Bull. Korean Math. Soc. **38** (2001), No. 3, pp. 605–610

IDENTICALLY DISTRIBUTED UNCORRELATED RANDOM VARIABLES NOT FULFILLING THE WLLN

DIETER LANDERS AND LOTHAR ROGGE

ABSTRACT. It is shown that for each $1 < p < 2$ there exist identically distributed uncorrelated random variables X_n with $E(|X_1|^p) < \infty$, not fulfilling the weak law of large numbers (WLLN). If, however, the random variables are moreover non-negative, the weaker integrability condition $E(X_1 \log X_1) < \infty$ already guarantees the strong law of large numbers.

Let X_n, $n \in \mathbb{N}$, be a sequence of identically distributed uncorrelated random variables. If $E(X_1^2) < \infty$ then it is well known that X_n, $n \in \mathbb{N}$, fulfills the strong law of large numbers (SLLN). Hence the question arises what happens if $E(X_1^2) < \infty$ is weakened to $E(|X_1|^p) < \infty$ for some $p < 2$.

For pairwise independent instead of uncorrelated random variables the SLLN holds according to the fundamental theorem of Etemadi (1981) even for $p = 1$. Interesting generalizations of this result were given by Soo Hak Sung [5]. He considered general strong laws of large numbers for pairwise independent identically distributed random variables. Moreover he obtained necessary and sufficient conditions for the strong law of large numbers. The following theorem shows that for uncorrelated random variables, however, for each $p < 2$ even the WLLN may fail.

THEOREM 1. *Let $1 \leq p < 2$. Then there exist identically distributed uncorrelated random variables X_n with $E(|X_1|^p) < \infty$, not fulfilling the weak law of large numbers.*

Received December 1, 2000.

2000 Mathematics Subject Classification: Primary 60F15; Secondary 60F05.

Key words and phrases: weak law of large numbers, strong law of large numbers, uncorrelated identically distributed random variables.

Proof. According to Lemma 4 it suffices to construct identically distributed random variables Y_n with

$$E(|Y_1|^p) < \infty, \quad E(Y_n Y_m) = -1 \quad \text{for} \quad n \neq m. \tag{1}$$

Since $1 \leq p < 2$ there exist $s \in \mathbb{N}$ with

$$s(2-p)/p \ > \ 1. \tag{2}$$

Let $X_{n,j},\ n, j \in \mathbb{N}$, be independent random variables with

$$P\{X_{n,j} = j^s\} = P\{X_{n,j} = -j^s\} = \frac{1}{2}\frac{1}{j^{2s}};$$
$$P\{X_{n,j} = 0\} = 1 - \frac{1}{j^{2s}}.$$

Then

$$E(X_{n,j}) = 0,\ E(X_{n,j}^2) = 1,\ E(|X_{n,j}|^p) = \frac{1}{j^{s(2-p)}}\,; \tag{3}$$

$$X_{n,j} \quad \text{and} \quad -X_{n,j} \quad \text{have the same distribution;} \tag{4}$$

$$\sum_{j=1}^{\infty} (E(|X_{n,j}|^p))^{\frac{1}{p}} \underset{(2),(3)}{<} \infty. \tag{5}$$

Put

$$Z_{n,j} := \begin{cases} X_{n,j} & \text{for} \quad j \leq n^2 \quad \text{or} \quad j \geq n^2 + n; \\ -X_{i,j} & \text{for} \quad j = n^2 + i \quad \text{with} \quad i < n \end{cases}$$

and

$$Y_n := \sum_{j=1}^{\infty} Z_{n,j}, \quad n \in \mathbb{N}.$$

Using (5) we obtain $E(|Y_n|^p) < \infty$. As $(Z_{n,j})_{j \in \mathbb{N}}$ has the same distribution for all $n \in \mathbb{N}$ (use (4)), the Y_n, $n \in \mathbb{N}$, are identically distributed. Now let $m, n \in \mathbb{N}$ with $m < n$ be given.

For (1) it remains to prove that $Y_n \cdot Y_m$ is integrable with $E(Y_n Y_m) = -1$. By definition of $Z_{n,j}$ we obtain for $j_0 = n^2 + m$ that

$$Z_{m,j_0} = -Z_{n,j_0}; \tag{6}$$

(7) $\qquad Z_{m,j},\ Z_{n,k}$ are independent if $\ j \neq j_0\ $ or $\ k \neq j_0$.

Since $|Y_n Y_m| \leq \sum_{j,k} |Z_{n,j}||Z_{m,k}|$ and $E(|Z_{n,j}|) = \frac{1}{j^s}$ we obtain the integrability of $Y_n Y_m$ by (6) and (7).

Using the theorem of Lebesgue we finally have

$$
\begin{aligned}
E(Y_n Y_m) &= \lim_{l \to \infty} E(\sum_{j,k \leq l} Z_{n,k} Z_{m,k}) \\
&\underset{(3),(7)}{=} E(Z_{n,j_0} Z_{m,j_0}) \\
&\underset{(3),(6)}{=} -1.
\end{aligned}
$$

□

The following considerations, Example 2 and Theorem 3 show that for nonnegative random variables we face a different situation; it is possible to obtain better results than suggested by Theorem 1.

If $X_n \geq 0$, $n \in \mathbb{N}$, are identically distributed uncorrelated random variables, then $E(X_1) < \infty$ implies the WLLN and $E(X_1^p) < \infty$ for some $p > 1$ implies the SLLN. There even holds a better result:

If $X_n \geq 0$, $n \in \mathbb{N}$, are uncorrelated-not necessarily identically distributed random variables then

$$\sup_{n \in \mathbb{N}} E(X_n (\log X_n)^{1+\varepsilon}) < \infty \quad \text{for some} \quad \varepsilon > 0$$

implies the SLLN (Landers-Rogge [3]). The following example shows that even for independent random variables $X_n \geq 0$ we cannot replace $\varepsilon > 0$ by $\varepsilon = 0$ in the condition above. Theorem 3 shows that for identically distributed uncorrelated random variables $X_n \geq 0$ we can replace $\varepsilon > 0$ by $\varepsilon = 0$, however.

EXAMPLE 2. There exist independent random variables $X_n \geq 0$, $n \in \mathbb{N}$ with $\sum_{n \in \mathbb{N}} E(X_n \log X_n) < \infty$, not fulfilling the SLLN.

Proof. Let X_n, $n \in \mathbb{N}$, be independent random variables with $P\{X_n = n\} = \frac{1}{n \log(n+2)}$ and $P\{X_n = 0\} = 1 - P\{X_n = n\}$. Then $\sup_{n \in \mathbb{N}} E(X_n \log X_n) \leq 1$ and $E(X_n) = \frac{1}{\log(n+2)}$. As $\sum_{n \in \mathbb{N}} P\{X_n \geq n\} = \infty$, the 0,1 law of Borel-Cantelli implies that

(1) $\qquad X_n(\omega) \geq n\ $ infinitely often for P-a.a. ω.

The SLLN would imply (with $a_n := \frac{1}{n}\sum_{i=1}^{n}(X_i - E(X_i))$) that

$$a_n - \frac{n-1}{n}a_{n-1} = \frac{1}{n}(X_n - E(X_n)) \to 0 \quad P\text{-a.e..}$$

As $E(X_n)/n \to 0$ this contradicts (1). Hence the SLLN cannot hold. □

THEOREM 3. Let X_n, $n \in \mathbb{N}$, be non-negative, identically distributed random variables with $E(X_iX_j) \le E(X_i)E(X_j)$ for $i \neq j$ and $E(X_1 \log X_1) < \infty$. Then

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - E(X_i)) \to 0 \quad P\text{-a.e..}$$

Proof. Let $X_{i,i} := X_i \cdot 1_{\{X_i \le i\}}$. According to the proof of Theorem 2() of Landers-Rogge (1997) it suffices to show

$$\sum_{i=1}^{\infty}\frac{1}{i^2}E(X_{i,i,}^2) < \infty; \tag{1}$$

$$\sum_{i=1}^{\infty}\frac{1}{i}E(X_i 1_{\{X_i > i\}}) < \infty. \tag{2}$$

As X_n, $n \in \mathbb{N}$, are identically distributed integrable random variables, (1) follows from the Lemma of Bose-Chandra [1].

Ad (2) : As X_n, $n \in \mathbb{N}$, are identically distributed, we have

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{i}E(X_i 1_{\{X_i > i\}}) &\le \sum_{i=1}^{\infty}\frac{1}{i}\sum_{k=i}^{\infty}(k+1)P\{k < X_1 \le k+1\} \\ &= \sum_{k=1}^{\infty}(k+1)P\{k < X_1 \le k+1\}\sum_{i=1}^{k}\frac{1}{i} \\ &\le c\sum_{k=1}^{\infty}(k+1)\log(k+1)P\{k < X_1 \le k+1\} \\ &< \infty,\end{aligned}$$

where the finiteness of the series follows from $E(X_1 \log X_1) < \infty$. $\square$

Schlömer [4] showed that $E(X_1 \log X_1) < \infty$ in Theorem 3 cannot be weakened to $E(X_1(\log X_1)^{1-\varepsilon}) < \infty$ for some $\varepsilon > 0$.

LEMMA 4. *Let $1 \le p < 2$. If there exist identically distributed random variables Y_n with $E(|Y_1|^p) < \infty$ and $E(Y_n Y_m) = -1$ for $n \neq m$, then there exist identically distributed uncorrelated random variables X_n with $E(|X_1|^p) < \infty$ not fulfilling the weak law of large numbers.*

Proof. Let $(\Omega, \mathcal{A}, P)$ be the original probability space on which the Y_n live. Let $\underline{\omega}, \overline{\omega}$ be different points not belonging to Ω. Put $\hat{\Omega} := \Omega \cup \{\underline{\omega}, \overline{\omega}\}$ and let $\hat{\mathcal{A}}$ be the σ-algebra on $\hat{\Omega}$ generated by $\mathcal{A} \cup \{\{\underline{\omega}\}\}$. Put

$$\hat{P}(\hat{A}) := \frac{1}{2} P(\hat{A} \cap \Omega) + \frac{1}{4}[\varepsilon_{\underline{\omega}}(\hat{A}) + \varepsilon_{\overline{\omega}}(\hat{A})] \quad \text{for} \quad \hat{A} \in \hat{\mathcal{A}}.$$

Let X_n on $\hat{\Omega}$ be defined by

$$X_n(\omega) := \begin{cases} -1 & \text{for} \quad \omega = \underline{\omega}; \\ Y_n(\omega) & \text{for} \quad \omega \in \Omega; \\ 1 & \text{for} \quad \omega = \overline{\omega}. \end{cases}$$

Then X_n are identically distributed uncorrelated random variables X_n with $E(|X_1|^p) < \infty$. Since

$$\frac{1}{n} \sum_{i=1}^{n} X_i(\underline{\omega}) = -1 \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} X_i(\overline{\omega}) = 1,$$

X_n do not fulfill the weak law of large numbers. $\square$

Let us remark that Theorem 1 also shows that the results of Soo Hak Sung [5] for pairwise independent random variables with $E(|X_1|^p) < \infty$ cannot be obtained for uncorrelated, identically distributed random variables.

References

[1] A. Bose and T. K. Chandra, *A note on the strong law of large numbers*, Calcutta Stat. Ass. Bull. 44 (1994), 115–122.

[2] N. Etemadi, *An elementary proof of the strong law of large numbers*, Z. Wahrsch. Verw. Geb. **55** (1981), 119–122.

[3] D. Landers and L. Rogge, *Laws of large numbers for uncorrelated Cesáro uniformly integrable random variables*, Sankhy ā **59** (1997), no. Sr. A, 301–310.

[4] R. Schlömer, *Starke Gesetze der grossen Zahlen unter modifizierten Unabhängigkeitsbedingungen*, Inaugural-Dissertation an der Universität zu Köln, 2000.

[5] Soo Hak Sung, *On the strong law of large numbers for pairwise i.i.d random variables*, Bull. Korean Math. Soc. **34** (1997), 517–526.

Dieter Landers, University of Cologne, Mathematisches Institut, Weyertal 86, D-50931 Köln

Lothar Rogge, University of Duisburg, Gerhard-Mercator-Universität Duisburg, Lotharstr. 65, D-47048 Duisburg
E-mail: rogge@math.uni-duisburg.de