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SPACE-LIKE SUBMANIFOLDS WITH CONSTANT SCALAR CURVATURE IN THE DE SITTER SPACES

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ABSTRACT. Let M^n be a space-like submanifold in a de Sitter space $M_p^{n+p}(c)$ with constant scalar curvature. We firstly extend Cheng-Yau's technique to higher codimensional cases. Then we study the rigidity problem for M^n with parallel normalized mean curvature vector field.

1. Introduction

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of constant curvature c whose index is p. It is called an indefinite space form of index p and simply a space form when $p = 0$. If $c > 0$, we call it as a de Sitter space of index p. Akutagawa [3] and Ramanathan [11] investigated space-like hypersurfaces in a de Sitter space and proved independently that a complete space-like hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the mean curvature H satisfies $H^2 \leq c$ when $n = 2$ and $n^2H^2 < 4(n-1)c$ when $n \geq 3$. Later, Cheng [4] generalized this result to general submanifolds in a de Sitter space.

To our best knowledge, there are almost no intrinsic rigidity results for the space-like submanifolds with constant scalar curvature in a de Sitter space until Zheng [15] obtained the following result.

THEOREM. *Let M^n be an n-dimensional compact space-like hypersurface in $M_1^{n+1}(c)$ with constant scalar curvature. If M^n satisfies*

(1) $K(M) > 0$,

(2) $Ric(M) \leq (n-1)c$,

(3) $R < c$,

where R is the normalized scalar curvature of M^n, then M^n is totally umbilical.

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In [5], Cheng-Yau firstly studied the rigidity problem for a hypersurface with constant scalar curvature in a space form by introducing a self-adjoint second order differential operator (See Theorems 1 and 2 in [5]). They proved that, for an M^n in $M^{n+1}(c)$, if R is constant and $R \geq c$, then $|\nabla\sigma|^2 \geq n^2|\nabla H|^2$ where σ and H denote the second fundamental form and the length of the mean curvature vector field of M^n respectively. By using Cheng-Yau's technique, Li [7] [8] studied the pinching problem and also proved some global rigidity theorems for hypersurfaces with constant scalar curvature.

In the present paper, we would like extend Cheng-Yau's technique to higher codimensional cases and use this result to study the rigidity problem for space-like submanifolds in a de Sitter space with constant scalar curvature.

2. Preliminaries

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional semi-Riemannian manifold of constant curvature c whose index is p. Let M^n be an n-dimensional Riemannian manifold immersed in $M_p^{n+p}(c)$. As the semi-Riemannian metric of $M_p^{n+p}(c)$ induces the Riemannian metric of M^n, M^n is called a space-like submanifold. We choose a local field of semi-Riemannian orthonormal frames $e_1, \ldots, e_{n+p}$ in $M_p^{n+p}(c)$ such that at each point of M^n, $e_1, \ldots, e_n$ span the tangent space of M^n and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+p; \quad 1 \leq i, j, k, \ldots \leq n; \quad n+1 \leq \alpha, \beta, \gamma \leq n+p.$$

Let $\omega_1, \ldots, \omega_{n+p}$ be its dual frame field so that the semi-Riemannian metric of $M_p^{n+p}(c)$ is given by $d\bar{s}^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \epsilon_A \omega_A^2$, where $\epsilon_i = 1$ and $\epsilon_\alpha = -1$. Then the structure equations of $M_p^{n+p}(c)$ are given by

$$d\omega_A = \sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{1}$$

$$d\omega_{AB} = \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D, \tag{2}$$

$$K_{ABCD} = c\epsilon_A\epsilon_B(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{3}$$

Restrict these form to M^n, we have

$$\omega_\alpha = 0, \quad n+1 \leq \alpha \leq n+p, \tag{4}$$

the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. From Cartan's lemma we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{5}$$

From these formulas, we obtain the structure equations of M^n:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{6}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} K_{ijkl} \omega_k \wedge \omega_l, \tag{7}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{8}$$

where R_{ijkl} are the components of the curvature tensor of M^n.

For indefinite Riemannian manifolds in detail, refer to O'Neill [9].

Denote $L_\alpha = (h_{ij}^\alpha)_{n\times n}$ and $H_\alpha = (1/n)\sum_i h_{ii}^\alpha$ for $\alpha = n+1, \cdots, n+p$. Then the mean curvature vector field ξ, the mean curvature H and the square of the length of the second fundamental form S are expressed as

$$\xi = \sum_\alpha H_\alpha e_\alpha, \quad H = |\xi|, \quad S = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2,$$

respectively. Moreover, the normal curvature tensor $\{R_{\alpha\beta kl}\}$, the Ricci curvature tensor $\{R_{ik}\}$ and the normalized scalar curvature R are expressed as

$$\begin{aligned} R_{\alpha\beta kl} &= \sum_m (h_{km}^\alpha h_{ml}^\beta - h_{lm}^\alpha h_{mk}^\beta), \\ R_{ik} &= (n-1)\, c\, \delta_{ik} - n \sum_\alpha (H_\alpha) h_{ik}^\alpha + \sum_{\alpha,j} h_{ij}^\alpha h_{jk}^\alpha, \\ R &= c + \frac{1}{n(n-1)}(S - n^2 H^2). \end{aligned} \tag{9}$$

Define the first and the second covariant derivatives of $\{h_{ij}^\alpha\}$, say $\{h_{ijk}^\alpha\}$ and $\{h_{ijkl}^\alpha\}$ by

$$\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum_k h_{kj}^\alpha \omega_{ki} + \sum_k h_{ik}^\alpha \omega_{kj} + \sum_\beta h_{ij}^\beta \omega_{\beta\alpha}, \tag{10}$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_m h^\alpha_{mjk}\omega_{mi} + \sum_m h^\alpha_{imk}\omega_{mj} + \sum_m h^\alpha_{ijm}\omega_{mk} + \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}. \tag{11}$$

Then, by exterior differentiation of (5), we obtain the Codazzi equation

$$h^\alpha_{ijk} = h^\alpha_{ikj}. \tag{12}$$

It follows from Ricci's identity that

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj}R_{mikl} + \sum_m h^\alpha_{im}R_{mjkl} + \sum_\beta h^\beta_{ij}R_{\beta\alpha kl}. \tag{13}$$

The Laplacian of h^α_{ij} is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. From (13), we have

$$\begin{aligned}\Delta h^\alpha_{ij} &= nH_{\alpha,ij} + \sum_{k,m} h^\alpha_{km}R_{mijk} + \sum_{k,m} h^\alpha_{im}R_{mkjk} + \sum_{k,\beta} h^\beta_{ik}R_{\beta\alpha jk} \\ &= nH_{\alpha,ij} + n\,c\,h^\alpha_{ij} - n\,c\,H_\alpha\delta_{ij} - n\sum_{\beta,m} H_\beta h^\alpha_{im}h^\beta_{mj} + \sum_\beta S_{\alpha\beta}h^\beta_{ij} \\ &\quad -2\sum_{\beta,k,m} h^\beta_{ik}h^\alpha_{km}h^\beta_{mj} + \sum_{m,k,\beta} h^\alpha_{im}h^\beta_{mk}h^\beta_{kj} + \sum_{\beta,k,m} h^\beta_{ik}h^\beta_{km}h^\alpha_{mj},\end{aligned}$$

where $S_{\alpha\beta} = \sum_{i,j} h^\alpha_{ij}h^\beta_{ij}$ for all α and β. Define $N(A) = \sum_{i,j} a^2_{ij}$ for any real matrix $A = (a_{ij})_{n\times n}$. Then we have

$$\begin{aligned}\sum_{i,j} h^\alpha_{ij}\Delta h^\alpha_{ij} &= n\sum_{i,j} H_{\alpha,ij}h^\alpha_{ij} + n\,c\,S_\alpha - c\,n^2H^2_\alpha - n\sum_\beta H_\beta Tr(L^2_\alpha L_\beta) \\ &\quad + \sum_\beta S^2_{\alpha\beta} + \sum_\beta N(L_\alpha L_\beta - L_\beta L_\alpha),\end{aligned} \tag{14}$$

where $S_\alpha = \sum_{i,j}(h^\alpha_{ij})^2$, for every α.

Suppose $H > 0$ on M^n and choose $e_{n+1} = \xi/H$. Then it follows that

$$H_{n+1} = H; \quad H_\alpha = 0, \qquad \alpha > n+1. \tag{15}$$

From (10) and (15) we can see

$$H_{n+1,k}\omega_k = dH, \quad H_{\alpha,k}\omega_k = H\omega_{n+1\alpha} \qquad \alpha > n+1. \tag{16}$$

From (11), (15) and (16) we have

$$H_{n+1,kl} = H_{kl} - \frac{1}{H}\sum_{\beta>n+1} H_{\beta,k}H_{\beta,l}, \tag{17}$$

where $dH = \sum_i H_i\omega_i$ and $\nabla H_k = \sum_l H_{kl}\omega_l \equiv dH_k + H_l\omega_{lk}$ for all k.

Using (14) and (17), we have

$$\begin{aligned}\sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} &= n\sum_{i,j} H_{ij}h_{ij}^{n+1} - \frac{n}{H}\sum_{i,j}\sum_{\beta>n+1} H_{\beta,i}H_{\beta,j}h_{ij}^{n+1} \\ &\quad + n\,c\,S_{n+1} - c\,n^2H^2 - nHf_{n+1} + S_{n+1}^2 + \sum_{\beta>n+1} S_{n+1\beta}^2 \\ &\quad + \sum_{\beta>n+1} N(L_{n+1}L_\beta - L_\beta L_{n+1}), \qquad (18)\end{aligned}$$

where $f_{n+1} = Tr(L_{n+1})^3$.

M. Okumura [10] established the following lemma (see also [2]).

LEMMA 2.1. *Let $\{a_i\}_{i=1}^n$ be a set of real numbers satisfying $\sum_i a_i = 0$, $\sum_i a_i^2 = t^2$, where $t \geq 0$. Then we have*

$$-\frac{n-2}{\sqrt{n(n-1)}}t^3 \leq \sum_i a_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}t^3,$$

and the equalities hold if and only if at least $(n-1)$ of the a_i are equal.

Denote the eigenvalues of L_{n+1} by $\{\lambda_i\}_{i=1}^n$. Then we have

$$nH = \sum_i \lambda_i, \quad S_{n+1} = \sum_i \lambda_i^2, \quad f_{n+1} = \sum_i \lambda_i^3. \qquad (19)$$

Set $\bar{L}_{n+1} = L_{n+1} - H\,I_n$, $\bar{f}_{n+1} = f_{n+1} - 3HS_{n+1} + 2nH^3$, $\bar{S}_{n+1} = S_{n+1} - nH^2$, and $\bar{\lambda}_i = \lambda_i - H$, where I_n denotes the identity matrix of degree n. Then (19) changes into

$$0 = \sum_i \bar{\lambda}_i, \quad \bar{S}_{n+1} = \sum_i \bar{\lambda}_i^2, \quad \bar{f}_{n+1} = \sum_i \bar{\lambda}_i^3. \qquad (20)$$

By applying Okumura's Lemma to $\bar{f}_{n+1}$, we have

$$\bar{f}_{n+1} \leq \frac{n-2}{\sqrt{n(n-1)}}\bar{S}_{n+1}\sqrt{\bar{S}_{n+1}} \iff$$

$$f_{n+1} \leq 3HS_{n+1} - 2nH^3 + \frac{n-2}{\sqrt{n(n-1)}}\bar{S}_{n+1}\sqrt{\bar{S}_{n+1}}.$$

So we have

$$\begin{aligned} & n\,c\,S_{n+1} - c\,n^2H^2 - nHf_{n+1} + S_{n+1}^2 \\ \geq\ & \bar{S}_{n+1}\{n\,c + \bar{S}_{n+1} - nH^2 - n(n-2)H\sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}}\}. \qquad (21)\end{aligned}$$

It follows from (15) that

$$\sum_{\beta>n+1} S_{n+1\beta}^2 = \sum_{\beta>n+1} \{\sum_{i,j}(h_{ij}^{n+1} - H\delta_{ij})h_{ij}^{\beta}\}^2. \tag{22}$$

Denote $S_I = \sum_{\beta>n+1} S_\beta$. From (22), we have

$$\sum_{\beta>n+1} S_{n+1\beta}^2 \leq \bar{S}_{n+1} S_I. \tag{23}$$

Let $T = \sum_{i,j} T_{ij}\omega_i\omega_j$ be a symmetric tensor on M^n defined by

$$T_{ij} = h_{ij}^{n+1} - nH\delta_{ij}. \tag{24}$$

We introduce an operator $\Box$ associated to T acting on $f \in C^2(M^n)$ by

$$\Box f = \sum_{i,j} T_{ij} f_{ij} = \sum_{i,j} h_{ij}^{n+1} f_{ij} - nH\Delta f,$$

where Δ is the Laplacian. Since (T_{ij}) is divergence-free, it follows from [5] that the operator $\Box$ is self-adjoint relative to the L^2-inner product of M^n.

Choosing $f = H$ in above expression, we have

$$\sum_{i,j} h_{ij}^{n+1} H_{ij} = \Box H + nH\Delta H. \tag{25}$$

Denote $\bar{S} = \bar{S}_{n+1} + S_I$. Substituting (21), (23) and (25) into (18), we get

$$\begin{aligned} \sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} \geq\ & n\Box H + \frac{1}{2}n^2\Delta(H^2) - n^2|\nabla H|^2 \\ & -\frac{n}{H}\sum_{\beta>n+1}\sum_{i,j} H_{\beta,i}H_{\beta,j}h_{ij}^{n+1} \\ & + \sum_{\beta>n+1} N(L_{n+1}L_\beta - L_\beta L_{n+1}) \\ & + \bar{S}_{n+1}\left\{n\,c - nH^2 + \bar{S}_{n+1} - n(n-2)H\sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}}\right\}. \end{aligned} \tag{26}$$

3. An extension of Cheng-Yau's technique

Cheng-Yau [5] gave a lower estimation for $|\nabla\sigma|^2$, the square of the length of the covariant derivative of σ, which plays an important role in their discussion. They proved that, for a hypersurface in a space form of constant scalar curvature c, if the normalized scalar curvature R is constant and $R \geq c$, then $|\nabla\sigma|^2 \geq n^2|\nabla H|^2$.

For the space-like submanifolds in a de Sitter space, we can prove the following

THEOREM 3.1. *Let M^n be a connected submanifold in $M_p^{n+p}(c)$ with nowhere zero mean curvature H. If R is constant and $R < c$, then*

$$|\nabla\sigma|^2 = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 \geq n^2|\nabla H|^2 \tag{27}$$

and the symmetric tensor T defined by (24) is negative semi-definite. Moreover, if the equality in (27) holds on M^n, then H is constant and T is negative definite.

Proof. From (9), we have $n^2H^2 - S = n(n-1)(c-R) > 0$. Taking the covariant derivative on both sides of this equality, we get

$$n^2H\,H_k = \sum_{i,j,\alpha} h_{ij}^\alpha\, h_{ijk}^\alpha, \quad k = 1, \cdots, n.$$

For every k, it follows from Cauchy-Schwarz's inequality that

$$n^4H^2H_k^2 = (\sum_{i,j,\alpha} h_{ij}^\alpha\, h_{ijk}^\alpha)^2 \leq S\sum_{i,j,\alpha}(h_{ijk}^\alpha)^2, \tag{28}$$

where the equality holds if and only if there exits a real function c_k such that

$$h_{ijk}^\alpha = c_k\, h_{ij}^\alpha \tag{29}$$

for all i, j and α. Taking sum on both sides of (28) with respect to k, we have

$$n^4H^2|\nabla H|^2 = n^4H^2\sum_k H_k^2 \leq S \sum_{(i,j,k,\alpha)} (h_{ijk}^\alpha)^2 \leq n^2H^2 \sum_{(i,j,k,\alpha)} (h_{ijk}^\alpha)^2. \tag{30}$$

Therefore (27) holds on M^n.

Denote the eigenvalues of L_{n+1} by $\{\lambda_i\}_{i=1}^n$. Then $(\lambda_i)^2 \leq S_{n+1} \leq S \leq n^2H^2$ for all i. Hence $|\lambda_i| \leq nH$ for all i. Therefore $T = (T_{ij}) = L_{n+1} - nHI_n$ is negative semi-definite.

Suppose that $\sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2 = n^2|\nabla H|^2$ holds on M^n. It follows from (30) that

$$0 \leq n^3(n-1)(c-R)|\nabla H|^2 \leq S\left(\sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2 - n^2|\nabla H|^2\right). \tag{31}$$

Hence $(c-R)|\nabla H|^2 = 0$ on M^n. Because $R < c$, $|\nabla H|^2 = 0$ on M^n. In this case, $|\lambda_i| \leq (S_{n+1})^{1/2} \leq S^{1/2} < nH$ for all i. Thus T is negative definite. This completes the proof of Theorem 3.1. □

4. Submanifolds with flat normal bundle

In this section, we propose to use the extension of Cheng-Yau's technique given in section 3 to study the rigidity problem for compact submanifolds in the de sitter space $M_p^{n+p}(c)$. We continue use the same notations as in section 2. Let M^n be a compact submanifold in $M_p^{n+p}(c)$ with nowhere zero mean curvature H. Suppose that ξ/H is parallel and choose $e_{n+1} = \xi/H$. Then $\omega_{n+1\alpha} = 0$ for all α. It follows from (11) and (16) that

$$H_{\alpha,k} = 0, \quad H_{\alpha,kl} = 0, \tag{32}$$

for all $\alpha > n+1$ and $k, l = 1, \cdots, n$.

Suppose in addition that the normal bundle of M^n is flat. Then

$$\Omega_{\alpha\beta} = -\frac{1}{2} R_{\alpha\beta kl}\omega_k \wedge \omega_l = 0, \tag{33}$$

for all α and β on M^n. For all α and β we have $L_\alpha L_\beta = L_\beta L_\alpha$, which is equivalent to that $\{L_\alpha\}_{\alpha=n+1}^{n+p}$ can be diagonized simultaneously.

We denote the eigenvalues of L_α by $\{\lambda_1^\alpha, \cdots, \lambda_n^\alpha\}$ for every α. It follows from [13] that

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + n\sum_{i,j,\alpha} H_{\alpha,ij} h_{ij}^\alpha + \sum_\alpha \sum_{i<j} K_{ij}(\lambda_i^\alpha - \lambda_j^\alpha)^2, \tag{34}$$

where $K_{ij} = c + \sum_\beta \lambda_i^\beta \lambda_j^\beta$ denotes the sectional curvature of M^n corresponding to the plane section spanned by $\{e_i, e_j\}$ for every pair of $i < j$.

Assume that R is constant and $R < c$. From (25) and (32), we have

$$\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + n\sum_{i,j,\alpha} H_{\alpha,ij} h_{ij}^\alpha = n\Box H + \frac{1}{2}\Delta(n^2H^2) + \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - n^2|\nabla H|^2.$$

Note that $\Delta S = \Delta(n^2H^2)$. Therefore (34) turns into

$$0 = n\Box H + \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - n^2|\nabla H|^2 + \sum_{i<j}\sum_\alpha K_{ij}(\lambda_i^\alpha - \lambda_j^\alpha)^2.$$

Integrating the both sides of above equality on M^n, we have

$$0 = \int_M \left(\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - n^2|\nabla H|^2 \right) * 1 + \sum_{i<j}\sum_\alpha \int_M K_{ij}(\lambda_i^\alpha - \lambda_j^\alpha)^2 * 1.$$

If $K_{ij} \geq 0$ on M^n, it follows from (27) and the above equality that

$$\sum_{(i,j,k,\alpha)} (h_{ijk}^\alpha)^2 \equiv n^2|\nabla H|^2; \qquad K_{ij}(\lambda_i^\alpha - \lambda_j^\alpha)^2 \equiv 0, \tag{35}$$

for every α and $i < j$. Hence we can prove the following theorem

THEOREM 4.1. *Let M^n be a compact submanifold with non-negative sectional curvature in $M_p^{n+p}(c)$. Suppose that the normal bundle $N(M)$ is flat and the normalized mean curvature vector is parallel. If R is constant and $R < c$, then M^n is totally umbilical.*

Proof. From the first equality of (35) and Theorem 3.1, we have that H is constant on M^n, then ξ is parallel. From Theorem 3 of [1] we know that M^n is totally umbilical. □

REMARK 4.1. In Theorem 4.1, we have used the assumptions that are different from that in Theorem 3 [1] to obtain the same result.

Also, we need the following

LEMMA 4.1 [12]. *Let A and B be $n \times n$-symmetric matrices satisfying $Tr\,A = 0$, $Tr\,B = 0$ and $AB - BA = 0$. Then*

$$(36)\quad -\frac{n-2}{\sqrt{n(n-1)}}(Tr\,A^2)(Tr\,B^2)^{1/2} \leq Tr\,A^2B \leq \frac{n-2}{\sqrt{n(n-1)}}(Tr\,A^2)(Tr\,B^2)^{1/2},$$

and the equality holds on the right (resp. left) hand side if and only if $n-1$ of the eigenvalues x_i of A and the corresponding eigenvalues y_i of B satisfy $|x_i| = \frac{(Tr\,A^2)^{1/2}}{\sqrt{n(n-1)}}$, $x_i x_j \geq 0$, $y_i = -\frac{(Tr\,B^2)^{1/2}}{\sqrt{n(n-1)}}$ *(resp.* $y_i = \frac{(Tr\,B^2)^{1/2}}{\sqrt{n(n-1)}}$*).*

Choose a suitable normal frame field $\{e_\beta\}_{\beta=n+2}^{n+p}$ such that $S_{\alpha\beta} = 0$ for all $\alpha \neq \beta$. Then

$$(37)\quad \sum_{\alpha,\beta>n+1} S_{\alpha\beta}^2 = \sum_{\beta>n+1} S_\beta^2 \leq S_I^2,$$

where the equality holds if and only if at least $p-2$ numbers of S_α's are zero.

Taking sum with respect to $\alpha > n+1$ on both-sides of (14), we have

$$(38)\quad \begin{aligned} \sum_{i,j,\alpha>n+1} h_{ij}^\alpha \Delta h_{ij}^\alpha &= (nc - nH^2)S_I - nH \sum_{\alpha>n+1} Tr(L_\alpha^2 \bar{L}_{n+1}) \\ &\quad + \sum_{\alpha>n+1} S_{n+1\alpha}^2 + \sum_{\alpha>n+1} S_\alpha^2. \end{aligned}$$

Using the left hand side of (36) to $Tr(L_\alpha^2 \bar{L}_{n+1})$, we have

$$Tr(L_\alpha^2 \bar{L}_{n+1}) \leq (n-2)S_\alpha \sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}}.$$

Substituting this into (38) and using (23) and (37), we have
(39)

$$\sum_{i,j,\alpha>n+1} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \geq S_I\left\{(n\,c-n\,H^2)-n(n-2)H\sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}}+\bar{S}_{n+1}\right\}.$$

Substituting (32) into (26), we have

$$\begin{aligned}\sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} &\geq n\Box H+\frac{1}{2}\Delta(n^2H^2)-n^2|\nabla H|^2\\ &+\bar{S}_{n+1}\left\{(n\,c-nH^2)-n(n-2)H\sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}}+\bar{S}_{n+1}\right\}.\end{aligned} \tag{40}$$

Note that $\Delta S=\Delta(n^2H^2)$ and

$$\frac{1}{2}\Delta S=\sum_{i,j,k,\alpha}\left(h_{ijk}^{\alpha}\right)^2+\sum_{i,j}h_{ij}^{n+1}\Delta h_{ij}^{n+1}+\sum_{i,j,\alpha>n+1}h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}. \tag{41}$$

From (39) and (40), we obtain

$$\begin{aligned}0 &\geq n\Box H+\sum_{(i,j,k,\alpha)}\left(h_{ijk}^{\alpha}\right)^2-n^2|\nabla H|^2\\ &+\bar{S}\left\{(n\,c-nH^2)-n(n-2)H\sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}}+\bar{S}_{n+1}\right\}.\end{aligned} \tag{42}$$

Consider the quadratic form $Q(u,t)=u^2-\frac{n-2}{\sqrt{n-1}}ut-t^2$. By the orthogonal transformation

$$\begin{cases}\bar{u}=\frac{1}{\sqrt{2n}}\{(1+\sqrt{n-1})u+(1-\sqrt{n-1})t\}\\ \bar{t}=\frac{1}{\sqrt{2n}}\{(\sqrt{n-1}-1)u+(\sqrt{n-1}+1)t\}\end{cases}$$

$Q(u,t)$ turns into $Q(u,t)=\frac{n}{2\sqrt{n-1}}(\bar{u}^2-\bar{t}^2)$, where $\bar{u}^2+\bar{t}^2=u^2+t^2$. Take $u=\sqrt{\bar{S}_{n+1}}$, $t=\sqrt{n}H$, then

$$\begin{aligned}&nc-nH^2-n(n-2)H\sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}}+\bar{S}_{n+1}\\ &=nc+Q(u,t)=nc+\frac{n(\bar{u}^2-\bar{t}^2)}{2\sqrt{n-1}}\\ &=nc+\frac{n(-\bar{u}^2-\bar{t}^2)}{2\sqrt{n-1}}+\frac{n\bar{u}^2}{\sqrt{n-1}}\\ &\geq nc-\frac{n\bar{S}_{n+1}}{2\sqrt{n-1}}\end{aligned} \tag{43}$$

Note that

$$\bar{S}_{n+1} \leq \bar{S}_{n+1} + S_I = \bar{S}. \tag{44}$$

From (43), (44) and (27) we have

$$0 \geq n\Box H + \bar{S}\left\{nc - \frac{n\bar{S}}{2\sqrt{n-1}}\right\}. \tag{45}$$

Integrating the both sides of (45) on M^n, we have

$$0 \geq \int_M \bar{S}\left\{nc - \frac{n\bar{S}}{2\sqrt{n-1}}\right\} * 1. \tag{46}$$

Therefore we can prove the following

THEOREM 4.2. *Let M^n $(n \geq 3)$ be a closed space-like submanifold with parallel normalized mean curvature vector field immersed into $M_p^{n+p}(c)$. Suppose that R is constant and $\bar{R} = c - R > 0$. If the normal bundle $N(M)$ is flat and*

$$S < nH^2 + 2\sqrt{n-1}c, \tag{47}$$

then $S = nH^2$ and M^n is umbilical (hence isometric to a sphere).

Proof. Denote $\bar{R} = c - R$. Then $\bar{S} = n(n-1)(H^2 - \bar{R})$ and $S = n\bar{R} + n^2(H^2 - \bar{R})$. Since $n \geq 3$, we have

$$nc - \frac{n\bar{S}}{2\sqrt{n-1}} = n(c - \frac{n(n-1)(H^2 - \bar{R}}{2\sqrt{n-1}}) = n(c - \frac{S - nH^2}{2\sqrt{n-1}}). \tag{48}$$

It is clear that the condition (47) is equivalent to

$$nc - \frac{\bar{S}}{2\sqrt{n-1}} > 0. \tag{49}$$

From (46) and (49) we have $\bar{S} = 0$ on M^n, so $H^2 = \bar{R}$ and $S = n\bar{R}$, that is $S = nH^2$. Since H is constant on M^n, hence ξ is parallel, from Theorem 3 of [1] we know that M^n is totally umbilical. □

References

[1] R. Aiyama, *Compact space-like submanifolds in a pseudo-Riemannian sphere $S_p^{m+p}(c)$*, Tokyo J. Math. **18** (1995), 81–90.

[2] H. Alencar and M. P. do Carmo, *Hypersurfaces with constant mean curvature in spheres*, Proc. Amer. Math. Soc. **120** (1994), 1223–1229.

[3] K. Akutagawa, *On space-like hypersurfaces with constant mean curvature in the de Sitter space*, Math. Z. **196** (1987), 13–19.

[4] Q. M. Cheng, *Complete space-like submanifolds in a de Sitter space with parallel mean curvature vector*, Math. Z. **206** (1991), 333–339.

[5] S. Y. Cheng and S. T. Yau, *Hypersurfaces with constant scalar curvature*, Math. Ann. **225** (1977), 195–204.
[6] Z. H. Hou, *Hypersurfaces in sphere with constant mean curvature*, Proc. Amer. Math. Soc. **125** (1997), 1193–1196.
[7] H. Li, *Hypersurfaces with constant scalar curvature in space forms*, Math. Ann. **305** (1996), 665–672.
[8] ———, *Global rigidity theorems of Hypersurfaces*, Ark. Mat. **35** (1997), 327–351.
[9] B. O'Neill, *Semi-Riemannian Geometry*, New York, Academic Press, 1983.
[10] M. Okumuru, *Hypersurfaces and a pinching problem on the second fundamental tensor*, Amer. J. Math. **96** (1974), 207–213.
[11] J. Ramanathan, *Complete space-like hypersurfaces of constant mean curvature in the de Sitter space*, Indiana Univ. Math. J. **36** (1987), 349-35.
[12] W. Santos, *Submanifolds with parallel mean curvature vector in spheres*, Tôhoku Math. J. **46** (1994), 403–415.
[13] K. Yano and S. Ishihara, *Submanifolds with parallel mean curvature vector*, J. Diff. Geom. **6** (1971), 95–118.
[14] S. T. Yau, *Submanifolds with constant mean curvature*, Amer. J. Math. **96** (1975), 346–366; **97** (1975), 76–100.
[15] Y. Zheng, *Space-like hypersurfaces with constant scalar curvature in the de Sitter spaces*, Diff. Goem. and its Appl. **6** (1996), 51–54.

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