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INFRA-NILMANIFOLDS AND THEIR FUNDAMENTAL GROUPS

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ABSTRACT. We present a survey of research results obtained for infra-nilmanifolds, their fundamental groups and some of their generalizations. This is presented from two different approaches and covers achievements obtained during the past four decades and showing a remarkable amount of mathematical interdisciplinarity. We go more in depth concerning the existence and construction of polynomial structures for these manifolds and groups, a direction where significant progress was made in the past few years. The bounded-degree polynomial structures developed by the authors triggered a number of challenging open problems. Also, their study already has lead to some interesting results concerning e.g. Anosov diffeomorphisms and expanding maps.

1. Flat, infra-nil and infra-solv: the Bieberbach-theorems-direction

Infra-nilmanifolds are natural generalizations of compact flat Riemannian manifolds. In fact, there is an even more general setting of which both are special cases. Let us describe this setup here.

For G a connected and simply connected Lie group we let $\mathrm{Aut}\,(G)$ denote the group of continuous automorphisms of G. The semi-direct product group $G \rtimes \mathrm{Aut}\,(G)$ acts naturally on G (on the left) via

$$\forall g, h \in G,\ \forall \alpha \in \mathrm{Aut}\,(G): \quad {}^{(g,\alpha)}h = g \cdot \alpha(h). \tag{1}$$

We call $G \rtimes \mathrm{Aut}\,(G)$ the affine group of G and write $\mathrm{Aff}(G)$ for it. In fact if one endows G with the linear connection defined by the left-invariant

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vector fields, then $\mathrm{Aut}\,(G)$ is exactly the group of connection-preserving diffeomorphisms of G ([41]).

Take C a maximal compact subgroup of $\mathrm{Aut}\,(G)$ (so C is unique up to conjugation in $\mathrm{Aut}\,(G)$). In this paper, we are interested in discrete and cocompact subgroups E of $G \rtimes C$. All such groups act properly discontinuously on G via (1). Moreover, when E is torsion free, this action is free and the quotient space $M = {}_E\backslash G$ is a manifold.

Let $A : G \to G$ be an automorphism of G which normalises E. Then A projects to a differentiable map of the manifold $M = {}_E\backslash G$ which is called an endomorphism of M. If $AEA^{-1} = E$, then E induces in that way an automorphism of M.

1.1. The abelian case: flat Riemannian manifolds

Now, let us specialize this situation to its simplest case, i.e. when G is abelian. Thus $G = \mathbb{R}^n$, for some n, $\mathrm{Aut}\,(G) = \mathrm{Gl}(n)$ and $\mathrm{Aff}(G) = \mathbb{R}^n \rtimes \mathrm{Gl}(n)$, the usual affine group. For C, we can choose the group $C = O(n)$, the orthogonal group. It follows that $G \rtimes C = \mathbb{R}^n \rtimes O(n) = \mathrm{Isom}(\mathbb{R}^n)$, the group of Euclidean isometries of $\mathbb{R}^n$. Observe that the action of $\mathrm{Aff}(\mathbb{R}^n)$ (resp. $\mathrm{Isom}(\mathbb{R}^n)$) on $\mathbb{R}^n$ as defined by (1) is the usual action of these groups.

Definition 1.1. A discrete and cocompact subgroup $E \subseteq \mathrm{Isom}(\mathbb{R}^n)$ is called an n-dimensional *crystallographic group*. If moreover, E is torsion free, then E is said to be a *Bieberbach group*.

It is obvious that when E is a Bieberbach group, the quotient manifold ${}_E\backslash\mathbb{R}^n$ inherits the flat Riemannian metric from the Euclidean space $\mathbb{R}^n$. Conversely, any compact flat Riemannian manifold is of the form ${}_E\backslash\mathbb{R}^n$, where E is a Bieberbach group. The Bieberbach groups, and therefore also the compact flat Riemannian manifolds, are quite well understood by the three Bieberbach Theorems:

Theorem 1.2 (1^{st} Bieberbach Theorem - 1910 ([8])). *Let E be a n-dimensional crystallographic group, then $\Gamma = E \cap \mathbb{R}^n$ is a lattice (i.e. a uniform discrete subgroup) of $\mathbb{R}^n$ and E/Γ is finite.*

It follows now that for an n-dimensional Bieberbach group E, $\Gamma = E \cap \mathbb{R}^n \cong \mathbb{Z}^n$ and that the quotient space ${}_\Gamma\backslash\mathbb{R}^n$ is a n-dimensional torus, while the total quotient space ${}_E\backslash\mathbb{R}^n = {}_{(E/\Gamma)}\backslash({}_\Gamma\backslash\mathbb{R}^n)$ is finitely covered by this torus ${}_{\mathbb{Z}^n}\backslash\mathbb{R}^n$.

The second Bieberbach theorem states that a compact flat Riemannian manifold is determined, up to affine equivalence, by its fundamental group.

THEOREM 1.3 (2nd Bieberbach Theorem - 1912 ([9])). *Let $\varphi : E \to E'$ be an isomorphism between two n-dimensional crystallographic groups, then there exists an element $\alpha \in \mathrm{Aff}(\mathbb{R}^n)$ such that*

$$\forall e \in E: \ \varphi(e) = \alpha e \alpha^{-1}.$$

In the last Bieberbach theorem, it is shown that there are, up to affine equivalence, only finitely many compact flat manifolds in each dimension. (This solves Hilbert's 18th problem.)

THEOREM 1.4 (3rd Bieberbach Theorem - 1910 ([8], [32])). *Up to isomorphism, there are only finitely many n-dimensional crystallographic groups for each n.*

Otherwise stated, a given torus T^n only covers finitely many compact flat Riemannian manifolds.

An explicit classification of all crystallographic (resp. Bieberbach) groups in dimensions ≤ 3 was achieved before 1900. For dimension 4 the isomorphism-type classification was one of the milestones of what is today called "computational group theory" (see [11]). Today this interesting database of groups is still expanding and a lot of information (e.g. a classification of Bieberbach groups up to dimension 6) can be interactively consulted on-line at URL

http://wwwb.math.rwth-aachen.de/carat/.

These three theorems show that flat manifolds are completely determined by the algebraic nature of their fundamental group (i.e. the corresponding Bieberbach group). H. Zassenhaus somehow completed these three Bieberbach theorems by given a complete algebraic characterization of the crystallographic groups.

THEOREM 1.5 (Algebraic characterization - 1948 ([63])). *Let E be a n-dimensional crystallographic group, then $\Gamma = E \cap \mathbb{R}^n$ is the unique normal, maximal abelian subgroup of E. Conversely, if E is a (abstract) group, containing a normal subgroup $\Gamma \cong \mathbb{Z}^n$ which is maximal abelian and of finite index in E, then there exists an injective morphism $\varphi : E \to \mathrm{Isom}(\mathbb{R}^n)$, such that $\varphi(E)$ is a crystallographic group.*

1.2. The nilpotent case: infra-nilmanifolds

The crystallographic group situation generalizes nicely to the case where G is nilpotent. Of crucial importance for this are the very good properties of lattices in connected, simply connected nilpotent Lie groups G. Take G such a Lie group and fix a maximal compact subgroup C of $\mathrm{Aut}\,(G)$.

DEFINITION 1.6. A discrete and cocompact subgroup $E \subseteq G \rtimes C$ is called an *almost-crystallographic group.* If moreover, E is torsion free, then E is called *almost-Bieberbach.*

Infra-nilmanifolds are precisely the quotient manifolds ${}_E\backslash G$, with E almost-Bieberbach; in case $E \subseteq G$, then ${}_E\backslash G$ is called a nilmanifold. Hence, nilmanifolds ${}_E\backslash G$ are the analogues of the tori.

During the past 40 years, the three (classical) Bieberbach theorems have been successfully generalized to the situation of infra-nilmanifolds.

THEOREM 1.7 (1-st Generalized Bieberbach Theorem-1960([2])). *If $E \subseteq G \rtimes C$ is an almost-crystallographic group, then $N = E \cap G$ is a lattice of G and E/Γ is finite.*

Being a lattice of a simply connected, connected nilpotent Lie group, N is finitely generated and torsion free. It is called the "translational" part of E. In fact, it is known that the group G is uniquely determined by N; G is called the Mal'cev completion of N. It follows also that any infra-nilmanifold M is of the form

$$M = {}_E\backslash G = {}_{(E/N)}\backslash({}_N\backslash G)$$

from which we see that M is finitely covered by the nilmanifold ${}_N\backslash G$.

The second Bieberbach theorem has also a straightforward generalization.

THEOREM 1.8 (2^{nd} Generalized Bieberbach Theorem - 1985 ([49])). *If $\varphi : E \to E'$ is an isomorphism between two almost-crystallographic groups, then E and E' are defined on the same simply connected, connected nilpotent Lie group G and, moreover, there exists an element $\alpha \in \mathrm{Aff}(G)$ such that*

$$\forall e \in E : \ \varphi(e) = \alpha e \alpha^{-1}.$$

Finding a correct statement to generalize the third theorem took considerably more effort. To make clear why, let us take a look at an instructive example. Take G the three dimensional Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \ \| \ x, y, z \in \mathbb{R} \right\}.$$

For each integer $k > 0$, there is a uniform lattice N_k in G, being the subgroup generated by

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 1 & 0 & \frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

These lattices define nilmanifolds $M_k = {}_{N_k}\backslash G$, which are pairwise non-homeomorphic, as can be seen from their first homology groups:

$$H_1(M_k) = H_1(N_k) = N_k/[N_k, N_k] = \mathbb{Z}^2 \oplus \mathbb{Z}_k.$$

So, already in dimension 3, there are infinitely many nilmanifolds, and, a fortiori, infinitely many infra-nilmanifolds. Therefore, claiming a straightforward statement like "in a given dimension n there are only finitely many almost-crystallographic groups (infra-nilmanifolds)" clearly fails. Even the analoguous statement saying that any torus only covers finitely many flat Riemannian manifolds, cannot be generalized directly. For this, observe that the groups N_k contain N_1 as a normal subgroup of index k, implying that M_1 is a k-fold covering of M_k; so there are already infinitely many nilmanifolds which are covered by M_1. It was K.B. Lee who introduced the notion of an essential covering to overcome these problems. A covering $M \to M'$ is said to be essential, if no element of the deck transformation group is homotopic to the identity. A covering $p : M = {}_N\backslash G \to M' = {}_E\backslash G$ of an infra-nilmanifold by a nilmanifold induces a map

$$p_* : \Pi_1(M) = N \to \Pi_1(M') = E.$$

It turns out that the covering is essential if and only if $p_*(N) = E \cap G$, i.e. N and E have the same translational part. One can now state a correct formulation to generalize the third Bieberbach theorem:

THEOREM 1.9 (3^{rd} Generalized Bieberbach Theorem - 1988 ([46])). *There are, up to affine equivalence, only finitely many infra-nilmanifolds, which are essentially covered by a fixed nilmanifold.*

This theorem was proved in the group theoretic language relying heavily upon the algebraic characterization of the almost-crystallographic groups (see also ([24])).

THEOREM 1.10 (Algebraic characterization - 1988 ([46])). *Let $E = G \rtimes C$ be an almost-crystallographic group, then $N = E \cap G$ is the unique normal, maximal nilpotent subgroup of E. Conversely, if E is a (abstract) group, containing a finitely generated, torsion free normal subgroup N which is maximal nilpotent and of finite index in E, then there exists an injective morphism $\varphi : E \to G \rtimes C$, where G is the Mal'cev completion of N, such that $\varphi(E)$ is an almost-crystallographic group.*

Inspired by these results, one defines a notion of essential group extensions.

DEFINITION 1.11. Let N be a finitely generated, torsion free nilpotent group and F a finite group. A group extension

$$1 \to N \to E \to F \to 1$$

is said to be essential if N is maximal nilpotent in E.

So, the algebraic characterization of almost-crystallographic groups says that any almost-crystallographic group E determines an essential extension

$$1 \to N = G \cap E \to E \to F = E/N \to 1,$$

and conversely, that any essential extension determines an almost-crystallographic group.

It is also true that a finite covering of an infra-nilmanifold by a nilmanifold is essential if and only if the induced short exact sequence on the level of the fundamental groups is an essential extension.

Observe that an almost-crystallographic group is a polycyclic-by-finite group and that the translational part N of E coincides with the Fitting subgroup $N = \text{Fitt}(E)$ of E. (The Fitting subgroup of a polycyclic-by-finite group Γ is the unique maximal normal and nilpotent subgroup of Γ).

Let us end this section by pointing out that the adjective "almost" has a geometrical origin, which arose earlier than the algebraic characterization given above. Indeed, the class of infra-nilmanifolds corresponds exactly to what M. Gromov ([36]) referred to as the almost flat manifolds. Recall that a compact Riemannian manifold M with diameter d is said to be ϵ-flat if the Riemannian sectional curvature satisfies $|K|d^2 \leq \epsilon$. It follows from the work of M. Gromov and E. Ruh ([56]) that any compact Riemannian manifold M which is sufficiently flat (where sufficient is determined in terms of the dimension of M) is an infra-nilmanifold. In a somewhat vague way, we can say that the infra-nilmanifolds are the only manifolds admitting an "almost" flat structure.

1.3. What about the solvable case?

There have been several efforts to extend the nice results obtained in the nilpotent case to the case of solvable Lie groups. This however, turns out to be quite disappointing, as even for those classes of solvable Lie groups sharing many properties with the class of nilpotent Lie groups (we think about so-called Lie groups of type (R) and type (E), ([34], [35])), a simple analogue of the first Bieberbach theorem seems not to be available ([27], [47]).

However, very recently, in a well written paper, B. Wilking ([62]) obtained results which can be seen as analogs to each of the three Bieberbach theorems in the solvable case. Their statements require a good deal of preliminary results and therefore the reader is advised to consult Wilking's paper for the mathematical details. Wilking obtains the following main results:

- *On the 2^{nd} Bieberbach theorem:* If $E \subseteq G \rtimes C$ is a lattice in a semidirect product where G is connected, simply connected solvable and C a compact subgroup of $\text{Aut}\,(G)$, then the action of E on G is metrically equivalent to an action of E on a supersolvable Lie group. This action is shown to be determined by E up to an affine diffeomorphism.
- *On the 1^{st} and 3^{rd} Bieberbach theorem:* A lattice $E \subseteq G \rtimes C$ is characterised algebraically as a "polycrystallographic group", i.e. a polycyclic-by-finite group containing no non-trivial finite normal

subgroups. Moreover, any such a polycrystallographic group can be realised as a lattice $E' \subseteq G_1 \rtimes F$, where G_1 is connected, simply connected solvable and F is a finite subgroup of Aut (G_1), whose order is bounded by a constant only depending on the dimension of G_1.

2. Crystallographic to polynomial crystallographic: *the representation-direction*

For a long time, it was also hoped that the infra-nilmanifolds could be equipped with another type of geometric structure, more precisely with a complete affinely flat structure. Recall that an affinely flat n-dimensional manifold is a manifold M, equipped with an atlas $\mathcal{A} = \{\mu_\alpha : U_\alpha \to \mathbb{R}^n \parallel \alpha \in I\}$ (where $\cup_{\alpha\in I} U_\alpha$ is an open covering of M) for which all transition functions between two overlapping charts are expressed by means of an affine transformation (i.e. an element of $\mathrm{Aff}(\mathbb{R}^n)$).

The simplest way to obtain such an affinely flat manifold M is by considering a subgroup $E \subseteq \mathrm{Aff}(\mathbb{R}^n)$ which acts freely and properly discontinuously on $\mathbb{R}^n$. In this case, the manifold $M = {}_E\backslash\mathbb{R}^n$ has an affinely flat structure. Moreover, in case one requires that the manifold is also complete (i.e. every geodesic $\gamma : [a, b] \to M$ can be extended to a complete geodesic $\gamma : \mathbb{R} \to M$) then the above construction yields all such manifolds.

Moreover, we will be especially interested in the case of compact manifolds.

DEFINITION 2.1. Let E be any group. An *affine structure* on the group E consists of a representation

$$\rho : E \to \mathrm{Aff}(\mathbb{R}^n)$$

letting E act properly discontinuously and cocompactly (i.e. with compact quotient) on $\mathbb{R}^n$ (for some n). A group E admitting an affine structure can also be called an *affine crystallographic group* (ACG).

Observe that, if E is a torsion-free group admitting an affine structure $\rho : E \to \mathrm{Aff}(\mathbb{R}^n)$, then ${}_{\rho(E)}\backslash\mathbb{R}^n$ is a compact and complete affinely flat manifold. Also, and in line with the above terminology, a (classical) crystallographic group could be called a *Euclidean crystallographic group* (ECG).

The main research in the area of affine structures for polycyclic-by-finite groups in general has been concentrated so-far around two main problems, being more or less each others inverse.

The Auslander Conjecture - 1964 ([3]): *This conjecture claims that any group appearing as the fundamental group of a compact and a complete affinely flat manifold is polycyclic-by-finite.*

In fact L. Auslander formulated his "conjecture" as a theorem in an even more general setting. He claimed that the fundamental group of any complete (but not necessarily compact) affinely flat manifold was polycyclic-by-finite. However, the proof given by Auslander was incorrect and in [53], G. Margulis constructed an example of a free non-abelian group acting properly discontinuously and affinely on $\mathbb{R}^3$. In this way he obtained a non-compact, but complete affinely flat manifold with a fundamental group which is not polycyclic-by-finite. On the other hand, Auslander's conjecture (so the compact case) – which is now known to hold up to dimension 6 ([1]) – is still open and is - in general - considered as a very hard problem.

Milnor's Problem - 1977 ([54]): *J. Milnor asked the following question: Is it true that any torsion-free polycyclic-by-finite group appears as the fundamental group of a compact and complete affinely flat manifold. In the present terminology, Milnor is asking whether each torsion-free polycyclic-by-finite group is affine crystallographic.*

2.1. Affine structures for infra-nilmanifolds

From our point of view we are more interested in Milnor's problem, i.e. the existence question. For a long time several people thought that (at least in the virtually nilpotent case, thus for almost-crystallographic groups) Milnor's problem would have a positive answer. This hope somehow relied upon the positive examples known, e.g.

- If N is a finitely generated, torsion free, nilpotent group, such that the Mal'cev completion G of N has a graded Lie algebra $\mathfrak{g}$, then N admits an affine structure.
- In 1974 ([57]), J. Scheuneman showed (using the Lie algebra language) that if N is a finitely generated, torsion free, 3-step nilpotent group, then N admits an affine structure.

- In 1983 ([45]), K.B. Lee could improve Scheuneman's result and showed that all finitely generated virtually 3-step nilpotent groups admit an affine structure.

After Lee's result, several people attempted to solve Milnor's problem by proving it in the general case, or at least for nilpotent groups. Somehow, it was rather unexpectedly that Y. Benoist ([5], [6]) constructed a nilpotent counter-example, i.e. he constructed an example of a 10-step nilpotent group (and of Hirsch length 11) not admitting any affine structure. This example was later generalized to a family of examples ([13]) and to examples of class 9 (and Hirsch length 10) ([12]), so far the counter-examples of smallest nilpotency class known, while it stays very hard to prove the existence of affine structures even for nilpotent groups of class 4 or higher. Anyhow, Milnor's problem is now known to have a negative answer even in the case of almost-crystallographic groups.

From the other side, the existence and explicit construction of affine structures on virtually nilpotent groups of small nilpotency class, and so on low dimensional almost-crystallographic groups, turned out to be fruitful. Indeed, an affine structure gives rise to a matrix representation of the group E, which is very helpful from the computational point. In fact, these representations turned out to be the indispensible tool in the formal computations, which lead us to the classification of all almost-Bieberbach groups in dimensions ≤ 4 ([16]). Besides an affine structure for these almost-Bieberbach groups, also a presentation by generators and relations and some cohomology computations (invariants) were obtained. These matrix representations, together with their presentations are now electronically available as a package (called "aclib") of the program "GAP" [33], and serve as a very interesting database for testing purposes.

2.2. Polynomial structures for infra-nilmanifolds

The negative answer to Milnor's problem immediately asks for a possible alternative to the affine structures, if possible for polycyclic-by-finite groups in general.

While, on the one hand, the group of affine transformations is not big enough to contain all almost crystallographic groups, it was known, on the other hand, that all polycyclic-by-finite groups act properly discontinuously and with compact quotient via C^∞-transformations. Therefore, it was natural to investigate the situation of genuine polynomial transformations.

We write $(\mathrm{P}(\mathbb{R}^n), \circ)$ for the group of polynomial diffeomorphisms of $\mathbb{R}^n$. To make clear, a map $\mu : \mathbb{R}^n \to \mathbb{R}^n$ belongs to $\mathrm{P}(\mathbb{R}^n)$ if and only if μ is a bijection and both μ and μ^{-1} are polynomial maps (in the usual coordinates of $\mathbb{R}^n$) and hence continuous. E.g. for each polynomial $q(y) \in \mathbb{R}[y]$, the map

$$\mu : \mathbb{R}^2 \to \mathbb{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + q(y) \\ y \end{pmatrix}$$

belongs to $\mathrm{P}(\mathbb{R}^2)$. From this example it follows immediately that, for $n \geq 2$, $\mathrm{P}(\mathbb{R}^n)$ is infinite dimensional. (One should observe that, for $n = 1$, $\mathrm{P}(\mathbb{R}) = \mathrm{Aff}(\mathbb{R})$.) Hence, $\mathrm{P}(\mathbb{R}^n)$ might contain substantially more groups acting properly discontinuously and with compact quotient, then $\mathrm{Aff}(\mathbb{R}^n)$ does. On the other hand, the algebraic structure of $\mathrm{P}(\mathbb{R}^n)$ is extremely difficult to grasp. In fact, only for dimension 2, one knows a satisfactory description of this algebraic structure, while the general case remains a mystery (see [43] and its references).

It is now natural to introduce the following definition.

DEFINITION 2.2. Let E be any group. A *polynomial structure* on the group E consists of a representation

$$\rho : E \to \mathrm{P}(\mathbb{R}^n)$$

letting E act properly discontinuously and with compact quotient on $\mathbb{R}^n$ (for some n). Such a group E will also be called a *polynomial crystallographic group* (PCG).

If the (total) degrees of the maps $\rho(e)$ (for $e \in E$) of a polynomial structure are bounded by some constant d, we speak about a polynomial structure of bounded degree or of degree $\leq d$.

In our search for polynomial structures, for polycyclic-by-finite groups and almost-crystallographic groups in particular, we explored two possible routes.

The *first* method was a trial to construct a polynomial structure by iteration. This was motivated by the approach of K.B. Lee used in his construction of affine structures on virtually 3-step nilpotent groups. Moreover, both Lee's construction (in the affine setting) and the construction of polynomial structures we are going to describe are applications of the more general theory of the so-called Seifert fibre space construction initiated by P. E. Conner and F. Raymond ([15]) and studied in a more general setting by K. B. Lee, F. Raymond and others ([42], [48]).

Let us recall the algebraic setup of this construction for polycyclic-by-finite groups. From the theory of polycyclic-by-finite groups ([58]) we know that any such a group Γ admits a filtration of characteristic subgroups Γ_i $(0 \leq i \leq c+1)$

$$\Gamma_* : \ \Gamma_0 = 1 \subset \Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_{c-1} \subset \Gamma_c \subset \Gamma_{c+1} = \Gamma$$

for which

$$\Gamma_i/\Gamma_{i-1} \cong \mathbb{Z}^{k_i} \text{ for } 1 \leq i \leq c \text{ and some } k_i \in \mathbb{N}_0 \text{ and } \Gamma/\Gamma_c \text{ is finite.}$$

Let us call such a filtration of Γ a *torsion free filtration.*

If E is an almost-crystallographic group, with Fitting subgroup N, we can e.g. take the following torsion free filtration

$$\begin{aligned} E_* : \ & Z_0(N) = 1 \subseteq Z_1(N) \\ & \qquad = Z(N) \subseteq Z_2(N) \subseteq \cdots Z_i(N) \subseteq \cdots \subseteq Z_c(N) \subseteq E, \end{aligned}$$

where $Z_i(N)$ is the i-th term in the upper central series of N. Working with the lower central series is not possible in general, since the filtration quotients there can have torsion. However, a slight adjustment of this filtration can be used ([23]).

Write K for the Hirsch length of Γ. Often, we will also use $K_i = k_i + k_{i+1} + \cdots + k_c$ and $K_{c+1} = 0$. It follows that $K = K_1$.

Having chosen and fixed a torsion free filtration Γ_*, we also fix a set of generators

$$\{a_{1,1}, a_{1,2}, \ldots, a_{1,k_1}, a_{2,1}, \ldots, a_{c,k_c}, \alpha_1, \ldots, \alpha_r\}$$

in a way which is compatible with Γ_*; i.e. we require that

$$\forall i \in \{1, 2, \ldots, c\} : \ a_{1,1}, a_{1,2}, \ldots, a_{i,k_i} \text{ generates } \Gamma_i.$$

The goal of the Seifert construction is to construct an action of a polycyclic-by-finite group Γ on some space $\mathbb{R}^n$ in an iterative way by lifting, step-by-step, the action constructed on Γ/Γ_{i+1} to an action of Γ/Γ_i (starting with $i = c$ and ending with $i = 0$, for which $\Gamma/\Gamma_0 = \Gamma$). The general mathematical objects to be introduced in this framework are the following: the real vector space (with addition defined pointwise)

$$\mathcal{M}(\mathbb{R}^K, \mathbb{R}^k) = \{\text{continuous maps } \lambda : \mathbb{R}^K \to \mathbb{R}^k\}$$

and the group (composition being the group law)

$$\mathcal{H}(\mathbb{R}^K) = \{\text{homeomorphisms } h : \mathbb{R}^K \to \mathbb{R}^K\}.$$

$\mathcal{M}(\mathbb{R}^K, \mathbb{R}^k)$ can be made into a $\mathrm{Gl}(k) \times \mathcal{H}(\mathbb{R}^K)$-module, by defining

$$^{(g,h)}\lambda = g \circ \lambda \circ h^{-1}, \ \forall \lambda \in \mathcal{M}(\mathbb{R}^K, \mathbb{R}^k), \ \forall g \in \mathrm{Gl}(k), \ \forall h \in \mathcal{H}(\mathbb{R}^K).$$

The resulting semi-direct product group $\mathcal{M}(\mathbb{R}^K, \mathbb{R}^k) \rtimes (\mathrm{Gl}(k) \times \mathcal{H}(\mathbb{R}^K))$ embeds nicely into $\mathcal{H}(\mathbb{R}^{k+K})$ if one defines

$$(2) \qquad (\lambda, g, h) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g(x) + \lambda h(y) \\ h(y) \end{pmatrix},$$

$$\forall x \in \mathbb{R}^k, \ \forall y \in \mathbb{R}^K, \ \forall \lambda \in \mathcal{M}(\mathbb{R}^K, \mathbb{R}^k), \ \forall g \in \mathrm{Gl}(k), \ \forall h \in \mathcal{H}(\mathbb{R}^K).$$

We are ready for the

DEFINITION 2.3. Let Γ be a polycyclic-by-finite group with a torsion free filtration Γ_*. A representation $\rho = \rho_0 : \Gamma \to \mathcal{H}(\mathbb{R}^K)$ will be called of *canonical type* w.r.t. Γ_* if and only if it induces a sequence of representations:

$$\rho_i : \Gamma/\Gamma_i \to \mathcal{H}(\mathbb{R}^{K_{i+1}}), \ \ (1 \leq i \leq c)$$

such that the following diagram commutes for all i:

(3)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^{k_i} \cong \Gamma_i/\Gamma_{i-1} & \longrightarrow & \Gamma/\Gamma_{i-1} & \longrightarrow & \Gamma/\Gamma_i & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \rho_{i-1}} & & \downarrow{\scriptstyle \psi_i \times \rho_i} & & \\ 1 & \longrightarrow & \mathcal{M}(\mathbb{R}^{K_{i+1}}, \mathbb{R}^{k_i}) & \longrightarrow & \mathcal{M} \rtimes (G \times \mathcal{H}) & \longrightarrow & \mathrm{Aut}\,(\mathbb{Z}^{k_i}) \times \mathcal{H}(\mathbb{R}^{K_{i+1}}) & \longrightarrow & 1 \end{array}$$

where

- $\mathcal{M} \rtimes (G \times \mathcal{H})$ is a shorthand for
$$\mathcal{M}(\mathbb{R}^{K_{i+1}}, \mathbb{R}^{k_i}) \rtimes (\mathrm{Aut}\,(\mathbb{Z}^{k_i}) \times \mathcal{H}(\mathbb{R}^{K_{i+1}})),$$
- $\forall z \in \mathbb{Z}^{k_i} : \ \ j(z) : \mathbb{R}^{K_{i+1}} \to \mathbb{R}^{k_i} : x \mapsto z$, and
- $\psi_i : \Gamma/\Gamma_i \to \mathrm{Aut}\,(\mathbb{Z}^{k_i})$ is the action of Γ/Γ_i induced on $\mathbb{Z}^{k_i}$ by conjugation in Γ/Γ_{i-1}.

Now, to establish a canonical type representation, we start with the trivial one $\rho_c : \Gamma/\Gamma_c \to \mathcal{H}(\mathbb{R}^0) = 1$, and we try to create ρ_i for i going from c down to 0. So at each i, we are facing an existence question: "*is it possible to lift the given action*" and one can also regard the uniqueness question "*in how many essentially different ways can we lift this action?*". The answers to both questions are stated in terms of group cohomology. Indeed, assume that the action ρ_i was constructed in this iterative way. One can check that this action lifts in the required way (i.e as indicated in diagram (3)) to the extension

$$1 \longrightarrow \mathbb{Z}^{k_i} \longrightarrow \Gamma/\Gamma_{i-1} \longrightarrow \Gamma/\Gamma_i \longrightarrow 1$$

if and only if the induced extension

$$(4) \qquad 1 \longrightarrow \mathcal{M}(\mathbb{R}^{K_{i+1}}, \mathbb{R}^{k_i}) \longrightarrow \Gamma/\Gamma_{i-1} \longrightarrow \Gamma/\Gamma_i \longrightarrow 1$$

is a split extension. This is always the case when $H^2(\Gamma/\Gamma_i,\ \mathcal{M}\ (\mathbb{R}^{K_{i+1}}, \mathbb{R}^{k_i})) = 0$. If moreover, $H^1(\Gamma/\Gamma_i,\ \mathcal{M}\ (\mathbb{R}^{K_{i+1}},\ \mathbb{R}^{k_i})) = 0$, then the splitting of the extension is unique (up to conjugation with an element of $\mathcal{M}(\mathbb{R}^{K_{i+1}}, \mathbb{R}^{k_i})$). In [14], it was shown that $H^j(\Gamma/\Gamma_i, \mathcal{M}(\mathbb{R}^{K_{i+1}}, \mathbb{R}^{k_i})) = 0$, for all $j > 0$, a result proving the major part of the following theorem

THEOREM 2.4. *Let Γ be a polycyclic-by-finite group with a fixed torsion free filtration Γ_*. Then, there exists a canonical type (w.r.t. Γ_*) representation $\rho : \Gamma \to \mathcal{H}(\mathbb{R}^K)$, which is unique up to conjugation inside $\mathcal{H}(\mathbb{R}^K)$. Moreover Γ acts properly discontinuous on $\mathbb{R}^K$, via ρ and the quotient space $_\Gamma\backslash\mathbb{R}^K$ is compact.*

Although the above theorem states that the general framework is always satisfying and although a similar result holds in the differentiable setting, we clearly want much nicer actions than just continuous or differentiable ones. Therefore, it is necessary to restrict the basic building blocks, to smaller ones, with higher computational or structural interest. Of course, this should be done in such a way that the iterative approach can be maintained. A general scheme for this uses the following tools: write $\mathcal{S}(\mathbb{R}^K, \mathbb{R}^k)$ for a subspace of $\mathcal{M}(\mathbb{R}^K, \mathbb{R}^k)$, containing the space of constant mappings $\mathbb{R}^k$ and restrict $\mathcal{H}(\mathbb{R}^K)$ to a subgroup $\mathcal{S}H(\mathbb{R}^K)$. It is needed that $\mathcal{S}(\mathbb{R}^K, \mathbb{R}^k)$ is a $(\mathrm{Gl}(k) \times \mathcal{S}H(\mathbb{R}^K))$-submodule and that there still is an embedding (induced by the formula (2)) $\mathcal{S}(\mathbb{R}^K, \mathbb{R}^k) \rtimes (\mathrm{Gl}(k) \times \mathcal{S}H(\mathbb{R}^K)) \hookrightarrow \mathcal{S}H(\mathbb{R}^{k+K})$.

In [45], K. B. Lee uses such a restriction process in order to obtain affine structures. Restricting $\mathcal{M}(\mathbb{R}^K, \mathbb{R}^k)$ to $\mathrm{Aff}(\mathbb{R}^K, \mathbb{R}^k)$ (the vector space of affine mappings) and $\mathcal{H}(\mathbb{R}^K)$ to $\mathrm{Aff}(\mathbb{R}^K)$, he succeeded to prove that any virtually 3-step nilpotent group admits an affine structure. It is, however, no longer true that all cohomology groups $H^j(\Gamma/\Gamma_i, \mathrm{Aff}(\mathbb{R}^{K_{i+1}}, \mathbb{R}^{k_i})) = 0$ for all $j > 0$, and therefore, this setup cannot be used to prove the existence of affine structures on all polycyclic-by-finite groups.

We studied a different restriction process, which finally led to the creating of polynomial structures. Let us write $\mathrm{P}(\mathbb{R}^K,\mathbb{R}^k)$ to refer to the vector space of polynomial mappings $p:\mathbb{R}^K\to\mathbb{R}^k$. Now take $\mathcal{S}(\mathbb{R}^K,\mathbb{R}^k)=\mathrm{P}(\mathbb{R}^K,\mathbb{R}^k)$ and $\mathcal{SH}(\mathbb{R}^K)=\mathrm{P}(\mathbb{R}^K)$. It is obvious that $\mathrm{P}(\mathbb{R}^K,\mathbb{R}^k)$ is a $\mathrm{Gl}(k)\times\mathrm{P}(\mathbb{R}^K)$-module and that the resulting semi-direct product group

$$\mathrm{P}(\mathbb{R}^K,\mathbb{R}^k)\rtimes(\mathrm{Gl}(k)\times\mathrm{P}(\mathbb{R}^K))\subseteq\mathrm{P}(\mathbb{R}^{K+k})$$

where the embedding is the restriction of (2), so

$$\forall(p,g,h)\in\mathrm{P}(\mathbb{R}^K,\mathbb{R}^k)\rtimes(\mathrm{Gl}(k)\times\mathrm{P}(\mathbb{R}^K)),\forall\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^{k+K},$$

$$(5)\qquad {}^{(p,g,h)}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}gx+p(h(y))\\h(y)\end{pmatrix}.$$

In [20], we obtained a cohomology vanishing theorem stating that, for all $j>0$, $H^j(\Gamma/\Gamma_i,\mathrm{P}(\mathbb{R}^{K_{i+1}},\mathbb{R}^{k_i}))=0$. This result implies the major part of the following theorem.

THEOREM 2.5. *Let Γ be a polycyclic-by-finite group, and assume that Γ_* is a torsion free filtration of Γ, then Γ admits a canonical type polynomial representation $\rho:\Gamma\to\mathrm{P}(\mathbb{R}^K)$. Moreover,*

1. *two such canonical type representations, w.r.t. the same filtration Γ_*, are conjugated to each other by an element of $\mathrm{P}(\mathbb{R}^K)$.*
2. *any canonical type polynomial representation is of bounded (but unknown) degree.*

This theorem gives a general and positive answer to the (polynomial) analogue of Milnor's question; in other words, any virtually polycyclic group is a PCG. Recently, Y.Benoist and one of the authors of this paper obtained a more general uniqueness result ([7]) concerning these polynomial structures. They proved that any two polynomial structures (not necessarily of canonical type) of bounded degree on a polycyclic-by-finite group are polynomially conjugated, thus generalizing a result which was already known for affine structures ([31]). Moreover, there is a strong feeling that any polynomial structure is of bounded degree; nevertheless examples of non bounded degree, properly discontinously, but not cocompactly acting polynomial representations are known.

For a *second* approach to polynomial structures we used a more direct method based on Mal'cev coordinates. Although this method is only applicable for almost-crystallographic groups (and therefore also for all finitely generated virtually nilpotent groups), we obtain a polynomial structure of bounded degree in which the bound is known and reasonable small. In fact, we were able to prove the following theorem ([21]):

THEOREM 2.6. *Let G be a connected and simply connected nilpotent group of nilpotency class c. Then there exists a polynomial action $\rho : \mathrm{Aff}(G) \to \mathrm{P}(\mathbb{R}^n)$ letting G act simply transitively on $\mathbb{R}^n$ and such that the action of each element of $\mathrm{Aff}(G)$ is of degree $\leq \max(1, c-1)$.*

This theorem immediately implies that any almost-crystallographic group E admits a polynomial structure of degree $\leq \max(1, c-1)$, where c is the nilpotency class of $\mathrm{Fitt}(E)$. This polynomial structure is just the restriction of the action of $\mathrm{Aff}(G)$ to E (where G is the Mal'cev completion of $\mathrm{Fitt}(E)$ and $E \subseteq \mathrm{Aff}(G)$).

2.3. Weighted polynomial structures: a valuable alternative to affine structures

Unfortunately, the group of polynomial diffeomorphisms $\mathrm{P}(\mathbb{R}^n)$ is infinite dimensional and still hiding almost every algebraic information. Recently ([22]) we were able to introduce finite dimensional subgroups of $\mathrm{P}(\mathbb{R}^n)$ which contain the polynomial structures obtained above. This is achieved by introducing a concept of weight for a polynomial (diffeomorphism); "weight" behaves in some sense like the usual degree, but some variables contribute more to it than others. Let us describe these very interesting groups using an approach which is different from the one followed in ([22]) (actually more straight forward and better matching the iterative set up of the previous section).

Let $K > 0$ be any positive integer and fix a finite sequence of integers $\kappa = (k_1, k_2, \ldots, k_n)$ with $k_i > 0$ $(i = 1, 2, \ldots, n)$ and such that $\sum_{i=1}^{n} k_i = K$. Also, choose any another n-tuple of positive integers $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$. Consider the ring

$$R = \mathbb{R}[x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}, x_{2,1}, x_{2,2}, \ldots, x_{2,k_2}, x_{3,1}, \ldots, x_{n,k_n}]$$

of real polynomials in the K variables $x_{i,j}$ $(1 \leq i \leq n$ and $1 \leq j \leq k_i)$. A typical element of this ring will be written as

$$p(x_{n,1}, x_{n,2}, \ldots, x_{n,k_n}, x_{n-1,1}, \ldots, x_{2,k_2}, x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}).$$

This particular numbering of the variables will be fruitful in the sequel. For the moment, we will write $\vec{x}$ for short to denote $(x_{n,1}, \ldots, x_{2,k_2}, x_{1,1}, \ldots, x_{1,k_1})$.

DEFINITION 2.7. Let κ and ω be as before and fixed, and take R as above. The *(κ,ω)-weight of a monomial* $x_{1,1}^{\alpha_{1,1}}x_{1,2}^{\alpha_{1,2}}\dots x_{n,k_n}^{\alpha_{n,k_n}}$ in R is defined as

$$\deg_{\kappa,\omega}(x_{1,1}^{\alpha_{1,1}}x_{1,2}^{\alpha_{1,2}}\dots x_{n,k_n}^{\alpha_{n,k_n}})=\sum_{i=1}^{n}\sum_{j=1}^{k_i}\omega_i\alpha_{i,j}.$$

More general, the *(κ,ω)-weight of a polynomial* $p(x_{n,1},\dots,x_{1,k_1})\in R$ is defined as the maximal weight of its terms. As in the case of the degree of a polynomial, we say that the (κ,ω)-weight of the zero polynomial is equal to $-\infty$. We denote the (κ,ω)-weight of the polynomial $p(\vec{x})$ by $\deg_{\kappa,\omega}(p(\vec{x}))$.

LEMMA 2.8. *Let κ, ω be fixed and R be as before. Assume that $p(\vec{x}),q(\vec{x})\in R$. It follows easily that*

1. $\deg p(\vec{x})\le\deg_{\kappa,\omega}(p(\vec{x}))$.
2. $\deg_{\kappa,\omega}(p(\vec{x}))=\deg p(y_{n,1}^{\omega_n},\dots,y_{2,1}^{\omega_2},\dots,y_{2,k_2}^{\omega_2},y_{1,1}^{\omega_1},y_{1,2}^{\omega_1},\dots,y_{1,k_1}^{\omega_1})$, *where the right hand side is seen as a polynomial in the K variables $y_{i,j}$ ($1\le i\le n$ and $1\le j\le k_i$).*
3. $\deg_{\kappa,\omega}(p(\vec{x})+q(\vec{x}))\le\mathrm{Max}\{\deg_{\kappa,\omega}(p(\vec{x})),\deg_{\kappa,\omega}(q(\vec{x}))\}$.
4. $\deg_{\kappa,\omega}(p(\vec{x})q(\vec{x}))=\deg_{\kappa,\omega}(p(\vec{x}))+\deg_{\kappa,\omega}(q(\vec{x}))$.

LEMMA 2.9. *Let $x_{1,1}^{\alpha_{1,1}}x_{1,2}^{\alpha_{1,2}}\dots x_{n,k_n}^{\alpha_{n,k_n}}$ be a monomial of (κ,ω)–weight $\le w$ for some w. If $\forall i\in\{1,\dots,n\}$, $\forall j\in\{1,\dots,k_i\}$, the polynomial $p_{i,j}\in\mathbb{R}[x_{1,1},\dots,x_{n,k_n}]$ is of (κ,ω)–weight $\le\omega_i$, then*

$$p_{1,1}^{\alpha_{1,1}}p_{1,2}^{\alpha_{1,2}}\dots p_{n,k_n}^{\alpha_{n,k_n}}$$

is (a polynomial) of (κ,ω)–weight $\le w$.

Proof. This is verified by a repeated application of the properties in Lemma 2.8. Indeed $p_{i,j}^{\alpha_{i,j}}$ can be seen as a product with $\alpha_{i,j}$ terms of (κ,ω)–weight $\le\omega_i$; hence, the total weight of $p_{i,j}^{\alpha_{i,j}}$ is $\le\alpha_{i,j}\omega_i$. Using the same argument, we find

$$\deg_{\kappa,\omega}(p_{1,1}^{\alpha_{1,1}}p_{1,2}^{\alpha_{1,2}}\dots p_{n,k_n}^{\alpha_{n,k_n}})\le\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{i,j}\omega_i\le w.\qquad\square$$

Now, let's return to diffeomorphisms. Any element $p\in\mathrm{P}(\mathbb{R}^{k_1+k_2+\cdots+k_n},\mathbb{R}^k)$, can be considered as consisting of k polynomials in R. In the sequel, when we speak of the weight of a polynomial mapping p, we will mean the maximum of the weights of the polynomials describing this map.

DEFINITION 2.10. Let $\kappa = (k_1, k_2, \ldots, k_n)$ be as above. The *blocked Jonquière group, J, of type $(k_n, k_{n-1}, \ldots, k_2, k_1)$* consists of all polynomial diffeomorphisms p of $\mathbb{R}^{k_1+k_2+\cdots+k_n}$ which are of the form:

(6)

$$p(\vec{x}) = \begin{pmatrix} p_n(x_{n,1}, x_{n,2}, \ldots, x_{n,k_n}, x_{n-1,1}, \ldots, x_{2,k_2}, x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}) \\ p_{n-1}(x_{n-1,1}, x_{n-1,2}, \ldots, x_{n-1,k_{n-1}}, \ldots, x_{2,k_2}, x_{1,1}, \ldots, x_{1,k_1}) \\ \vdots \\ p_2(x_{2,1}, x_{2,2}, \ldots, x_{2,k_2}, x_{1,1}, \ldots, x_{1,k_1}) \\ p_1(x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}) \end{pmatrix}$$

where, for each i, the i-th last block is given by

$$p_i(x_{i,1}, x_{i,2}, \ldots, x_{1,k_1}) = A_i \begin{pmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,k_i} \end{pmatrix} + q_i(x_{i-1,1}, \ldots, x_{1,1}, \ldots, x_{1,k_1})$$

with $A_i \in \mathrm{Gl}(k_i, \mathbb{R})$ and $q_i \in \mathrm{P}(\mathbb{R}^{k_1+k_2+\cdots+k_{i-1}}, \mathbb{R}^{k_i})$.

The iterative set-up described earlier and in particular the existence of canonical type polynomial structures (Theorem 2.5) together with a close inspection of the iterated embedding explained in (5), should convince the reader that this blocked Joncquière group is exactly the group of polynomial diffeomorphisms "containing" the canonical type structures. In fact, the iterative built up can also be seen even more explicitly as follows.

PROPOSITION 2.11. *Let J be the blocked Jonquière group of type $(k_n, \ldots, k_1)$, then J can be seen as the semi-direct product group*

$$J = \mathrm{P}(\mathbb{R}^{k_1+k_2+\cdots+k_{n-1}}, \mathbb{R}^{k_n}) \rtimes (\mathrm{Gl}(k_n, \mathbb{R}) \times J'),$$

where J' is the blocked Jonquière group of type $(k_{n-1}, \ldots, k_2, k_1)$.

Proof. This isomorphism is obtained by mapping the polynomial transformation $p \in J$ of the form (6) to the triple $(q_n \circ \bar{p}^{-1}, A_n, \bar{p})$, where $\bar{p}$ is the polynomial diffeomorphism of $\mathbb{R}^{k_1+k_2+\cdots+k_{n-1}}$ given by (7)

$$\bar{p}\begin{pmatrix} x_{n-1,1} \\ \vdots \\ x_{1,1} \\ \vdots \\ x_{1,k_1} \end{pmatrix} = \begin{pmatrix} p_{n-1}(x_{n-1,1}, x_{n-1,2}, \ldots, x_{n-1,k_{n-1}}, \ldots, x_{2,k_2}, x_{1,1}, \ldots, x_{1,k_1}) \\ \vdots \\ p_2(x_{2,1}, x_{2,2}, \ldots, x_{2,k_2}, x_{1,1}, \ldots, x_{1,k_1}) \\ p_1(x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}) \end{pmatrix}. \qquad \square$$

We can now introduce

DEFINITION 2.12. Let n be a positive integer; fix $\kappa = (k_1, k_2, \ldots, k_n)$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, two n-tuples of positive integers and write $K = \sum_i k_i$. We define $\mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\cdots+k_n})$ to be the subset of the blocked Jonquière group of type $(k_n, \ldots, k_2, k_1)$ consisting of those transformations p of $\mathbb{R}^K$ of the form (6) where for each $i \in \{1, 2, \ldots, n\}$, the polynomial $p_i(x_{i,1}, x_{i,2}, \ldots, x_{1,k_1})$ is of (κ, ω)–weight ω_i. (Obviously, this is equivalent to saying that the polynomial q_i is of (κ, ω)–weight $\leq \omega_i$.)

Note that, to be precise, we can consider the (κ, ω)–weight of the k_i polynomials in $\mathbb{R}[x_{1,1}, x_{1,2}, \ldots, x_{i,k_i}]$ describing p_i, using the natural embedding $\mathbb{R}[x_{1,1}, x_{1,2}, \ldots, x_{i,k_i}] \hookrightarrow \mathbb{R}[x_{1,1}, \ldots, x_{n,k_n}]$.

Let us illustrate this definition by means of two examples.

EXAMPLE 2.13. Take $\kappa = (1, 1, 1)$ and $\omega = (1, 2, 1)$. Then $\mathrm{P}_{\kappa,\omega}(\mathbb{R}^3)$ consists of the diffeomorphisms of the form

$$p\begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} gx_3 + hx_1 + i \\ cx_2 + dx_1^2 + ex_1 + f \\ ax_1 + b \end{pmatrix}, \tag{8}$$

with $a, b, \ldots, i \in \mathbb{R}$ and a, c and g non-zero.

EXAMPLE 2.14. When $\kappa = (1,2,1)$ and $\omega = (1,2,3)$, then $\mathrm{P}_{\kappa,\omega}(\mathbb{R}^4)$ consists of all diffeomorphisms of the form

(9)
$$p\begin{pmatrix} x_3 \\ x_{2,1} \\ x_{2,2} \\ x_1 \end{pmatrix} = \begin{pmatrix} gx_3 + hx_1x_{2,1} + ix_1x_{2,2} + jx_1^3 + kx_{2,1} + \ell x_{2,2} + mx_1^2 + nx_1 + o \\ \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}\begin{pmatrix} x_{2,1} \\ x_{2,2} \end{pmatrix} + \begin{pmatrix} d_1x_1^2 + e_1x_1 + f_1 \\ d_2x_1^2 + e_2x_1 + f_2 \end{pmatrix} \\ ax_1 + b \end{pmatrix},$$

with $a, b, c_{1,1}, \ldots, m, n, o \in \mathbb{R}$, a and g non-zero and matrix $\begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$ invertible.

REMARK 2.15. Fix $\kappa = (k_1, k_2, \ldots, k_n)$. It is not too hard to see that, for any $\lambda \in \mathbb{N}_0$ $\mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\cdots+k_n})$ equals $\mathrm{P}_{\kappa,\lambda\omega}(\mathbb{R}^{k_1+k_2+\cdots+k_n})$ $(\lambda\omega = (\lambda\omega_1, \ldots, \lambda\omega_n))$. Hence, we can always assume that the sequence ω consists of relatively prime integers. In particular, if $n = 1$, for any ω, $\mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1}) = \mathrm{Aff}(\mathbb{R}^{k_1})$. Also, for any n and any κ, and for $\omega = (1, 1, \ldots 1)$, observe that $\mathrm{P}_{\kappa,\omega}(\mathbb{R}^n) \subseteq \mathrm{Aff}(\mathbb{R}^n)$.

A very important feature is the following

THEOREM 2.16. *$\mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\cdots+k_n})$ is a subgroup of the blocked Jonquière group of type $(k_n, \ldots, k_2, k_1)$. It follows that there is an iterative way of building up $\mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\cdots+k_n})$ as a semi-direct product group*

$$\mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\cdots+k_{n-1}}, \mathbb{R}^{k_n}) \rtimes \left(\mathrm{Gl}(k_n, \mathbb{R}) \times \mathrm{P}_{\bar{\kappa},\bar{\omega}}(\mathbb{R}^{k_1+k_2+\cdots+k_{n-1}})\right),$$

where

1. $\bar{\kappa} = (k_1, k_2, \ldots, k_{n-1})$,
2. $\bar{\omega} = (\omega_1, \omega_2, \ldots, \omega_{n-1})$,
3. $\mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\cdots+k_{n-1}}, \mathbb{R}^{k_n})$ *is the subspace of* $\mathrm{P}(\mathbb{R}^{k_1+k_2+\cdots+k_{n-1}}, \mathbb{R}^{k_n})$ *of polynomial maps of* $(\bar{\kappa}, \bar{\omega})$*–weight* $\leq \omega_n$.

Proof. Proceed by induction on n. For $n = 1$, the blocked Jonquière group of type (k_1) coincides with the group of affine motions of $\mathbb{R}^{k_1}$. Hence, also $\mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1})$ coincides with the group of affine motions and the theorem is proved. Remark that there is a semi-direct product decomposition

$$\mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1}) = \mathrm{P}_{\kappa,\omega}(\mathbb{R}^0, \mathbb{R}^{k_1}) \rtimes (\mathrm{Gl}(k_1, \mathbb{R}) \times \mathrm{P}_{\bar{\kappa},\bar{\omega}}(\mathbb{R}^0)) = \mathbb{R}^{k_1} \rtimes \mathrm{Gl}(k_1, \mathbb{R}).$$

Next, assume that $n > 1$ and that the theorem holds for smaller values of n. Consider a general element p of the blocked Jonquière group of type $(k_n, \dots, k_2, k_1)$. Then p can be treated as a triple

$$p = (q_n', A_n, \bar{p}) \in \mathrm{P}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}}, \mathbb{R}^{k_n}) \rtimes \left(\mathrm{Gl}(k_n, \mathbb{R}) \times J'\right)$$

(Here $\bar{p}$ is as in the proof of Proposition 2.11.).
It is clear at once that

$$p \in \mathrm{P}_{\kappa,\omega}(\mathbb{R}^K) \quad \Leftrightarrow \quad \begin{cases} (1) & q_n' \circ \bar{p} \in \mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}}, \mathbb{R}^{k_n}) \\ (2) & \bar{p} \in \mathrm{P}_{\bar{\kappa},\bar{\omega}}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}}). \end{cases}$$

Of key importance for the proof is the following

Observation. *Let*

$$\bar{p} \in \mathrm{P}_{\bar{\kappa},\bar{\omega}}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}}) \text{ and } q_n' \in \mathrm{P}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}}, \mathbb{R}^{k_n}).$$

Then

$$q_n' \circ \bar{p} \in \mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}}, \mathbb{R}^{k_n}) \Leftrightarrow q_n' \in \mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}}, \mathbb{R}^{k_n}).$$

To prove this, let us first assume that $q_n' \in \mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}}, \mathbb{R}^{k_n})$. To compute $q_n' \circ \bar{p}$ we have to replace each appearance of a variable $x_{i,j}$ in q_n' by a polynomial of weight $\leq \omega_i$. It follows from Lemma 2.9 that the weight of $q_n' \circ \bar{p} \leq$ the weight of q_n', which was to be shown.

Conversely, assume that $q_n' \circ \bar{p} \in \mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}}, \mathbb{R}^{k_n})$. By the induction hypothesis, we know that $\bar{p}^{-1} \in \mathrm{P}_{\bar{\kappa},\bar{\omega}}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}})$. Therefore, using the previous argument it follows that $q_n' \circ \bar{p} \circ \bar{p}^{-1} = q_n' \in \mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}}, \mathbb{R}^{k_n})$.

With this observation, it now is clear that

$$p \in \mathrm{P}_{\kappa,\omega}(\mathbb{R}^K) \quad \Leftrightarrow \quad \begin{cases} (1) & q_n' \in \mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}}, \mathbb{R}^{k_n}) \\ (2) & \bar{p} \in \mathrm{P}_{\bar{\kappa},\bar{\omega}}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}}). \end{cases}$$

Hence, as a set we find that

$$\begin{aligned} \mathrm{P}_{\kappa,\omega}(\mathbb{R}^K) &= \mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}}, \mathbb{R}^{k_n}) \\ &\quad \times \left(\mathrm{Gl}(k_n, \mathbb{R}) \times \mathrm{P}_{\bar{\kappa},\bar{\omega}}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}})\right). \end{aligned}$$

To finish the proof, it suffices to show that $\mathrm{P}_{\kappa,\omega}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}}, \mathbb{R}^{k_n})$ is a module over the group $\mathrm{Gl}(k_n, \mathbb{R}) \times \mathrm{P}_{\bar{\kappa},\bar{\omega}}(\mathbb{R}^{k_1+k_2+\dots+k_{n-1}})$. But this, again, follows immediately from the observation we just proved. □

For a given and fixed κ, these weighted subgroups $\mathrm{P}_{\kappa,\omega}(\mathbb{R}^K)$ form a directed system of subgroups (for the inclusion) of $\mathrm{P}(\mathbb{R}^K)$, and their union - over all sets of weights ω - equals the blocked Joncquière group J. In fact, each of the $\mathrm{P}_{\kappa,\omega}(\mathbb{R}^K)$ is a *semi-direct* product group of a nilpotent group and a product of general linear groups.

In [22] we obtained the following key-result.

THEOREM 2.17. *Any finitely generated subgroup of a blocked Jonquière group of type* $(k_n, \ldots, k_1)$ *is contained in a subgroup* $\mathrm{P}_{\kappa,\omega}(\mathbb{R}^K)$, *for some* $\omega = (\omega_1, \ldots, \omega_n)$.

Combining this result with the earlier theorem (2.5) it is now clear that any polycyclic-by-finite group admits a polynomial structure via a representation inside one (and hence several) of these weighted groups $\mathrm{P}_{\kappa,\omega}(\mathbb{R}^K)$. More specific sets of weights in the case of virtually nilpotent groups, are also given in [22].

3. Anosov diffeomorphisms and expanding maps

In this section we shortly discuss two topics where recent results revealed an interesting or maybe rather surprising link with affine and polynomial structures.

3.1. Anosov diffeomorphisms on infra-nilmanifolds

An Anosov diffeomorphism of a compact differentiable manifold M is a diffeomorphism f of M such that, at any point of M the tangent space decomposes as a direct sum of a contracting and an expanding part.

That is, for any Riemannian metric on M, there are constants c and λ ($c > 0$, $0 < \lambda < 1$) such that, at any point $m \in M$, the tangent space TM_m decomposes as a direct sum $E^s \oplus E^u$ where, for all $r \in \mathbb{N}_0$, $\|Tf^r(v)\| \leq c\lambda^r \|v\|$ (for all $v \in E^s$) and $\|Tf^{-r}(w)\| \leq c\lambda^r \|w\|$ (for all $w \in E^u$). As usual in this context, s and u stand for *stable* (the s-dimensional contracting part of TM_m) and *unstable* (the u-dimensional expanding part of TM_m). Such maps are studied in e.g. [30], [40], [52]. For an elementary example, think of the transformation $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, on the 2-torus.

Anosov diffeomorphisms play an important role in dynamical systems as being examples of hyperbolic (and so structurally stable) systems. Standard references are [59] and [61]. Not every smooth compact manifold admits an Anosov diffeomorphism. For example, there are no Anosov diffeomorphisms on the Klein bottle. This highlights the long standing problem to classify the manifolds that carry Anosov diffeomorphisms. At this moment, the only known examples are algebraic ones (tori, flat Riemannian manifolds, nilmanifolds and infra-nilmanifolds). In literature there are some "classical" examples of manifolds admitting an Anosov diffeomorphism, e.g. M a non-toral nilmanifold ([61]), another class of nilmanifolds ([4]), M is a 4-dimensional flat Riemannian manifold which is not a torus ([59]), 6-dimensional (non-flat) infra-nilmanifold which is not a nilmanifold ([59] and [51]).

The usual procedure to construct Anosov diffeomorphisms on (infra-nil)manifolds goes as follows. Let A be an automorphism of the universal cover G of an infra-nilmanifold M and let $dA \in \mathrm{Aut}(\mathfrak{g})$ be the corresponding automorphism on the Lie algebra $\mathfrak{g}$ of G. Such an automorphism A is said to be hyperbolic if and only if dA has no eigenvalues of modulus 1. If A induces an automorphism of M, then this is called a hyperbolic infra-nilmanifold automorphism and it is in fact an Anosov diffeomorphism.

In this algebraic context, any Anosov diffeomorphism is in fact known to be topologically conjugate to a hyperbolic automorphism of the manifold ([30], [52]). It has been conjectured that only (manifolds homeomorphic to) infra-nilmanifolds support Anosov diffeomorphisms. In fact, in [10], it is proved that every Anosov diffeomorphism satisfying a certain spectral condition is topologically conjugate to a hyperbolic infra-nilmanifold automorphism. The general question however is still left open and hence, it is interesting to study the existence and classification of hyperbolic infra-nilmanifold automorphisms.

Of course, the absence of hyperbolic automorphisms on the covering nilmanifold of an infra-nilmanifold M is a natural obstruction to the existence of Anosov diffeomorphisms on M. This certainly highlights the primary importance of classifying the nilmanifolds carrying Anosov diffeomorphisms, or equivalently, describing the finitely generated, torsion-free, nilpotent groups admitting hyperbolic automorphisms.

In [50], this pure algebraic problem has been explored up to rank 6 and the following complete classification was obtained:

THEOREM 3.1. *A nilmanifold of dimension at most six admits an Anosov diffeomorphism if and only if its fundamental group N is one of the following (we refer the reader to [38] for the notion of discriminant):*

- $N \cong \mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z}^4, \mathbb{Z}^5$ *or* $\mathbb{Z}^6$,
- $N \in T(4,2)$ *and the discriminant of* N *is a natural number which is not a perfect square,*
- $N \in T(3,3)$.

Recall that a group N belongs to $T(n,m)$ if and only if N is torsion-free, 2-step nilpotent such that $\Gamma = \sqrt[N]{[N,N]} \cong \mathbb{Z}^m$ and $N/\Gamma \cong \mathbb{Z}^n$, i.e. $0 \to \mathbb{Z}^m \to N \to \mathbb{Z}^n \to 0$ is a short exact sequence.

Of course, we only have to study non-abelian cases since all tori (except the circle) support Anosov diffeomorphisms. Such examples only arise from dimension 6 onwards and hence, the problem is reduced to classifying all torsion–free, nilpotent groups of rank 6 which admit a hyperbolic automorphism.

In [28], the particular situation of $T(3,3)$-groups was generalized. We restricted to nilpotent groups of class 2 whose commutator subgroup is of maximal torsion-free rank (in other words, $T(n, \frac{n(n-1)}{2})$-groups) and proved

THEOREM 3.2. *A* $T(n, \frac{n(n-1)}{2})$*-group admits a hyperbolic automorphism if and only if* $n \geq 3$. *Equivalently, a nilmanifold with a* $T(n, \frac{n(n-1)}{2})$*-group as fundamental group admits an Anosov diffeomorphism if and only if* $n \geq 3$.

It is remarkable and interesting to note that a better understanding of certain properties of the groups of weighted diffeomorphisms, has led to a further generalization of this Anosov-diffeomorphism result - i.e. for nilpotent groups of any nilpotency class - in [18]. Let us focus for a short while on a result which was a key observation in this perspective.

Although we limited ourselves to $\mathrm{P}(\mathbb{R}^n)$ and some of its subgroups so far, it is perfectly natural to consider also $\mathrm{P}(F^n)$ for any subfield F of $\mathbb{C}$. So $\mathrm{P}(F^n)$ consists of all bijections $\mu : F^n \to F^n$ such that both μ and μ^{-1} are expressed by polynomials with coefficients in F. Another, but essentially the same, way to view these groups is to define $\mathrm{P}(F^n)$ as consisting of those $\mu \in \mathrm{P}(\mathbb{C}^n)$, such that $\mu(F^n) \subseteq F^n$ (and then automatically $\mu(F^n) = F^n$). A little bit more care is needed to define also the group $\mathrm{P}(\mathbb{Z}^n)$.

DEFINITION 3.3.

$$\mathrm{P}(\mathbb{Z}^n) = \{\mu \in \mathrm{P}(\mathbb{C}^n) \mid \mu(\mathbb{Z}^n) = \mathbb{Z}^n)\}\,.$$

Note that $\mu \in \mathrm{P}(\mathbb{Z}^n)$ does not imply that μ has integral coefficients, e.g.

$$\mu : \mathbb{C}^2 \to \mathbb{C}^2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \frac{y^2}{2} - \frac{y}{2} \\ y \end{pmatrix}$$

belongs to (or, if you prefer, restricts to an element of) $\mathrm{P}(\mathbb{Z}^2)$.

Also, be aware that $\mu(\mathbb{Z}^n) \subseteq \mu(\mathbb{Z}^n)$ does not imply that $\mu(\mathbb{Z}^n) = \mathbb{Z}^n$, e.g. consider $\mu : \mathbb{C} \to \mathbb{C} : x \mapsto 2x$.

We can also define $\mathrm{P}(F^K, F^k)$ and $\mathrm{P}(\mathbb{Z}^K, \mathbb{Z}^k)$, where this last free abelian group consists of all maps $\mu \in \mathrm{P}(\mathbb{C}^K, \mathbb{C}^k)$, with $\mu(\mathbb{Z}^K) \subseteq \mathbb{Z}^k$.

Of course, it is now also possible to define blocked Jonquière groups and weighted groups for all F and for $\mathbb{Z}$ and all these groups can be seen as being iteratively built as semi-direct products, e.g.

$$\mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_n}) = \mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}}, \mathbb{Z}^{k_n}) \rtimes \Big(\mathrm{Gl}(k_n, \mathbb{Z}) \times \mathrm{P}_{\bar{\kappa},\bar{\omega}}(\mathbb{Z}^{k_1+k_2+\cdots+k_{n-1}})\Big).$$

THEOREM 3.4. *Let (κ, ω) be two fixed n–tuples of positive integers. There exists an $m \in \mathbb{N}$ and a faithful representation*

$$\chi : \mathrm{P}_{\kappa,\omega}(\mathbb{C}^{k_1+k_2+\cdots+k_n}) \to \mathrm{Gl}(m, \mathbb{C})$$

such that for all subfields F of $\mathbb{C}$ we have that

$$\chi(\mathrm{P}_{\kappa,\omega}(F^{k_1+k_2+\cdots+k_n})) = \chi(\mathrm{P}_{\kappa,\omega}(\mathbb{C}^{k_1+k_2+\cdots+k_n})) \cap \mathrm{Gl}(m, F)$$

and

$$\chi(\mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_n})) = \chi(\mathrm{P}_{\kappa,\omega}(\mathbb{C}^{k_1+k_2+\cdots+k_n})) \cap \mathrm{Gl}(m, \mathbb{Z}).$$

Proof. The proof of this theorem is a straightforward generalization of the proof of Theorem 2.11 in [18]. □

This theorem turned out to be relevant in the study of automorphisms of torsion free finitely generated nilpotent groups N. Indeed, let N be such a group and consider a descending central series of characteristic subgroups of N with torsion free filtration quotients:

$$N = N_1 \supset N_2 \supset N_3 \supset \cdots \supset N_n \text{ with } N_i/N_{i+1} \cong \mathbb{Z}^{k_i}.$$

We now fix a set of generators

$$a_{n,1}, a_{n,2}, \ldots, a_{n,k_n}, a_{n-1,1}, \ldots, a_{2,k_2}, a_{1,1}, a_{1,2}, \ldots, a_{1,k_1}$$

in such a way that the elements corresponding to $a_{i,1}, a_{i,2}, \ldots, a_{i,k_i}$ freely generate N_i/N_{i+1}. Having fixed this set of generators, any element of $n \in N$ can be uniquely expressed as a product

$$n = a_{n,1}^{z_{n,1}} a_{n,2}^{z_{n,2}} \ldots a_{2,1}^{z_{2,1}} a_{2,2}^{z_{2,2}} \ldots a_{2,k_2}^{z_{2,k_2}} a_{1,1}^{z_{1,1}} a_{1,2}^{z_{1,2}} \ldots a_{1,k_1}^{z_{1,k_1}}, \quad \text{with } z_{i,j} \in \mathbb{Z}.$$

So, we can identify N with $\mathbb{Z}^{k_1+k_2+\cdots+k_n}$. Using this identification it is easy to verify that any automorphism of N corresponds to an element of $\mathrm{P}(\mathbb{Z}^{k_1+k_2+\cdots+k_n})$. In fact such an automorphism belongs to the blocked Jonquière group of type $(k_n, \ldots, k_2, k_1)$. As $\mathrm{Aut}\,(N)$ is a finitely generated group ([58, page 122]), we can find a n-tuple of integers ω such that $\mathrm{Aut}\,(N) \subseteq \mathrm{P}_{\kappa,\omega}(\mathbb{Z}^{k_1+k_2+\cdots+k_n})$. Now, using the embedding χ from Theorem 3.4, we obtain a faithful $\mathbb{Z}$-linear representation of $\mathrm{Aut}\,(N)$. Moreover, the same route can be followed for the Mal'cev completion G of N and for the divisible closure (or rational Mal'cev completion) $N_{\mathbb{Q}}$ of N. In this way we find an embedding (the restriction of χ again) of $\mathrm{Aut}\,(G)$ in some $\mathrm{Gl}(m, \mathbb{R})$. This embedding has the very nice property that

$$\chi(\mathrm{Aut}\,(N_{\mathbb{Q}})) = \chi(\mathrm{Aut}\,(G)) \cap \mathrm{Gl}(m, \mathbb{Q})$$

and

$$\chi(\mathrm{Aut}\,(N)) = \chi(\mathrm{Aut}\,(G)) \cap \mathrm{Gl}(m, \mathbb{Z}).$$

This embedding was used in studying Anosov diffeomorphisms in [18], where we showed that the study of the existence of hyperbolic automorphisms on a torsion free finitely generated group N could be translated to the level of its divisible closure $N_{\mathbb{Q}}$. This shows that the existence of a hyperbolic automorphism on such a group N, can be reduced to any (suitable) finite index subgroup of N. We were able to effectively use such a reduction strategy in case the Mal'cev completion of N is a free nilpotent Lie group (i.e. the corresponding Lie algebra is free nilpotent). This substantially generalizes the situation of $T(n, \frac{n(n-1)}{2})$-groups and provides lots of examples of nilpotent groups (read nilmanifolds) of arbitrary nilpotency class admitting hyperbolic automorphisms (read Anosov diffeomorphisms).

For those infra-nilmanifolds M with a covering nilmanifold which supports hyperbolic automorphisms, it is certainly a relevant question to ask under what additional condition(s) this is sufficient to obtain an Anosov diffeomorphism on M. In this perspective, in [51], we successfully extended the "only if" part of an interesting algebraic characterization due to Porteous for flat manifolds admitting Anosov diffeomorphisms ([55]) and, in this way, deduced an obstruction to the existence of Anosov diffeomorphisms on a given infra-nilmanifold M, in terms of the associated holonomy representation. Unfortunately, in this more general context, this necessary condition turns out to be insufficient to decide that M indeed supports an Anosov diffeomorphism. Anyhow, the number of possible holonomy groups for infra-nilmanifolds supporting Anosov diffeomorphisms is substantially reduced by this observation.

3.2. Expanding maps on infra-nilmanifolds

An expanding map of a compact differentiable manifold M is a differentiable map $f : M \to M$ such that, for some Riemannian metric on M, there exists constants $c > 0$ and $\lambda > 1$ such that $\|Tf^r(v)\| \geq c\lambda^r\|v\|$, for all $v \in TM$ and all positive integers $r > 0$. For more detailed information on expanding maps, we refer to [26], [29], [39], [44] and [60]. Similar as for Anosov diffeomorphisms on infra-nilmanifolds, there is a straightforward algebraic construction for expanding maps. An automorphism A of the universal cover G of an infra-nilmanifold M is expanding if and only if all eigenvalues of dA are of modulus > 1. If A induces an endomorphism on M, then this is an expanding map, also referred to as an expanding endomorphism of M.

In contrast with the situation for Anosov diffeomorphisms, for expanding maps it is known that they indeed only occur on infra - nilmanifolds. In fact, any expanding map of an arbitrary closed smooth manifold is topologically conjugate to an expanding infra-nilmanifold endomorphism ([37]).

In [26], some real progress was made concerning the existence of expanding maps on infra-nilmanifolds. The main result of that paper shows that any infra-nilmanifold of homogeneous type admits an expanding map. This class contains all infra-nilmanifolds, having a virtually 2-step nilpotent fundamental group. Again, this result was obtained using polynomial diffeomorphisms.

To close the discussion concerning Anosov and expanding maps on infra-nilmanifolds, we formulate the following remarkable fact making a strong link between the existence of an affine structure on an infra-nilmanifold M and the existence of an Anosov or expanding map on M. This is proved in [25].

THEOREM 3.5. *Let M be an infra-nilmanifold admitting an expanding map or an Anosov diffeomorphism, then M also admits an affine structure.*

4. Open problems

By Theorem 2.5, we know that any polycyclic-by-finite group admits a polynomial structure of bounded (but unknown) degree. In [17], it was shown that an upper-bound on this degree can be given in terms of the Hirsch length of the group alone. Moreover, in [21] we were able to show that any polycyclic-by-finite group has a subgroup Γ' of finite index admitting a polynomial structure of degree $\leq \max(1, h(\Gamma') - 1)$ (where $h(\Gamma')$ stands for the Hirsch length of Γ'). One is therefore led to the following open problem:

OPEN PROBLEM 4.1. *Is it true that any polycyclic-by-finite group Γ admits a polynomial structure of degree $\leq \max(1, h(\Gamma) - 1)$?*

The closest we can get to an answer concerning this problem is the following:

THEOREM 4.2. *Let Γ be a polycyclic-by-finite group. Then Γ admits a polynomial structure ρ, such that there exists a finite index subgroup Γ' of Γ, for which the restriction of ρ to Γ' is of degree $\leq \max(1, h(\Gamma)-1)$.*

Proof. By [21] There exists a subgroup Γ' of Γ admitting a polynomial structure $\rho_1 : \Gamma' \to \mathrm{P}(\mathbb{R}^n)$ of degree $\leq \max(1, h(\Gamma) - 1)$ (note that $h(\Gamma) = h(\Gamma')$). By Theorem 2.5, there is also a polynomial structure $\rho_2 : \Gamma \to \mathrm{P}(\mathbb{R}^n)$ of bounded degree. By the uniqueness result of [7], we know that ρ_1 and $\rho_{2|\Gamma'}$ are polynomially conjugated. i.e. there exists a $p \in \mathrm{P}(\mathbb{R}^n)$ such that

$$\rho_1(\gamma) = p \circ \rho_2(\gamma) \circ p^{-1}, \ \forall \gamma \in \Gamma'.$$

We can now finish the proof by taking $\rho = p \circ \rho_2 \circ p^{-1}$. □

At different points already, we were talking about polynomial structures of bounded degree. In fact there are no known examples of polynomial structures of unbounded degree. Moreover, in [19], it was shown that all polynomial structures on a polycyclic-by-finite group of Hirsch length 2 (so in dimension 2) are of bounded degree. Hence, OPEN PROBLEM 4.3. *Is it true that any polynomial structure (on a polycyclic-by-finite group) is of bounded degree.*

Also, we can investigate the analogue of Auslander's conjecture for polynomial crystallographic groups:

OPEN PROBLEM 4.4 (Generalized Auslander Conjecture). *Is it true that any polynomial crystallographic group is polycyclic-by-finite?*

Again in [19], we made the first steps in attacking this problem and proved that in dimension 2 all polynomial crystallographic groups of bounded degree are polycyclic-by-finite.

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