least ordinal α such that $H_\alpha = \mathbf{1}$. If the lattice L does not have h–dimension, we write $h(L) = \infty$.

As one can see, Krause's definition varies slightly from the usual one. These differences are due to the starting points $L = L^0$ or $L = H_{-1}$ and the choice of $\alpha + 1$ or α in determining the dimensions $k(L)$ and $h(L)$, respectively.

For example, let L be a chain of order type $(\omega^\omega+1)^*$. For any positive integer $n \geq 1$ the congruence $\approx^n$ identifies $x, y \in L$ with $x < y$ whenever $k(y/x) \leq n-1$. Hence $\approx^\omega = \bigcup_{n<\omega} \approx^n$ identifies $x, y \in L$ with $x < y$ whenever $k(y/x) < \omega$. Thus $L/\approx^\omega \neq \mathbf{1}$; however, $L/\approx^{\omega+1} = \mathbf{1}$. By the definition of $k(L)$, we have $k(L) = \omega$, but by the definition of $h(L)$, we have $h(L) = \omega + 1$.

Indeed, by [8, Lemma 1.1 (3)], $h(L)$ can never be a limit ordinal. This is true because a limit ordinal can never be the least ordinal γ such that $0 \asymp_\gamma 1$. Since $h(L)$ and $k(L)$ are always defined basically from the same congruence relation $\approx^\gamma$, this forces the dimensions to vary by 1 for lattices L with $k(L) \geq \omega$.

Explicitly, we have the following formulas:

$$\begin{aligned} h(L) &= k(L) \text{ for } k(L) < \omega, \\ h(L) &= k(L) + 1 \text{ for } k(L) \geq \omega, \\ h(L) &= \infty \text{ when } k(L) \text{ does not exist.} \end{aligned}$$

The real thrust of the work of Krause [8] is in the representation theory of algebras, where modules of finite length play a very important role. We define a dimension, which we denote by $f(L)$ ("f" for finite) for a lattice L: $f(L) = k_\Gamma(L)$, where $\Gamma = \{\omega, \omega^*\}$. In particular, $f(L) = -1$ in case $L = \mathbf{1}$; $f(L) = 0$ if $L \neq \mathbf{1}$ and L has finite length (i.e., L is both Noetherian and Artinian.) Higher dimensions are defined inductively as in Definition 1.6 to measure how close the lattice is to being of finite length. We note that our definition is analogous to one given for modules in [14].

In [8], Krause introduces a dimension similar to $f(L)$; it is denoted by $\dim_{\mathcal{L}}(L)$, where $\mathcal{L}$ denotes the class of all finite length lattices. The definition of $\dim_{\mathcal{L}}(L)$ is based on the same principal as $h(L)$, which uses the Artinian lattices instead of the finite length lattices. So, by reasoning similar to the Krull dimension case, we have

$$\dim_{\mathcal{L}}(L) = f(L) \text{ for } f(L) < \omega,$$
$$\dim_{\mathcal{L}}(L) = f(L) + 1 \text{ for } f(L) \geq \omega,$$
$$\dim_{\mathcal{L}}(L) = \infty \text{ when } f(L) \text{ does not exist.}$$

An easy transfinite induction shows that $f(L)$ is precisely the $\mathbf{2}$–dimension $\dim_{\mathbf{2}}(L)$ ("$\mathbf{2}$" for two–point lattice) or the m–dimension m–$\dim(L)$ ("m" for minimal congruence) of L considered in [13].

2. Localization of Γ-deviation

The aim of this section is to deal with the localization of Γ–deviation for two important cases. To do this, we determine when chains can be lifted; i.e., if L is a modular lattice and $\sim$ is a congruence relation on L, when can a chain of order type γ in $L/\sim$ be lifted to a chain of type γ in L? Throughout this section we shall use freely some notation, terminology, and facts from [2], [3], and [4].

Proposition 2.1. *Let $\sim$ be a congruence relation on a lattice $L \in \mathcal{M}$. Assume that each congruence class of $\sim$ has a unique maximal (minimal) element. Then for each chain C' in $L' = L/\sim$, there exists a chain C in L such that $C \cong C'$ and each member of C is in a distinct congruence class of $\sim$.*

Proof. Let $C' = \{c'_\alpha\}$ be a chain in L'. Let c_α be the unique maximal (minimal) element of c'_α. Then $C = \{c_\alpha\}$ is a chain in L with $C \cong C'$ and $c_\alpha \in c'_\alpha$. □

Recall from [4] that if $\mathcal{X}$ be a nonempty class of posets and Γ is a nonempty set of linear order types, then a poset P is said to be $\mathcal{X} - \Gamma$ if, for any chain C of P of order type γ with $\gamma \in \Gamma$, there exists $a < b$ in C such that b/a, considered as an interval in P, belongs to $\mathcal{X}$.

Proposition 2.2. *Let $\mathcal{X}$ be a Serre class of lattices for a lattice $L \in \mathcal{M}$ such that L has an $\mathcal{X}$–closure operator $L \longrightarrow \mathrm{Sat}_{\mathcal{X}}(L)$, $x \longmapsto \overline{x}$, and let Γ be a nonempty class of linear order types. Then the following* *statements are equivalent.*

(1) *The lattice L is $\mathcal{X}-\Gamma$.*

(2) $k_\Gamma(\mathrm{Sat}_{\mathcal{X}}(L)) \leq 0$.

(3) $k_\Gamma(L/\sim_{\mathcal{X}}) \leq 0$.

Proof. For simplicity, let $L' = \mathrm{Sat}_{\mathcal{X}}(L)$. Clearly (2) $\Longleftrightarrow$ (3) by Lemma 1.1.

(1) $\Longrightarrow$ (2): Assume that L' contains a chain C' of order type γ for a certain $\gamma \in \Gamma$. Then there exist $x' < y'$ in C' such that $y'/x' \in \mathcal{X}$. Since $x' \in L'$, it follows that $y' = x'$, which is a contradiction. Thus $k_\Gamma(\mathrm{Sat}_{\mathcal{X}}(L)) \leq 0$.

(2) $\Longrightarrow$ (1): Assume that $k_\Gamma(\mathrm{Sat}_{\mathcal{X}}(L)) \leq 0$. We are going to show that L is $\mathcal{X}-\Gamma$. To do so, let C be a chain of L of order type γ for some $\gamma \in \Gamma$, and consider the set

$$C' := \{\, \overline{x} \mid x \in C \,\}.$$

Then clearly C' is a chain in L'.

Observe that the map

$$C \longrightarrow C', \ \ c \longmapsto \overline{c}$$

cannot be injective, for otherwise L' would contain a chain of order type γ, which contradicts the fact that $k_\Gamma(\mathrm{Sat}_{\mathcal{X}}(L)) \leq 0$. Hence there exist $x \neq y$ in C such that $\overline{x} = \overline{y}$. Since C is a chain, we may assume that $x < y$. We have

$$y/x \subseteq \overline{y}/x = \overline{x}/x \in \mathcal{X}.$$

This proves that L is $\mathcal{X}-\Gamma$. □

If all the order types in Γ are countable, then we can avoid the assumption that L has an $\mathcal{X}$–closure operator. To see this, we need the following lemma.

LEMMA 2.3. *Let $\sim$ be a congruence on a lattice L, let $L' = L/\sim$ be the quotient lattice and let $p : L \longrightarrow L'$ be the canonical surjection. If C' is a countable chain in L', then there exists a chain C in L such that $C \simeq C'$ via p.*

Proof. We notice first that for every $a \in L$ the set

$$L_a \ = \ \{\, x \in L \mid x \text{ comparable to } a \,\}$$

is a sublattice of L. Since C' is countable, let us consider a bijective map $\mathbb{N} \longrightarrow C'$, $i \mapsto C'(i)$. We will construct inductively pairs (T_n, C_n) with T_n sublattice in L, C_n a chain in T_n and such that

(i) $p(T_n) \supseteq C'$,
(ii) p restricted to C_n is an injective map,
(iii) $p(C_n) = \{ C'(i) \mid 0 \leq i \leq n \}$,
(iv) x and y are comparable for every $x \in T_n$ and for every $y \in C_n$.

For $n = 0$, let $y_0 \in C'(0)$. Thus, for $T_0 = L_{y_0}$ and $C_0 = \{y_0\}$, the pair (T_0, C_0) clearly verifies (ii), (iii) and (iv). In order to verify (i), we notice that, for $x \in C'(i)$, since C' is a chain, there are just two cases:

a) if $C'(i) \leq p(y_0)$, then for $x' = x \wedge y_0$, since T_0 is sublattice, we have $x' \in T_0$ and $p(x') = p(x) \wedge p(y_0) = C'(i)$;
b) if $C'(i) > p(y_0)$, then for $x' = x \vee y_0$, since T_0 is sublattice, we have $x' \in T_0$ and $p(x') = p(x) \vee p(y_0) = C'(i)$.

In both cases there exists $x' \in T_0$ such that $p(x') = C'(i)$. Therefore (i) holds for T_0.

Assume now that we have constructed (T_n, C_n) with properties (i), (ii), (iii) and (iv). From (i) there exists $y_{n+1} \in T_n \cap C'(n+1)$. Set $T_{n+1} = T_n \cap L_{y_{n+1}}$; it is clear that T_{n+1} is a sublattice of L. Using (iv), we have $C_n \subseteq L_{y_{n+1}}$ and so $C_n \subseteq T_{n+1}$. Set $C_{n+1} = C_n \cup \{y_{n+1}\}$. By definition, (T_{n+1}, C_{n+1}) verifies (ii), (iii) and (iv). Since T_{n+1} is a sublattice, using (i) for T_n in the same argument as in the case $n = 0$, we obtain that (T_{n+1}, C_{n+1}) also verifies (i).

We have constructed inductively the desired family of pairs. We notice that $C_n \subseteq C_{n+1}$ for every $n \in \mathbb{N}$; hence $C = \bigcup_{n \in \mathbb{N}} C_n$ is a chain in L. Set $f : C \longrightarrow C'$ by $f(y) = p(y)$. Since (ii) and (iii) hold for every C_n, we deduce that f is a bijective map. Since p is increasing, f is also increasing. Since C and C' are chains, f is a chain isomorphism. □

Proposition 2.4. *Let $\mathcal{X}$ be a Serre class of lattices for a lattice $L \in \mathcal{M}$, and let Γ be a nonempty set of countable linear order types. Then the following statements are equivalent.*

(1) *The lattice L is $\mathcal{X} - \Gamma$.*
(2) $k_\Gamma(\mathrm{Sat}_{\mathcal{X}}(L)) \leq 0$.
(3) $k_\Gamma(L/\sim_{\mathcal{X}}) \leq 0$.

Proof. (1) $\Longrightarrow$ (3). Assume that there exists a chain C' of $L/\sim_{\mathcal{X}}$ of order type γ for a certain $\gamma \in \Gamma$. By Lemma 2.3, there also exists a chain C of L of order type γ, and each element of C is in a different congruence class of $\sim_{\mathcal{X}}$. By (1) there exist $x < y$ in C such that $y/x \in \mathcal{X}$. Hence $\widehat{x} = \widehat{y}$ in $L/\sim_{\mathcal{X}}$, which is a contradiction. Thus $k(L/\sim_{\mathcal{X}}) \leq 0$.

The other implications follow as in the proof of Proposition 2.2. □

Denote by $\mathcal{K}_\Gamma$ the class of all modular lattices having Γ–deviation. For each ordinal $\alpha \geq 0$, set

$$\mathcal{K}_\Gamma^\alpha := \{ L \in \mathcal{K}_\Gamma \mid k_\Gamma(L) < \alpha \} .$$

Then Proposition 4.6 from [4] can be reformulated more compactly as follows:

PROPOSITION 2.5. *For any ordinal $\alpha \geq 0$ and any set Γ consisting only of indecomposable linear order types, the class $\mathcal{K}_\Gamma^\alpha$ is a Serre class of lattices.*

In particular, the theory for the localization of modular lattices developed in [2] and [3] applies to Γ–deviation, and hence Propositions 2.2, 2.4, and 2.5 combine to give the following result.

THEOREM 2.6. *Let $\alpha \geq 0$, $L \in \mathcal{M}$ and Γ a nonempty set of indecomposable linear order types. Denote by L_Γ^α the quotient lattice of L with respect to the Serre class of lattices $\mathcal{K}_\Gamma^\alpha$. In case either L has an $\mathcal{K}_\Gamma^\alpha$–closure operator or Γ contains only countable order types, then*

$$k_\Gamma(L) \leq \alpha \iff k_\Gamma(L_\Gamma^\alpha) \leq 0 .$$

3. Existence of Γ-deviation

In this section, we explore some conjectures made in [4] and [9] about the existence of Γ–deviation. In one of the two cases that we explore, the localization techniques of the previous section are very useful.

Let η denote the order type of the rational numbers. It was shown [10, Theorem 5] that a poset fails to have ω–deviation if and only if it contains a chain of order type η; this is due to the density of η. More generally, we consider a set Γ of η linear order types and call a nontrivial subset S of P Γ–*dense* if $x, y \in S$ with $x < y$ implies that there is a chain C of type γ for some $\gamma \in \Gamma$ in S with $C \subseteq y/x$. Conjectures have been made in [4] and [9] that Γ–density is connected to the failure of a poset to have Γ–deviation, and results in [4] and [9] provide evidence of this.

First, we note a preliminary result that relates Γ–density and Γ–deviation. Proposition 3.1 and its proof are completely analogous to [9, Theorem 1.1] and [4, Lemma 3.22].

PROPOSITION 3.1. *If a poset P contains a Γ–dense set, then the Γ–deviation of P fails to exist.*

If case P is linearly ordered, the converse of Proposition 3.1 is true [9, Theorem 1.4]. The converse has been conjectured to hold for modular lattices [4]. Our next result will provide evidence that the conjecture of [4] is true.

A lattice L is known to be modular if and only if L contains no pentagon sublattice. Equivalently, L is modular if and only if either

(1) $a, b, c \in L,\ a < b$, and a, b incomparable to c imply $a \wedge c < b \wedge c$,

or

(2) $a, b, c \in L,\ a < b$, and a, b incomparable to c imply $a \vee c < b \vee c$

hold.

We define a lattice to be *strongly modular* if both (1) and (2) hold.

In [4], a property denoted by (†) is introduced as a natural extension of a condition used in [9]. In particular. [4, Lemma 3.23] states that if a poset P fails to have Γ–deviation, then there exists a $\gamma \in \Gamma$ and a chain C in P of order type γ having the following property:

(†) *for any $a < b$ in C, the interval b/a in P does not have Γ–deviation.*

We use property (†) as a tool in the proofs of Theorems 3.2 and 3.6 below. Our proof of Theorem 3.2 uses the method of [9, Theorem 1.4].

THEOREM 3.2. *Let Γ be a set of indecomposable order types, and let P be a strongly modular lattice. Then P fails to have Γ–deviation if and only if P contains a Γ–dense set.*

Proof. In view of Proposition 3.1, we only need to prove the "only if" part. We assume that $k_\Gamma(P)$ does not exist and construct a Γ–dense subset of C of P. For convenience, we assume throughout the proof that all intervals considered are intervals of P. Let $C_0 = \emptyset$. By [4, Lemma 3.23] P contains a chain C_1 of order type $\gamma \in \Gamma$ with property (†). We wish to construct sets C_i $(0 \le i < \omega)$ with the following properties:

(1) For each $x, y \in \bigcup_{i<n} C_i$ with $x < y$, there exists a chain of order type $\gamma \in \Gamma$ in C_n, all of whose members lie in the interval y/x.
(2) $\bigcup_{i\le n} C_i$ has property (†).

Assume for induction that $C_0, C_1, \ldots, C_n$ have been constructed. In order to construct C_{n+1}, we use transfinite induction. Let

$$S = \{\, (a,b) \mid a < b\,;\; a, b \in \bigcup_{i \leq n} C_i \,\},$$

and let $\{\, (a_\alpha, b_\alpha) \mid \alpha < \tau\}$ be a well-ordering of S. By [4, Lemma 3.23] we can find a chain $C(a_0, b_0)$ of type $\gamma \in \Gamma$ such that $C(a_0, b_0) \subseteq b_0/a_0$ and $C(a_0, b_0)$ has property (†).

We wish to modify $C(a_0, b_0)$ to an isomorphic chain in b_0/a_0 to ensure that $D = (\bigcup_{i \leq n} C_i) \bigcup C(a_0, b_0)$ has property (†). Let $x, y \in D$ with $x < y$. From our construction, $k_\Gamma(y/x)$ fails to exist if $x, y \in \bigcup_{i \leq n} C_i$ or if $x, y \in C(a_0, b_0)$.

Now suppose that instead, $x \in \bigcup_{i \leq n} C_i$ and $y \in C(a_0, b_0)$. If there is a $z \in D$ such that $x < z < y$, then by [4, Proposition 4.6] either z/x or y/z fails to have Γ–deviation by our choices of $\bigcup_{i \leq n} C_i$ or $C(a_0, b_0)$, respectively. So we assume that there is no $z \in D$ with $x < z < y$. If there is an element $w \in C(a_0, b_0)$ such that $w < y$ but x is incomparable to w, then suppose for contradiction that y/x has Γ–deviation. Since $w, y \in C(a_0, b_0)$, then y/w does not have Γ–deviation, and hence neither does $y/(w \wedge x)$. By [4, Proposition 4.6] and our assumption that y/x has Γ–deviation, we see that $x/(w \wedge x)$ does not have Γ–deviation. Thus there exists $p \in P$ with $w \wedge x < p < x$; this forces $w \wedge x = p \wedge w$, which contradicts the strong modularity of P. Therefore, y/x cannot have Γ–deviation in this case. Thus, the only remaining case to consider is $w \in C(a_0, b_0)$ and $x < w$ imply $y \leq w$. But then, if y/x has Γ–deviation, we may replace y by x in our chain $C(a_0, b_0)$ without changing the order type of $C(a_0, b_0)$.

Similarly, if $x \in C(a_0, b_0)$ and $y \in \bigcup_{i \leq n} C_i$, we can replace x by y, whenever $k_\Gamma(y/x)$ exists.

We note that, by [4, Prop. 4.6], no two elements of $C(a_0, b_0)$ are ever replaced by the same element of $\bigcup_{i \leq n} C_i$. After this adjustment of $C(a_0, b_0)$ to an isomorphic chain in b_0/a_0 is completed, D has property (†) and $C(a_0, b_0)$ is still a chain of type γ contained in b_0/a_0.

Assume that for $\alpha < \beta < \tau$, we have defined chains $C(a_\alpha, b_\alpha)$ such that $C(a_\alpha, b_\alpha) \subseteq b_\alpha/a_\alpha$ and

$$\left(\bigcup_{i \leq n} C_i \right) \bigcup \left(\bigcup_{\alpha < \beta} C(a_\alpha, b_\alpha) \right)$$

has property (†). By [4, Lemma 3.23] we can find a chain $C(a_\beta, b_\beta)$ of order type $\gamma \in \Gamma$ that $C(a_\beta, b_\beta) \subseteq b_\alpha/a_\alpha$ and $C(a_\beta, b_\beta)$ has property (†). Using the technique of the case $\beta = 0$, we can adjust $C(a_\beta, b_\beta)$ to an isomorphic chain in b_α/a_α such that $(\bigcup_{i\leq n} C_i) \bigcup (\bigcup_{\alpha\leq\beta} C(a_\alpha, b_\alpha))$ has property (†).

By transfinite induction, it now follows that $C_{n+1} = \bigcup_{\alpha<\tau} C(a_\alpha, b_\alpha)$ satisfies the analogues of (1) and (2).

Let $C = \bigcup_{i<\omega} C_i$. Then it follows from (1) that C is Γ–dense.

In [9, Theorem 1.6] it is shown that the converse of Proposition 3.1 holds whenever $\Gamma = \{\gamma\}$ and γ does not contain a dense set (i.e., a set of order type η.) We have the following result for order types that contain a dense set.

Corollary 3.3. *Let γ be a countable linear order type that contains a dense set, and let P be a strongly modular lattice. Then P fails to have γ–deviation if and only if P contains a γ–dense set.*

Proof. Any such γ is indecomposable; so the result follows from Theorem 3.2. □

Next we use the results of Section 2 to extend [9, Theorem 1.6] to posets that behave like the lattice of modules over a ring.

Recall that an element c of a complete lattice L is said to be *compact* if whenever $c < \bigvee_{y\in S} y$ for $S \subseteq L$, there exists a finite subset $\{y_1, y_2, \ldots, y_n\} \subseteq S$ such that $x \leq y_1 \vee y_2 \vee \ldots \vee y_n$.

Let Γ be a set of indecomposable linear order types, and let L be a modular lattice. Then $\mathcal{K}_\Gamma$ is a Serre class of lattices by [4, Proposition 4.6]. Let $\sim_\Gamma$ be the congruence relation by on L determined by $\mathcal{K}_\Gamma$. In this situation, we have the following result for a modular lattice L.

Proposition 3.4. *The following assertions hold for a lattice $L \in \mathcal{M}$ and a set Γ of indecomposable linear order types.*

1. *No proper interval of $L/{\sim_\Gamma}$ has Γ–deviation.*
2. *If L is complete and every element of L is compact, then each congruence class of $\sim_\Gamma$ has a maximal element.*

Proof. (1) Since L is a set, we can find an ordinal α such that

$$\sup\{k_\Gamma(P) | P \text{ is a sublattice of } L\} \leq \alpha.$$

For contradiction, suppose that $L/\sim_\Gamma$ has a proper interval with Γ–deviation. As in the argument of [9, Theorem 1.1], we can find a proper interval $\overline{y}/\overline{x}$ of $L/\sim_\Gamma$ with $k_\Gamma(\overline{y}/\overline{x}) = 0$. For any chain C of type $\gamma \in \Gamma$ in the interval y/x, the natural mapping of C to a chain $\overline{C}$ in $L/\sim_\Gamma$ cannot be one-to-one. Hence C has an interval b/a in some congruence class of $\sim_\Gamma$, and thus $k_\Gamma(b/a) \leq \alpha$. Therefore, $k_\Gamma(y/x) \leq \alpha + 1$. So by the definition of $\sim_\Gamma, \overline{x} = \overline{y}$, which is a contradiction to our choice of $\overline{x}$ and $\overline{y}$.

(2) By [6, p. 14], a complete lattice L is Noetherian if and only if every element of L is compact. The result now follows at once. □

REMARK 3.5. Proposition 3.4 (2) shows that the Serre class $\mathcal{K}_\Gamma$ is an example of a localizing class for any Noetherian complete lattice. In particular, [3, Proposition 2.2] implies that $\mathcal{K}_\Gamma$ is a localizing class for a complete lattice L if and only if every congruence class of $\sim_\Gamma$ has a maximal element; when $\mathcal{K}_\Gamma$ is a localizing class, then L has a $\mathcal{K}_\Gamma$-closure operator by [3, Corollary 2.3].

We now give our second partial converse to Proposition 3.1.

THEOREM 3.6. *Let L be a modular lattice, let Γ be a set of indecomposable order types, and assume each congruence class of $\sim_\Gamma$ has a maximal element. Then L fails to have Γ–deviation if and only if L contains a Γ–dense set.*

Proof. Again, we only need to show the "only if" part; so we assume that the Γ–deviation of L does not exist and construct a Γ–dense set in a manner similar to the proof of Theorem 3.2. Let $C_0 = \emptyset$. By [4, Lemma 3.23], there is a chain B_1 of order type $\gamma \in \Gamma$ with property (†). Each element of B_1 is in a different congruence class of $\sim_\Gamma$. By hypothesis and [4, Propostion 4.6], we can replace B_1 by a chain C_1 consisting of the maximal elements of the congruence classes of the elements in B_1 and still have a chain of type $\gamma \in \Gamma$ with property (†). By using the maximal elements from congruence classes of $\sim_\Gamma$, we can now proceed inductively as in the proof of Theorem 3.2 to construct sets $C_n (0 \leq n \leq \omega)$ such that $\bigcup_{n<\omega} C_n$ is a Γ–dense set. We note that the construction is, in fact, easier than in Theorem 3.2, for in this case the interval between any two points in C will automatically fail to have Γ–deviation, as distinct elements of C come from distinct congruence classes of $\sim_\Gamma$. □

4. Further directions

The following related topics may provide future directions for research.

- When does $k_\Gamma(L) \leq 0$ imply that $k_{\Gamma^0}(L) \leq 0$; i.e., for which sets of order types Γ and for which modular lattices L does a so called Γ–*Hopkins–Levitzki Theorem* hold? As we already noticed, this can fail by [9, p. 261] for any Γ consisting of a single initial ordinal ω_ξ with $\xi > 0$ an arbitrary ordinal. See [2] and [3] for other relevant information on this problem.
- When can results found in [2], [3] and/or [9] be extended from usual Krull dimension and/or from chains to modular lattices with Γ–Krull dimension ?
- Does a modular lattice L fail to have Γ–deviation if and if L contains Γ–dense set ?

Acknowledgements. This work was started during the stay of the first author at the University of Wisconsin–Milwaukee as a Visiting Professor in the academic year 1997–1998, and continued while he was a Visiting Professor in the academic year 1998–1999 at the Ohio State University, Columbus and at the University of California, Santa Barbara. In the final stage of its preparation he was partially supported by Grant # 7D/2000 awarded by the Consiliul Naţional al Cercetării Ştiinţifice din Învăţământul Superior, România. He would like to thank all these institutions for their hospitality and financial support. He is indebted to Henning Krause for helpful discussions.

The authors would also like to thank M. Iosif for providing the authors with a shorter proof of Lemma 2.3.

References

[1] T. Albu and C. Năstăsescu, *Relative Finiteness in Module Theory*, Marcel Dekker, Inc., New York and Basel, 1984.

[2] T. Albu and P. F. Smith, *Localization of modular lattices, Krull dimension, and the Hopkins-Levitzki Theorem (I)*, Math. Proc. Cambridge Philos. Soc. **120** (1996), 87–101.

[3] ———, *Localization of modular lattices, Krull dimension, and the Hopkins-Levitzki Theorem (II)*, Comm. Algebra **25** (1997), 1111–1128.

[4] T. Albu and M. L. Teply, *Generalized deviation of posets and modular lattices*, Discrete Math. **214** (2000), 1–19.

[5] ———, *The double infinite chain condition and generalized deviations of posets and modules*, in "Algebra and its Application", edited by D. V. Huynh, S. K.

Jain, and S. R. López-Permouth, Contemporary Math., Vol. 259, Amer. Math. Soc., Providence, pp. 13–43 (2000).
[6] P. Crawley and R. P. Dilworth, *Algebraic Theory of Lattices*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1973.
[7] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.
[8] H. Krause, *Generic modules over Artin algebras*, Proc. London Math. Soc. **76** (1998), 276–306.
[9] W. G. Lau, M. L. Teply and A. K. Boyle, *The deviation, density, and depth of partially ordered sets*, J. Pure Appl. Algebra **60** (1989), 253–268.
[10] B. Lemonnier, *Déviation des ensembles et groupes abéliens totalement ordonnés*, Bull. Sci. Math. **96** (1972), 289–303.
[11] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, John Wiley & Sons, Chichester, 1987.
[12] M. Pouzet et N. Zaguia, *Dimension de Krull des ensembles ordonnés*, Discrete Math. **53** (1985), 173–192.
[13] M. Prest, *Model Theory and Modules*, Cambridge University Press, Cambridge, 1988.
[14] F. F. Raggi C. and C. J. E. Signoret P., *On* $\{\omega^*,\omega\}$ *-deviation of a module*, Comm. Algebra **21** (1993), 2155–2174.
[15] J. G. Rosenstein, *Linear Orderings*, Academic Press, New York, 1982.
[16] B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin Heidelberg New York, 1975.

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