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DECAY ESTIMATES FOR STEADY MAGNETOHYDRODYNAMICS EQUATIONS IN A SEMI-INFINITE CYLINDER

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ABSTRACT. This paper establishes exponential decay estimates for the solution of a stationary magnetohydrodynamics equations in a semi-infinite pipe flow when homogeneous lateral surface boundary conditions are applied.

1. Introduction

The model equations of magnetohydrodynamics are discussed in the book of Ames and Straughan [1, p.154]. In this paper we investigate exponential decay results for a solution of the model equations of magnetohydrodynamics. Under appropriate homogeneous boundary conditions on the lateral surface of the cylinder we establish Saint-Venant type decay of solutions as the distance from the finite end of the cylinder tends to infinity.

Several papers in the literature have dealt with spatial exponential decay in the Navier-Stokes equations (see Ames and Payne [3], Ames *et al.* [2], and Horgan and Wheeler [10]). For a survey of Saint-Venant type spatial decay results see Horgan and Knowles [8] and Horgan [6, 7]. More recent work on spatial decay in porous medium problems has been carried out by Payne and Song [12, 13], and Chadam and Qin [5].

We consider that a stationary magnetohydrodynamics flow occupies the interior of a semi-infinite cylindrical pipe of arbitrary cross section and generators parallel to the x_3 axis. Denoting the cross section of the pipe by D and the interior half cylinder by R, we introduce the notation:

$$R_z = \{(x_1, x_2, x_3)|(x_1, x_2) \in D,\ x_3 > z \geq 0\},$$
$$D_z = \{(x_1, x_2, x_3)|(x_1, x_2) \in D,\ x_3 = z\},$$

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where z is a running variable along the x_3 axis. Clearly, $R_0 \equiv R$ and $D_0 \equiv D$.

Throughout this paper we adopt the summation convention of summing in any term over a repeated index, Latin subscripts ranging from 1 to 3 and Greek subscripts from 1 to 2. A comma is used to denote partial differentiation.

Let v_i, h_i, and p denote the velocity, magnetic field, and pressure in the semi-infinite pipe, respectively. The governing equations for the stationary magnetohydrodynamics are

$$\begin{aligned} v_j v_{i,j} - h_j h_{i,j} &= -p_{,i} + \Delta v_i \quad \text{in } R, \\ v_j h_{i,j} - h_j v_{i,j} &= \Delta h_i \quad \text{in } R, \\ v_{j,j} = 0, \quad h_{j,j} &= 0 \quad \text{in } R. \end{aligned} \tag{1.1}$$

The symbol Δ denotes the Laplace operator. The boundary conditions are given by

$$\begin{aligned} v_i = 0, \quad h_i &= 0 \quad \text{on } \partial R \backslash D_0, \\ v_i &= f_i(x_1, x_2) \quad \text{on } D_0, \\ h_i &= h_i(x_1, x_2) \quad \text{on } D_0. \end{aligned} \tag{1.2}$$

The prescribed entrance profiles f_i and h_i are assumed to be zero on ∂D_0. As $x_3 \to \infty$, v_i is assumed to tend to the fully developed velocity field $\hat{v}\delta_{3i}$, i.e., $(0, 0, \hat{v}(x_1, x_2))$, corresponding to the net flow

$$\int_{D_0} f_3 dA \equiv F, \tag{1.3}$$

with the magnetic field corresponding to the fully developed velocity field vanishing. Thus $\hat{v}(x_1, x_2)$ may be characterized as the solution of the boundary value problem

$$\begin{aligned} \hat{v}_{,\alpha\alpha} &= \hat{p}_{,3} \quad \text{in } D_z, \\ \hat{v} &= 0 \quad \text{on } \partial D_z, \\ \int_{D_z} \hat{v} dA &= F. \end{aligned} \tag{1.4}$$

The gradient of the pressure $\hat{p}$ in $(1.4)_1$ has the form $\hat{p}_{,i} = -P\delta_{3i}$ where P is a positive constant and δ_{ij} is the Kronecker delta symbol.

In order to render our appropriate inequalities more explicit, we point out that the solution of (1.4) is nonnegative (by the maximum principle) and is just a multiple of the well studied torsion problem (where P is replaced by 2), and F is just an appropriate multiple of the torsional rigidity. Also bounds for the maximum value of the solution can be found in the literature (see [3]).

We define

$$u_i = v_i - \hat{v}\delta_{3i}, \quad q = p - \hat{p}. \tag{1.5}$$

Then the boundary value problem for u_i and h_i is

$$\begin{aligned} \Delta u_i &= q_{,i} + (u_j + \hat{v}\delta_{3j})u_{i,j} + u_\alpha \hat{v}_{,\alpha}\delta_{3i} - h_j h_{i,j} \quad \text{in } R, \\ \Delta h_i &= (u_j + \hat{v}\delta_{3j})h_{i,j} - h_j u_{i,j} - h_\alpha \hat{v}_{,\alpha}\delta_{3i} \quad \text{in } R, \\ u_{i,i} &= 0, \quad h_{i,i} = 0 \quad \text{in } R, \\ u_i &= 0, \quad h_i = 0 \quad \text{on } \partial R \backslash D_0, \\ u_i &= f_i - \hat{v}\delta_{3i} \quad \text{on } D_0, \\ h_i &= g_i \quad \text{on } D_0, \end{aligned} \tag{1.6}$$

$$u_i,\ h_i,\ u_{i,j},\ h_{i,j},\ q = o(x_3^{-1}) \quad \text{uniformly in } (x_1, x_2) \text{ as } x_3 \to \infty. \tag{1.7}$$

In view of (1.3) and $(1.4)_3$ and from the divergence theorem, it follows that

$$\int_{D_z} u_3 dA = 0 \quad \text{for all } z \in [0, \infty). \tag{1.8}$$

Our goal in this paper is to derive an integro-differential inequality for an energy integral $E(z)$ defined in (3.1) of the form

$$\frac{dE}{dz} + M \int_z^\infty E(\xi)d\xi \leq KE(z), \tag{1.9}$$

where for positive constants M and K are given in (3.27) and (3.28), respectively. We shall show that E decays to zero exponentially as $z \to \infty$ provided (3.29) is met.

In the next section we record some auxiliary inequalities that we shall use. In section 3 we derive the differential inequality for E, which integrates to yield exponential decay and in the final section we establish a bound for $E(0)$.

2. Auxiliary inequalities

We shall make frequent use of Schwarz's inequality and the arithmetic mean-geometric mean inequality in our derivation of the first-order integro-differential inequality we seek. In addition, we need the following Babuška-Aziz inequality (2.1) and inequalities (2.2), (2.4).

LEMMA. *Let R be a bounded three-dimensional region with Lipschitz boundary and let χ be a bounded function in R of mean value zero. Then* *there exists a vector function ω_i and a constant C depending only on the shape R such that*

$$\omega_{i,i} = \chi \quad \text{in } R, \quad \omega_i = 0 \quad \text{on } \partial R,$$

and

$$\int_R \omega_{i,j}\omega_{i,j}\,dx \le C \int_R \chi^2\,dx. \tag{2.1}$$

A proof of this result is given in Ladyzhenskaya and Solonnikov [11] *(see also Velte* [15], *and Horgan and Payne* [9]*).*

We now record here the following Poincaré inequality. Let D be a plane domain with sufficiently smooth boundary ∂D, and let v be a sufficiently smooth function defined on the closure $\overline{D}$ of D. If $v = 0$ on ∂D, then

$$\lambda_1 \int_D v^2\,dA \le \int_D v_{,\alpha}v_{,\alpha}\,dA, \tag{2.2}$$

where λ is the smallest positive eigenvalue of

$$w_{,\alpha\alpha} + \lambda w = 0 \quad \text{in } D, \quad w = 0 \quad \text{on } \partial D.$$

Second, if $\partial v/\partial n == 0$ on ∂D and $\int_D v\,dA = 0$, then

$$\mu_2 \int_D v^2\,dA \le \int_D v_{,\alpha}v_{,\alpha}\,dA, \tag{2.3}$$

where μ_2 is the smallest positive eigenvalue of

$$\Delta w + \mu w = 0 \quad \text{in } D, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial D, \quad \int_D w\,dA = 0.$$

The inequality (2.2) has been well studied (see, e.g., Bandle [4]), whereas the inequality (2.1) was extensively used for energy decay estimates for Darcy and Brinkman flows investigated by Payne and Song [12].

Finally, we have the following Sobolev inequality [10, see Eq.(A.11)]

$$\int_{R_z} v^4 dx \le \frac{1}{\sqrt{\lambda_1}} \left(\int_{R_z} v_{,j}v_{,j} dx \right)^2, \tag{2.4}$$

for sufficiently smooth functions v vanishing on $\partial R_z \backslash D_z$.

3. Decay estimates

In this section in order to derive an inequality which will imply exponential decay, we first consider the energy integral

$$E(z) = \int_{R_z} (u_{i,j}u_{i,j} + h_{i,j}h_{i,j}) dx. \tag{3.1}$$

We integrate by parts and use boundary conditions for u_i and h_i to obtain

$$E(z) = -\int_{D_z} (u_{i,3}u_i + h_{i,3}h_i)dA - \int_{R_z} (u_i\Delta u_i + h_i\Delta h_i)dx. \tag{3.2}$$

From the definition of $E(z)$, it follows that

$$E'(z) = -\int_{D_z} (u_{i,j}u_{i,j} + h_{i,j}h_{i,j})dA, \tag{3.3}$$

where the prime denotes differentiation with respect to z. To establish the desired integro-differential inequality (1.9), we first form

$$E'(z) + \gamma\int_z^\infty E(\xi)d\xi = -I_1 + I_2, \tag{3.4}$$

where we have used the differential equations for u_i and h_i,

$$I_1 = \int_{D_z} (u_{i,j}u_{i,j} + h_{i,j}h_{i,j})dA - \frac{\gamma}{2}\int_{D_z} (u_iu_i + h_ih_i)dA, \tag{3.5}$$

$$\begin{aligned} I_2 = &-\gamma\int_z^\infty\int_{R_\xi} u_i[q_{,i} + (u_j + \hat{v}\delta_{3j})u_{i,j} + u_\alpha\hat{v}_{,\alpha}\delta_{3i} - h_jh_{i,j}]\,dxd\xi \\ &-\gamma\int_z^\infty\int_{R_\xi} h_i[(u_j + \hat{v}\delta_{3j})h_{i,j} - h_ju_{i,j} - h_\alpha\hat{v}_{,\alpha}\delta_{3i}]\,dxd\xi, \end{aligned} \tag{3.6}$$

and γ is a positive parameter to be chosen. If we choose $\gamma = 2\lambda_1$, we drop the term I_1 which is nonnegative in view of (2.2).

In estimating the term I_2 we integrate by parts and note that the term $\int_z^\infty\int_{R_\xi} u_{i,j}h_ih_j\,dxd\xi$ cancels. Then we see

$$\begin{aligned} \frac{I_2}{\gamma} = &\int_{R_z} u_3q\,dx + \frac{1}{2}\int_{R_z} u_iu_i(u_3 + \hat{v})dx + \int_z^\infty\int_{R_\xi} \hat{v}u_{3,j}u_j\,dxd\xi \\ &+ \int_{R_z} \hat{v}u_3^2dx - \int_{R_z} u_ih_3h_idx + \frac{1}{2}\int_{R_z} (u_3 + \hat{v})h_ih_idx \\ &- \int_z^\infty\int_{R_\xi} \hat{v}h_{3,j}h_j\,dxd\xi - \int_{R_z} \hat{v}h_3^2dx \\ = &\sum_{n=1}^{8} J_n. \end{aligned} \tag{3.7}$$

We have thus written I_2/γ as the sum of eight integrals J_n's each of which we must now bound in terms of $E(z)$. Let us consider the sum of two integrals J_2 and J_6. Using Schwarz's inequality repeatedly and inequalities

(2.2), (2.4) successively, we find

$$
\begin{aligned}
& J_2 + J_6 \\
& \leq \frac{1}{2}\left(\int_{R_z} u_3^2 dx\right)^{1/2}\left[\left\{\int_{R_z} (u_i u_i)^2 dx\right\}^{1/2}\right. \\
& \quad \left. + \left\{\int_{R_z} (h_i h_i)^2 dx\right\}^{1/2}\right] + \frac{1}{2}\frac{\hat{v}_s}{\lambda_1}\int_{R_z} (u_{i,j}u_{i,j} + h_{i,j}h_{i,j}) dx \\
& \leq \frac{\lambda_1^{-3/4}}{2}\left(\int_{R_z} u_{3,\alpha}u_{3,\alpha} dx\right)^{1/2}\left(\int_{R_z} u_{i,j}u_{i,j} dx + \int_{R_z} h_{i,j}h_{i,j} dx\right) \\
& \quad + \frac{1}{2}\frac{\hat{v}_s}{\lambda_1}\int_{R_z} (u_{i,j}u_{i,j} + h_{i,j}h_{i,j}) dx \\
& \leq \frac{\lambda_1^{-3/4}}{2}\sqrt{E(0)}E(z) + \frac{1}{2}\frac{\hat{v}_s}{\lambda_1}E(z),
\end{aligned}
\tag{3.8}
$$

where $\hat{v}_s = \max_{D_z} v(x_1, x_2)$ and the last step following from the monotonicity of $E(z)$. To make our inequalities more explicit, we mention here the bound for $\hat{v}_s$ obtained by Ames and Payne [3, see eq.(7.8)].

We apply Schwarz's inequality and the inequality (2.2) to the sum of two integrals J_3 and J_7 to obtain

$$
\begin{aligned}
J_3 + J_7 &\leq \frac{\hat{v}_s}{\sqrt{\lambda_1}}\left\{\int_z^\infty \int_{R_\xi} (u_{i,j}u_{i,j} + h_{i,j}h_{i,j}) dx d\xi\right\} \\
&= \frac{\hat{v}_s}{\sqrt{\lambda_1}}\int_z^\infty E(\xi) d\xi.
\end{aligned}
\tag{3.9}
$$

As for J_5, an application of Schwarz's inequality, inequalities (2.2), (2.4), and the arithmetic mean-geometric mean inequality yields the bound

$$
\begin{aligned}
J_5 &\leq \left(\int_{R_z} u_i u_i dx\right)^{1/2}\left\{\int_{R_z} (h_i h_i)^2 dx \int_{R_z} h_3^4 dx\right\}^{1/4} \\
&\leq \frac{1}{\lambda_1^{1/2}}\left(\int_{R_z} u_{i,\alpha}u_{i,\alpha} dx\right)^{1/2} \\
&\quad \times \frac{1}{\lambda_1^{1/8}}\left\{\int_{R_z} h_{i,j}h_{i,j} dx\right\}^{1/2}\frac{1}{\lambda_1^{1/8}}\left\{\int_{R_z} h_{3,j}h_{3,j} dx\right\}^{1/2} \\
&\leq \frac{\lambda_1^{-3/4}}{2}\left(\int_{R_z} u_{i,j}u_{i,j} dx + \int_{R_z} h_{i,j}h_{i,j} dx\right)\left(\int_{R_z} h_{3,j}h_{3,j} dx\right)^{1/2} \\
&\leq \frac{\lambda_1^{-3/4}}{2}\sqrt{E(0)}E(z).
\end{aligned}
\tag{3.10}
$$

An application of the inequality (2.2) to the sum of two integrals J_4 and J_8 leads to the bound

$$\begin{aligned} J_4 + J_8 &\le \int_{R_z} |\hat{v}|(u_3^2 + h_3^2)dx \\ &\le \frac{\hat{v}_s}{\lambda_1}\int_{R_z}(u_{3,\alpha}u_{3,\alpha} + h_{3,\alpha}h_{3,\alpha})dx \\ &\le \frac{\hat{v}_s}{\lambda_1}E(z). \end{aligned} \tag{3.11}$$

Finally, to complete the derivation of the integro-differential inequality (1.9), we need to eliminate the pressure term q in the J_1 integral. This can be accomplished with the aid of (2.1) by using the arguments of Horgan and Payne [9], and Payne and Song [12]. Thus we rewrite the integral J_1 as

$$\int_{R_z} u_3 q\, dx = \sum_{n=0}^{\infty} \int_{z+nL}^{z+(n+1)L} \int_{D_\xi} u_3 q\, dA d\xi, \tag{3.12}$$

for some positive constant L at our disposal. We introduce the further notation

$$R_z^n = D \times \{z + nL < \xi < z + (n+1)L\}, \tag{3.13}$$

so that we may write

$$\int_{R_z} u_3 q\, dx = \sum_{n=0}^{\infty} \int_{R_z^n} u_3 q\, dx. \tag{3.14}$$

We may proceed to bound each term on the right separately. In fact, we show how to bound an arbitrary term in terms of quantities which do not depend on z so that the same type of bound holds for each term. Recall that for each value of $z(\ge 0)$

$$\int_{D_z} u_3\, dA = 0. \tag{3.15}$$

This allows us to define a vector field ω_i satisfying

$$\begin{aligned} \omega_{i,i} &= u_3 \quad \text{in } R_z^n, \\ \omega_i &= 0 \quad \text{on } \partial R_z^n. \end{aligned} \tag{3.16}$$

Using the vector function ω_i, it follows that

$$
(3.17)\quad
\begin{aligned}
\int_{R_z^n} u_3 q dx &= \int_{R_z^n} \omega_{i,i} q dx = -\int_{R_z^n} \omega_i q_{,i} dx \\
&= \int_{R_z^n} \omega_i [-\Delta u_i + (u_j + \hat{v}\delta_{3j}) u_{i,j} + u_\alpha \hat{v}_{,\alpha}\delta_{3i} - h_j h_{i,j}] dx \\
&= \int_{R_z^n} \omega_{i,j} u_{i,j} dx - \int_{R_z^n} \omega_{i,j}(u_j + \hat{v}\delta_{3j}) u_i dx \\
&\quad + \int_{R_z^n} \omega_3 u_\alpha \hat{v}_{,\alpha} dx + \int_{R_z^n} \omega_{i,j} h_j h_i dx \\
&\leq \int_{R_z^n} \omega_{i,j} u_{i,j} dx + \int_{R_z^n} |\omega_{i,j}|(u_i u_j + h_i h_j) dx \\
&\quad - \int_{R_z^n} \hat{v}\omega_{i,3} u_i dx + \int_{R_z^n} \omega_3 u_\alpha \hat{v}_{,\alpha} dx \\
&= P_n^{(1)} + P_n^{(2)} + P_n^{(3)} + P_n^{(4)}.
\end{aligned}
$$

Now by means of Schwarz's inequality and inequalities (2.1), (2.2), we have

$$
(3.18)\quad
\begin{aligned}
P_n^{(1)} &\leq \left(\int_{R_z^n} \omega_{i,j}\omega_{i,j} dx \right)^{1/2} \left(\int_{R_z^n} u_{i,j} u_{i,j} dx \right)^{1/2} \\
&\leq \left(C \int_{R_z^n} u_3^2 dx \right)^{1/2} \left(\int_{R_z^n} u_{i,j} u_{i,j} dx \right)^{1/2} \\
&\leq \left(\frac{C}{\lambda_1} \int_{R_z^n} u_{3,\alpha} u_{3,\alpha} dx \right)^{1/2} \left(\int_{R_z^n} u_{i,j} u_{i,j} dx \right)^{1/2} \\
&\leq \sqrt{\frac{C}{\lambda_1}} e_n(z),
\end{aligned}
$$

where $e_n(z) = \int_{R_z^n} (u_{i,j} u_{i,j} + h_{i,j} h_{i,j}) dx$. Using Schwarz's inequality repeatedly, inequalities (2.1), (2.4), and the monotonicity of $e_n(z)$, for $P_n^{(2)}$ we obtain

$$
\begin{aligned}
P_n^{(2)} &\leq \left(\int_{R_z^n} \omega_{i,j}\omega_{i,j} dx \right)^{1/2} \\
&\quad \times \left[\left\{ \int_{R_z^n} (u_i u_i)^2 dx \right\}^{1/2} + \left\{ \int_{R_z^n} (h_i h_i)^2 dx \right\}^{1/2} \right] \\
&\leq \frac{1}{\lambda_1^{1/4}} \left(C \int_{R_z^n} u_3^2 dx \right)^{1/2} \\
&\quad \times \left(\int_{R_z^n} u_{i,j}u_{i,j} dx + \int_{R_z^n} h_{i,j}h_{i,j} dx \right) \\
&\leq \frac{C^{1/2}}{\lambda_1^{3/4}} \sqrt{e_n(0)} e_n(z).
\end{aligned}
\tag{3.19}
$$

A bound for $P_n^{(3)}$ is established through the use of Schwarz's inequality and inequalities (2.1), (2.2). We have

$$
\begin{aligned}
P_n^{(3)} &\leq \hat{v}_s \left(\int_{R_z^n} \omega_{i,3}\omega_{i,3} dx \right)^{1/2} \left(\int_{R_z^n} u_i u_i dx \right)^{1/2} \\
&\leq \hat{v}_s \left(C \int_{R_z^n} u_3^2 dx \right)^{1/2} \left(\frac{1}{\lambda_1} \int_{R_z^n} u_{i,\alpha}u_{i,\alpha} dx \right)^{1/2} \\
&\leq \frac{\hat{v}_s C^{1/2}}{\lambda_1} e_n(z).
\end{aligned}
\tag{3.20}
$$

We turn now to the bound for $P_n^{(4)}$. On integrating by parts we use Schwarz's inequality since $u_{\alpha,\alpha} = -u_{3,3}$ and find

$$
\begin{aligned}
P_n^{(4)} &= -\int_{R_z^n} \hat{v}\omega_{3,\alpha} u_\alpha dx - \int_{R_z^n} \hat{v}\omega_3 u_{\alpha,\alpha} dx \\
&\leq \hat{v}_s \left(\int_{R_z^n} \omega_{3,\alpha}\omega_{3,\alpha} dx \right)^{1/2} \left(\int_{R_z^n} u_\alpha u_\alpha dx \right)^{1/2} \\
&\quad + \hat{v}_s \left(\int_{R_z^n} \omega_3^2 dx \right)^{1/2} \left(\int_{R_z^n} u_{3,3}^2 dx \right)^{1/2} .
\end{aligned}
\tag{3.21}
$$

It then follows by applying inequalities (2.1) and (2.2) that

$$
\begin{aligned}
P_n^{(4)} &\leq \frac{\hat{v}_s}{\sqrt{\lambda_1}} \left(\int_{R_z^n} \omega_{3,\alpha}\omega_{3,\alpha} dx \right)^{1/2} \left(\int_{R_z^n} u_{\alpha,\beta}u_{\alpha,\beta} dx \right)^{1/2} \\
&\quad + \frac{\hat{v}_s}{\sqrt{\lambda_1}} \left(\int_{R_z^n} \omega_{3,\alpha}\omega_{3,\alpha} dx \right)^{1/2} \left(\int_{R_z^n} u_{3,3}^2 dx \right)^{1/2} \\
&\leq \frac{\hat{v}_s}{\sqrt{\lambda_1}} \left(\int_{R_z^n} \omega_{i,j}\omega_{i,j} dx \right)^{1/2} \left(\int_{R_z^n} u_{\alpha,\beta}u_{\alpha,\beta} dx \right)^{1/2} \\
&\quad + \frac{\hat{v}_s}{\sqrt{\lambda_1}} \left(\int_{R_z^n} \omega_{i,j}\omega_{i,j} dx \right)^{1/2} \left(\int_{R_z^n} u_{3,3}^2 dx \right)^{1/2} \\
&\leq \frac{\hat{v}_s\sqrt{C}}{\sqrt{\lambda_1}} \left(\int_{R_z^n} u_3^2 dx \right)^{1/2} \\
&\quad \times \left[\left(\int_{R_z^n} u_{\alpha,\beta}u_{\alpha,\beta} dx \right)^{1/2} + \left(\int_{R_z^n} u_{3,3}^2 dx \right)^{1/2} \right].
\end{aligned} \tag{3.22}
$$

Using the inequality (2.2), we have

$$
\begin{aligned}
P_n^{(4)} &\leq \frac{\hat{v}_s\sqrt{C}}{\lambda_1} \left(\int_{R_z^n} u_{3,\alpha}u_{3,\alpha} dx \right)^{1/2} \left(\int_{R_z^n} u_{i,j}u_{i,j} dx \right)^{1/2} \\
&\leq \frac{\hat{v}_s\sqrt{C}}{\lambda_1} e_n(z).
\end{aligned} \tag{3.23}
$$

Since C and λ_1 are independent of z and

$$
\sum_{n=0}^{\infty} e_n(z) = E(z), \tag{3.24}
$$

we may sum over n to bound J_1 in terms of $E(z)$. Then we have

$$
J_1 \leq \left\{ \sqrt{\frac{C}{\lambda_1}} + \frac{C^{1/2}}{\lambda_1^{3/4}} \sqrt{E(0)} + 2\hat{v}_s \frac{C^{1/2}}{\lambda_1} \right\} E(z). \tag{3.25}
$$

Substituting the bounds for the J_n's into (3.7), inserting the bound for I_2, and dropping the nonnegative term I_1 in (3.4), we obtain the desired integro-differential inequality for the energy integral

$$
E'(z) + M \int_z^{\infty} E(\xi) d\xi \leq K E(z), \tag{3.26}
$$

where

$$M = 2\lambda_1 \left(1 - \frac{\hat{v}_s}{\sqrt{\lambda_1}}\right), \tag{3.27}$$

$$K = 2\lambda_1 \left[\frac{3}{2}\frac{\hat{v}_s}{\lambda_1} + \sqrt{\frac{C}{\lambda_1}} + 2\hat{v}_s\frac{C^{1/2}}{\lambda_1} + \lambda_1^{-3/4}(1+\sqrt{C})\sqrt{E(0)}\right]. \tag{3.28}$$

To ensure decay, we require $M > 0$, or

$$\hat{v}_s < \sqrt{\lambda_1}. \tag{3.29}$$

This condition yields a restriction on the flow. We shall subsequently require somewhat stronger restrictions (4.26) in bounding the total energy $E(0)$.

To investigate (3.26) we refer to Horgan and Wheeler [10, see eq.(4.15)]. It follows then that

$$E(z) \le aE(0)e^{-bz} \quad \text{for all } z \in [0,\infty), \tag{3.30}$$

where

$$a = \frac{\left(K^2+4M\right)^{1/2}}{b}, \quad b = \frac{1}{2}\left\{\left(K^2+4M\right)^{1/2} - K\right\}. \tag{3.31}$$

To make (3.30) explicit we need a bound for $E(0)$ in terms of data, which we derive in the next section.

4. A bound for $E(0)$

To bound $E(0)$, we introduce auxiliary functions which are

$$\begin{aligned}
\Delta\zeta_i &= \hat{q}_{,i} \quad \text{in } R,\\
\Delta H_i &= s_{,i} \quad \text{in } R,\\
\zeta_{i,i} &= 0, \quad H_{i,i} = 0 \quad \text{in } R,\\
\zeta_i &= 0, \quad H_i = 0 \quad \text{on } \partial R\backslash D_0,\\
\zeta_i &= f_i - \hat{v}\delta_{3i} \quad \text{on } D_0,\\
H_i &= g_i \quad \text{on } D_0,
\end{aligned} \tag{4.1}$$

where $\hat{q}$ and s are functions that are not prescribed *a priori* but are determined up to constants by the fact that all of the equations (4.1) are to be satisfied.

We define

$$X = \int_{R_0} u_{i,j}u_{i,j}dx, \quad Y = \int_{R_0} h_{i,j}h_{i,j}dx,$$
$$A = \int_{R_0} \zeta_{i,j}\zeta_{i,j}dx, \quad B = \int_{R_0} H_{i,j}H_{i,j}dx. \tag{4.2}$$

From the triangle inequality, it follows that

$$E(0) = X + Y \leq 2\left\{\int_{R_0}(u_i - \zeta_i)_{,j}(u_i - \zeta_i)_{,j}dx\right\} + 2A$$
$$+ 2\left\{\int_{R_0}(h_i - H_i)_{,j}(h_i - H_i)_{,j}dx\right\} + 2B. \tag{4.3}$$

First, we consider the leading term on the right side of (4.3). From the divergence theorem we see

$$\int_{R_0}(u_i - \zeta_i)_{,j}(u_i - \zeta_i)_{,j}dx = -\int_{R_0}(u_i - \zeta_i)\Delta(u_i - \zeta_i)dx$$
$$= -\int_{R_0}(u_i - \zeta_i)[(q-\hat{q})_{,i} + (u_j + \hat{v}\delta_{3j})u_{i,j} + u_\alpha \hat{v}_{,\alpha}\delta_{3i} - h_j h_{i,j}]dx. \tag{4.4}$$

We integrate by parts in the first and second term and obtain

$$-\int_{R_0}(u_i - \zeta_i)[(q - \hat{q})_{,i} + (u_j + \hat{v}\delta_{3j})u_{i,j}]dx$$
$$= \frac{1}{2}\int_{D_0}(f_3 - \hat{v})(f_i - \hat{v}\delta_{3i})(f_i - \hat{v}\delta_{3i})dA$$
$$+ \frac{1}{2}\int_{D_0}\hat{v}(f_i - \hat{v}\delta_{3i})(f_i - \hat{v}\delta_{3i})dA$$
$$- \int_{R_0}\zeta_{i,j}u_j u_i dx + \int_{R_0}\hat{v}\zeta_i u_{i,3}dx. \tag{4.5}$$

We bound the last two integrals by the appropriate integral inequalities

$$-\int_{R_0}\zeta_{i,j}u_j u_i dx \leq \frac{\sqrt{A}}{\lambda_1^{1/4}}\int_{R_0}u_{i,j}u_{i,j}dx, \tag{4.6}$$

$$\int_{R_0}\hat{v}\zeta_i u_{i,3}dx \leq \frac{\epsilon_1}{2}\frac{\hat{v}_s}{\sqrt{\lambda_1}}\int_{R_0}u_{i,j}u_{i,j}dx + \frac{\epsilon_1^{-1}}{2}\frac{\hat{v}_s}{\sqrt{\lambda_1}}A. \tag{4.7}$$

An integration by parts in the third term of (4.4) results in

$$
(4.8)\qquad \begin{aligned}
-\int_{R_0}(u_3-\zeta_3)u_\alpha\hat{v}_{,\alpha}dx &= \int_{R_0}\hat{v}u_{3,\alpha}u_\alpha dx+\int_{R_0}\hat{v}u_3u_{\alpha,\alpha}dx\\
&\quad -\int_{R_0}\hat{v}\zeta_{3,\alpha}u_\alpha dx-\int_{R_0}\hat{v}\zeta_3u_{\alpha,\alpha}dx.
\end{aligned}
$$

For the second integral on the right side of (4.8), integrating by parts since $u_{\alpha,\alpha}=-u_{3,3}$, we have

$$
(4.9)\qquad \begin{aligned}
\int_{R_0}\hat{v}u_3u_{\alpha,\alpha}dx &= -\int_{R_0}\hat{v}u_3u_{3,3}dx\\
&=\frac{1}{2}\int_{D_0}\hat{v}(f_3-\hat{v})^2dA.
\end{aligned}
$$

As for the remaining integrals on the right side of (4.8), applications of the appropriate inequalities enable us to conclude

$$
(4.10)\qquad \int_{R_0}\hat{v}u_{3,\alpha}u_\alpha dx\le\frac{\hat{v}_s}{\sqrt{\lambda_1}}\int_{R_0}u_{i,j}u_{i,j}dx,
$$

$$
(4.11)\qquad -\int_{R_0}\hat{v}\zeta_{3,\alpha}u_\alpha dx\le\frac{\epsilon_2}{2}\frac{\hat{v}_s}{\sqrt{\lambda_1}}\int_{R_0}u_{i,j}u_{i,j}dx+\frac{\epsilon_2^{-1}}{2}\frac{\hat{v}_s}{\sqrt{\lambda_1}}A,
$$

$$
(4.12)\qquad \begin{aligned}
-\int_{R_0}\hat{v}\zeta_3u_{\alpha,\alpha}dx &\le \hat{v}_s\left(\int_{R_0}\zeta_3^2dx\right)^{1/2}\left\{\int_{R_0}(u_{\alpha,\alpha})^2dx\right\}^{1/2}\\
&\le\frac{\hat{v}_s}{\sqrt{\lambda_1}}\left(\int_{R_0}\zeta_{3,\alpha}\zeta_{3,\alpha}dx\int_{R_0}u_{i,j}u_{i,j}dx\right)^{1/2}\\
&\le\frac{\epsilon_3}{2}\frac{\hat{v}_s}{\sqrt{\lambda_1}}\int_{R_0}u_{i,j}u_{i,j}dx+\frac{\epsilon_3^{-1}}{2}\frac{\hat{v}_s}{\sqrt{\lambda_1}}A.
\end{aligned}
$$

Integrating by parts in the last term of (4.4), we apply Schwarz's inequality and the inequality (2.4) to obtain

$$
(4.13)\qquad \begin{aligned}
\int_{R_0}(u_i-\zeta_i)h_jh_{i,j}dx &= -\int_{R_0}u_{i,j}h_jh_idx++\int_{R_0}\zeta_{i,j}h_jh_idx\\
&\le-\int_{R_0}u_{i,j}h_jh_idx++\frac{\sqrt{A}}{\lambda_1^{1/4}}\int_{R_0}h_{i,j}h_{i,j}dx.
\end{aligned}
$$

Consider the third term on the right side of (4.3). On integrating by parts, we observe

$$\begin{aligned}\int_{R_0}(h_i-H_i)_{,j}(h_i-H_i)_{,j}dx &= -\int_{R_0}(h_i-H_i)\Delta(h_i-H_i)dx\\ &= -\int_{R_0}(h_i-H_i)[(u_j+\hat{v}\delta_{3j})h_{i,j}-h_ju_{i,j}-h_\alpha\hat{v}_{,\alpha}\delta_{3i}-s_{,i}]dx.\end{aligned} \tag{4.14}$$

A further integration by parts in the leading term on the right side of (4.14) yields

$$-\int_{R_0}h_{i,j}u_jh_idx=\frac{1}{2}\int_{D_0}g_ig_i(f_3-\hat{v})dA, \tag{4.15}$$

$$-\int_{R_0}h_i\hat{v}\delta_{3j}h_{i,j}dx=\frac{1}{2}\int_{D_0}\hat{v}g_ig_idA. \tag{4.16}$$

As for some of the remaining integral on the right side of (4.14), a straightforward application of Schwarz's inequality, the inequalities (2.2), (2.4) and the arithmetic mean-geometric mean inequality gives

$$\int_{R_0}H_iu_jh_{i,j}dx\le\frac{B^{1/2}\lambda_1^{-1/4}}{2}(X+Y), \tag{4.17}$$

$$\int_{R_0}(h_i-H_i)h_ju_{i,j}dx\le\int_{R_0}h_ih_ju_{i,j}dx++\frac{B^{1/2}\lambda_1^{-1/4}}{2}(X+Y). \tag{4.18}$$

On integrating by parts in the penultimate term in (4.14), we are led to

$$\begin{aligned}&\int_{R_0}(h_3-H_3)h_\alpha\hat{v}_{,\alpha}dx\\ &\qquad=-\int_{R_0}\hat{v}(h_{3,\alpha}-H_{3,\alpha})h_\alpha dx-\int_{R_0}\hat{v}(h_3-H_3)h_{\alpha,\alpha}dx.\end{aligned} \tag{4.19}$$

In a manner similar to the computation of each of integrals on the right side of (4.8), we arrive at

$$\begin{aligned}-\int_{R_0}\hat{v}h_{3,\alpha}h_\alpha dx &\le \hat{v}_s\left(\int_{R_0}h_{3,\alpha}h_{3,\alpha}dx\int_{R_0}h_\alpha h_\alpha dx\right)^{1/2}\\ &\le\frac{\hat{v}_s}{\sqrt{\lambda_1}}\int_{R_0}h_{i,j}h_{i,j}dx,\end{aligned} \tag{4.20}$$

$$\begin{aligned}\int_{R_0} \hat{v} H_{3,\alpha} h_\alpha dx &\le \hat{v}_s \left(\int_{R_0} H_{3,\alpha} H_{3,\alpha} dx \int_{R_0} h_\alpha h_\alpha dx \right)^{1/2} \\ &\le \frac{\hat{v}_s}{\sqrt{\lambda_1}} \left(\int_{R_0} H_{3,\alpha} H_{3,\alpha} dx \int_{R_0} h_{\alpha,\beta} h_{\alpha,\beta} dx \right)^{1/2} \\ &\le \frac{\epsilon_4}{2} \frac{\hat{v}_s}{\sqrt{\lambda_1}} \int_{R_0} h_{i,j} h_{i,j} dx + \frac{\epsilon_4^{-1}}{2} \frac{\hat{v}_s}{\sqrt{\lambda_1}} B,\end{aligned} \tag{4.21}$$

$$-\int_{R_0} \hat{v} h_3 h_{\alpha,\alpha} dx = \int_{R_0} \hat{v} h_3 h_{3,3} dx = -\frac{1}{2} \int_{D_0} \hat{v} g_3^2 dA, \tag{4.22}$$

$$\begin{aligned}-\int_{R_0} \hat{v} H_3 h_{3,3} dx &\le \hat{v}_s \left(\int_{R_0} H_3^2 dx \int_{R_0} h_{3,3}^2 dx \right)^{1/2} \\ &\le \frac{\epsilon_5}{2} \frac{\hat{v}_s}{\sqrt{\lambda_1}} \int_{R_0} h_{i,j} h_{i,j} dx + \frac{\epsilon_5^{-1}}{2} \frac{\hat{v}_s}{\sqrt{\lambda_1}} B.\end{aligned} \tag{4.23}$$

Observing that the term $\int_{R_0} h_i h_j u_{i,j} dx$ cancels in adding (4.13) and (4.18), using the computable data A and B given in Ames *et al.* [2], Horgan and Wheeler [10] and Song [14], and combining our appropriate bounds, we may write (4.3) as

$$\mathcal{D}_1 X + \mathcal{D}_2 Y \le \text{data}, \tag{4.24}$$

where

$$\begin{aligned}\mathcal{D}_1 &= 1 - 2\frac{\hat{v}_s}{\lambda_1^{1/2}} - 2\frac{A^{1/2}}{\lambda_1^{1/4}} - 2\frac{B^{1/2}}{\lambda_1^{1/4}} - \frac{(\epsilon_1 + \epsilon_2 + \epsilon_3)\hat{v}_s}{\lambda_1^{1/2}}, \\ \mathcal{D}_2 &= 1 - 2\frac{\hat{v}_s}{\lambda_1^{1/2}} - 2\frac{A^{1/2}}{\lambda_1^{1/4}} - 2\frac{B^{1/2}}{\lambda_1^{1/4}} - \frac{(\epsilon_4 + \epsilon_5)\hat{v}_s}{\lambda_1^{1/2}}.\end{aligned} \tag{4.25}$$

Inequality (4.24) is the desired bound for $E(0)$ provided

$$\mathcal{D}_1 > 0, \quad \mathcal{D}_2 > 0, \tag{4.26}$$

for some sufficiently small positive constants ϵ_n's and restrictions on $\hat{v}_s$, A and B resulting from the flow.

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