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COMPLETE SYSTEM OF FINITE ORDER FOR CR MAPPINGS BETWEEN REAL ANALYTIC HYPERSURFACES OF DEGENERATE LEVI FORM

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ABSTRACT. We prove that the germ of a CR mapping f between real analytic real hypersurfaces has a holomorphic extension and satisfies a complete system of finite order if the source is of finite type in the sense of Bloom-Graham and the target is k-nondegenerate under certain generic assumptions on f.

Introduction

This paper is concerned with construction of a complete system for CR mappings and with the real analyticity and the finiteness of CR mappings between real analytic CR manifolds of degenerate Levi form.

Let M and M' be germs of real analytic(C^ω) real hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, $1 \le n \le N$, respectively, and $F = (f^1, \cdots, f^{N+1}) : M \to M'$ be a continuously differentiable CR mapping. Then F is a solution of an overdetermined system

$$(1) \qquad \begin{cases} \overline{L}_i f^j = 0 \quad i = 1, ..., n \ , \ j = 1, ..., N+1 \\ r' \circ F = 0 \end{cases},$$

where $\{L_i\}_{i=1,\dots,n}$ is a basis of the CR structure bundle $H^{1,0}(M) := T^{1,0}(\mathbb{C}^{n+1}) \cap \mathbb{C}T(M)$ of M and r' is a C^ω defining function of M'.

It is well known that if M and M' are Levi-nondegenerate hypersurfaces in $\mathbb{C}^{n+1}$ and $F : M \to M'$ is a CR equivalence, then F extends holomorphically to a neighborhood of M([12], [14], [16]).

Moreover, F is determined by 2-jet at a point. This follows from the fact that F preserves the complete set of Chern-Moser invariants and thus

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F satisfies the complete system of third order in the sense of Definition 5, see [5] and [7].

Let r be a C^ω defining function of M such that $dr \neq 0$ on M and let $\{L_j\}_{j=1,2,...,n}$ be a C^ω basis of $H^{1,0}(M)$. For an n-tuple of integers $\alpha = (\alpha_1, ..., \alpha_n)$ let $L^\alpha := L_1^{\alpha_1} \cdots L_n^{\alpha_n}$. We say that M is k-nondegenerate at $p \in M$ if the vectors $\{\overline{L}^\alpha r_Z(p) : |\alpha| \leq k\}$ span $\mathbb{C}^{n+1}$, where $r_Z = \left(\dfrac{\partial r}{\partial z_1}, ..., \dfrac{\partial r}{\partial z_{n+1}}\right)$.

The smallest such integer k does not depend on the choice of the basis $L_1, ..., L_n$ and the defining function r. M is 1-nondegenerate at p if and only if M is of nondegenerate Levi form at p.

In this paper we study the analyticity and finite determination of CR mappings to C^ω hypersurface which is k-nondegenerate at a reference point. Our main results are the following:

THEOREM 1. *Let M and M' be C^ω real hypersurfaces through the origin of $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, $1 \leq n \leq N$, respectively, and let $F : M \to M'$ be a CR mapping such that $F(0) = 0$. Let $\{L_j\}_{j=1,...,n}$ be a C^ω basis of $H^{1,0}(M)$. Suppose that M is of finite type at 0 in the sense of Bloom-Graham and M' is k-nondegenerate at 0. Suppose further that there exists a positive integer K such that*

$$\left\{L^\gamma \left(r'_{\overline{Z}} \circ F\right)(0) : |\gamma| \leq K\right\} \tag{2}$$

span $\mathbb{C}^{N+1}$. Then F extends holomorphically to a neighborhood of $0 \in M$ if $F \in C^K$.

THEOREM 2. *Let M and M' be C^ω real hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, $1 \leq n \leq N$, respectively as in Theorem 1 and let $F : M \to M'$ be a CR mapping as in Theorem 1. Then F is determined by $4K$-jet at 0. Moreover, F satisfies a complete system of order $4K + 1$.*

If M and M' are of same dimension and k-nondegenerate, then a CR equivalence F between M and M' extends holomorphically to a neighborhood of M if F is sufficiently differentiable([6],[2]) and is determined by $(k^2 + k)$-jet at a point([7]). A basic idea in [7] is to construct, by differentiating (1) repeatedly, a complete system of finite order, which determines all the derivatives of F of order greater than or equal to $k^2 + k + 1$. More recently Zaitsev showed that F is determined by $4k$-jet at a point by using the Segre varieties([17]).

Suppose M and M' are in normal coordinates at 0(see §2). Then (2) span $\mathbb{C}^{N+1}$ if and only if the image

$$\left\{\left(a_1^K(z), \cdots, a_N^K(z)\right) : z \in \mathbb{C}^n\right\} \tag{3}$$

is not contained in a hyperplane of $\mathbb{C}^N$, where a_j^K, $j = 1, ..., N$, are K-th order Taylor series expansion of $\frac{\partial r'}{\partial \overline{z}_j}\left(F(z,0), \overline{F(0)}\right)$, $j = 1, ..., N$.

In [10], Hayashimoto showed using the method of complete system that if M and M' are real hypersurfaces in $\mathbb{C}^{n+1}$ and if M' is of nondegenerate Levi form, then F extends holomorphically to a neighborhood of M and is determined by a finite jet at a point under the condition that the image

$$\left\{\left(a_1^K(z), \cdots, a_n^K(z)\right) : z \in \mathbb{C}^n\right\} \tag{4}$$

is not contained in a hyperplane of $\mathbb{C}^n$, which is equivalent to our hypotheses in Theorem 1.

In [2], Baouendi, Jacobowitz and Treves replace the holomorphic structure on a neighborhood of M by a new one whose real analytic structure is the same as the standard one. Then they extend each f^j as a collection of holomorphic functions(in one variable in the case of hypersurface) to a wedge with edge M using some identity that involves CR vector fields and a defining function of M. By the edge of the wedge theorem F is real analytic on M and hence extends holomorphically to a neighborhood of M under the original holomorphic structure.

In this paper, we express F in terms of the derivatives of $\overline{F}$ on M. We use this identity to prove Theorem 1 by the same argument as in §3 of [2]. To prove Theorem 2 we use the method of Segre variety as in [15], [1] and [17].

Holomorphic continuation of a CR mapping to a neighborhood of C^ω CR submanifold has been studied by many authors. In [3], Baouendi and Rothschild showed the holomorphic continuation of a CR mapping between C^ω real hypersurfaces of same dimension under certain nondegeneracy conditions.

To state their result we fix notations and definitions first:

Let $M = \{r = 0\} \subset \mathbb{C}^{n+1}$ be in normal coordinates. We can write $r\left((z,0),(\overline{z},0)\right) = \sum_\alpha a_\alpha(z)\overline{z}^\alpha$, where $z \in \mathbb{C}^n$. Then M is said to be essentially finite at 0 if the $\mathbb{C}$-vector space $\mathcal{O}[z]/\left(a_\alpha(z)\right)$ is of finite dimension, where $\left(a_\alpha(z)\right)$ is the ideal generated by $\{a_\alpha(z)\}$ in $\mathcal{O}[z]$. The essential type of M at 0 is the dimension of the complex vector space $\mathcal{O}[z]/\left(a_\alpha(z)\right)$.

Suppose that $F : M \to M'$ is a C^K, $K \in \mathbb{N}\cup\{\infty\}$, CR mapping between C^∞ real hypersurfaces in $\mathbb{C}^{n+1}$. Then there exists a (formal) holomorphic change of coordinates on a neighborhood of M such that $F = J(Z) + O(|Z|^{K+1})$ if $K < \infty$ and $F = J(Z) + O(|Z|^{l+1})$ for all l if $K = \infty$, where

$Z = (z, z_{n+1}) \in \mathbb{C}^{n+1}$ and $J(Z) = (j_1(Z), ..., j_{n+1}(Z))$ is an $(n+1)$-tuple of (formal) holomorphic functions in Z. We say that F is of finite multiplicity at 0 if $\mathcal{O}[z]/(J(z,0))$ is of finite dimension. The multiplicity of F at 0 is defined by the dimension of the complex vector space $\mathcal{O}[z]/(J(z,0))$.

THEOREM 3. ([3]) *Let $F : M \to M'$ be a smooth CR mapping, where M and M' are C^ω hypersurfaces in $\mathbb{C}^{n+1}$. Let $0 \in M$ and $F(0) = 0$. If either one of the following two conditions is satisfied, then F is the restriction of a holomorphic mapping from a neighborhood of 0 in $\mathbb{C}^{n+1}$ into $\mathbb{C}^{n+1}$.*

i) *The mapping H is of finite multiplicity at 0, and M' is essentially finite at 0.*
ii) *M is essentially finite at 0 and F satisfies*

$$dF(\mathbb{C}T_0M) \not\subseteq H_0^{1,0}(M') \oplus H_0^{0,1}(M') \quad (\textit{Hopf Lemma property}).$$

From Theorem 1 and Theorem 2 we have the following

COROLLARY 4. *Let $F : M \to M'$ be a CR mapping, where M and M' are C^ω hypersurfaces in $\mathbb{C}^{n+1}$. Let $F(0) = 0$. Suppose M' is k-nondegenerate at 0. Then F satisfies a complete system of finite order if one of the following conditions is satisfied:*

i) *The mapping F is of finite multiplicity at 0.*
ii) *M is essentially finite at 0 and F satisfies*

$$dF(\mathbb{C}T_0M) \not\subseteq H_0^{1,0}(M') \oplus H_0^{0,1}(M'). \tag{5}$$

In case i) F satisfies a complete system of order $4k \cdot (\text{mult } F_0) + 1$ and in case ii) F satisfies a complete system of order $4k \cdot (\text{ess type } M_0) + 1$, where $(\text{mult } F_0)$ is the multiplicity of F at 0 and $(\text{ess type } M_0)$ is the essential type of M at 0.

After finishing this paper, the author was informed of the B. Lamel's result[11], in which he proved the real analyticity of F in Theorem 1 in more general situation(generic CR manifolds) using ideas similar to ours.

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1. E. Cartan's equivalence problem and the complete systems

In this section we briefly explain E. Cartan's equivalence problem and the notion of complete system.

For a C^∞ manifold M with a geometric structure, construct a principal fiber bundle P with the structure group G over M such that any structure preserving map f lifts to $\widetilde{f}$ for which the following diagram commutes:

$$\begin{array}{ccc} P_1 & \xrightarrow{\widetilde{f}} & P_2 \\ {\scriptstyle \pi_{M_1}}\downarrow & & {\scriptstyle \pi_{M_2}}\downarrow \\ M_1 & \xrightarrow{f} & M_2 \end{array} . \tag{6}$$

E. Cartan's equivalence problem is to find necessary and sufficient conditions for the existence of $\widetilde{f}$.

Suppose there exists a unique torsion-free connection ω on M. Then there is a unique vector-valued 1-form

$$\varpi : T(P) \to \mathbb{R}^K \tag{7}$$

which is an isomorphism at each point ,where $K = dimM + dimG = dimP$, such that there exists a local structure preserving map $f : M_1 \to M_2$ if and only if $\widetilde{f}_*(\varpi_2) = \varpi_1$. Such ϖ is called a complete set of invariants for the equivalence problem. In this case, f satisfies

$$\frac{\partial^2 f^a}{\partial x^i \partial x^j} = h^a_{ij}\left(x, f, \frac{\partial f^b}{\partial x^k} : b, k = 1, ..., n\right) \tag{8}$$

for all $i, j = 1, ..., n$, where h^a_{ij} is a C^∞ function in its arguments.

The concept of complete system is the generalization of the equation (3). We define the notion of complete system in jet-theoretical setting using the same notations as in [13].

Let $J^q(M, \mathbb{R}^N)$ be the q-th order jet space of $M \times \mathbb{R}^N$. Consider a system of differential equations of order q for unknown functions $u = (u^1, \cdots, u^N)$ of independent variables $x = (x^1, \cdots, x^n)$

$$\Delta_\lambda(x, u^{(q)}) = 0, \ \lambda = 1, ..., l, \tag{9}$$

where $u^{(q)}$ is the q-th jet of u.

A complete system of order k is defined as follows.

Definition 5. We say that (4) satisfies a complete system of order k if there exist C^∞ functions $H^a_J(x, u^{(p)} : p < k)$ in their arguments such that for any C^k solution u of (4),

$$u^a_J = H^a_J(x, u^{(p)} : p < k) \tag{10}$$

for all $a = 1, ..., N$ and for all multi-indices J with $|J| = k$.

Let $\phi_I^a = du_I^a - \sum_{j=1}^n u_{I,j}^a dx^j$, $a = 1, ..., N$, $|I| \leq k-2$, be the contact 1-forms defined on $J^{k-1}(M, \mathbb{R}^N)$ and $\mathcal{S}_\Delta \subsetneqq J^{k-1}(M, \mathbb{R}^N)$ be the zero set of (4) and the derivatives of (4) in the space of partial derivatives of u up to order $k-1$. If (4) satisfies a complete system of order k, then f is a solution of (4) if and only if $x \to (\frac{\partial^{|I|} f}{\partial x^I}(x), |I| \leq k-1)$ is a maximal integral manifold of the distribution

$$\phi_I^a = 0 \ , \ a = 1, ..., N \ , \ |I| \leq k-2$$

and

$$du_I^a - \sum_{j=1}^n H_{I,j}^a dx^j = 0 \ , \ |I| = k-1,$$

where $H_{I,j}^a = D_j H_I^a$. In particular, we have

PROPOSITION 6. *Suppose (4) satisfies a complete system of order k, then a solution f of (4) is uniquely determined by $(k-1)$-jet at a point and is C^∞ if $f \in C^k$. Furthermore, if (4) is C^ω, then each H_J^a is C^ω and $f \in C^\omega$.*

2. Proof of theorems and corollary

Let M, M' and F be as in Theorem 1.

In this section we use $\alpha, \beta, \gamma, \cdots$ for n-tuples of integers and $\alpha', \beta', \gamma' \cdots$ for N-tuples of integers.

We say that M is in normal coordinates if M is defined by

$$z_{n+1} = R(z, \overline{z}) + \overline{z}_{n+1} P(z, \overline{z}, \overline{z}_{n+1}) \tag{11}$$

where $z \in \mathbb{C}^n$ and R, P are holomorphic in their arguments such that

$$R(z, 0) \equiv R(0, \overline{z}) \equiv 0$$

and

$$P(z, 0, \overline{z}_{n+1}) \equiv P(0, \overline{z}, \overline{z}_{n+1}) \equiv 1. \text{ ([3])}$$

Since the smallest integer K which satisfies the hypotheses of Theorem 1 is independent of choice of $\{L_i\}_{i=1,...,n}$ and defining function r', we may assume that M and M' are in normal coordinates.

Now assume that M' is defined by

$$\zeta_{N+1} = R'(\zeta, \overline{\zeta}) + \overline{\zeta}_{N+1} P'(\zeta, \overline{\zeta}, \overline{\zeta}_{N+1}), \tag{12}$$

where $\zeta \in \mathbb{C}^N$. Write

$$R'(\zeta,\overline{\zeta}) = \sum_{j=1}^{N} a_j(\overline{\zeta})\zeta_j + \sum_{|\alpha'|\geq 2} a_{\alpha'}(\overline{\zeta})\zeta^{\alpha'}. \tag{13}$$

LEMMA 7. *There exist Φ_j, $j = 1, ..., N+1$, which are holomorphic in their arguments such that*

$$f^j = \Phi_j(\overline{L}^\gamma \overline{F}, |\gamma| \leq K) \tag{14}$$

for all $j = 1, ..., N+1$.

Proof. Let $F = (f, g) = (f^1, \cdots, f^N, g)$. Then we have

$$g = \sum_{j=1}^{N} a_j(\overline{f})f^j + \sum_{|\alpha'|\geq 2} a_{\alpha'}(\overline{f})f^{\alpha'} + \overline{g}P'(f,\overline{f},\overline{g}) \tag{15}$$

Applying $\overline{L}^\gamma$, $|\gamma| > 0$, to (15) we have

$$0 = \sum_{j=1}^{N} \overline{L}^\gamma a_j(\overline{f})f^j + \sum_{|\alpha'|\geq 2} \overline{L}^\gamma a_{\alpha'}(\overline{f})f^{\alpha'} + \overline{L}^\gamma \left(\overline{g}P(f,\overline{f},\overline{g})\right). \tag{16}$$

Since $\overline{L}^\gamma \overline{g}(0) = 0$ for all γ, we have

$$\overline{L}^\gamma \left(r'_Z \circ F\right)(0) = \left(\overline{L}^\gamma a_1(\overline{f})(0), \cdots, \overline{L}^\gamma a_N(\overline{f})(0), 0\right) \tag{17}$$

for all γ with $|\gamma| > 0$.

By the hypothesis of Theorem 1, there exist γ_l, $l = 1, ..., N$, such that $|\gamma_l| \leq K$ and $\{\overline{L}^{\gamma_l}(r'_Z \circ F)(0)\}_{l=1,...,N}$ together with $r'_Z \circ F(0) = (0, \cdots, 0, 1)$ span $\mathbb{C}^{N+1}$. Then by the implicit function theorem we can solve the system

$$g = \sum_{j=1}^{N} a_j(\overline{f})f^j + \sum_{|\alpha'|\geq 2} a_{\alpha'}(\overline{f})f^{\alpha'} + \overline{g}P(f,\overline{f},\overline{g})$$

$$0 = \sum_{j=1}^{N} \overline{L}^{\gamma_l} a_j(\overline{f})f^j + \sum_{|\alpha'|\geq 2} \overline{L}^{\gamma_l} a_{\alpha'}(\overline{f})f^{\alpha'} + \overline{L}^{\gamma_l} \left(\overline{g}P(f,\overline{f},\overline{g})\right),$$

$l = 1, ..., N$, for f^j, $j = 1, ..., N$, and $g = f^{N+1}$ in terms of $\overline{L}^\gamma \overline{F}$, $|\gamma| \leq K$. This implies that there exist Φ_j, $j = 1, ..., N+1$, which are holomorphic in their arguments such that

$$f^j = \Phi_j(\overline{L}^\gamma \overline{F}, |\gamma| \leq K) \tag{18}$$

for all $j = 1, ..., N+1$. □

Proof of Theorem 1

In [4], Baouendi and Treves showed that if M is of finite type in the sense of Bloom-Graham, then there is one side of M to which every CR distribution extends as a holomorphic function. Then by Lemma 7 together with Lemma 2.2 and Lemma 2.4 of [2] F is C^ω on M and hence extends holomorphically to a neighborhood of M.

Proof of Theorem 2

Let $\Phi = (\Phi_1, \cdots, \Phi_{N+1})$ and $Q(z,\overline{z},\overline{z}_{n+1}) = R(z,\overline{z})+\overline{z}_{n+1}P(z,\overline{z},\overline{z}_{n+1})$. Since F is holomorphic on a neighborhood of M, we can write (14) as

$$\begin{aligned} F\left(z, Q(z,\overline{z},\overline{z}_{n+1})\right) &= \Phi\left(j^K\overline{F}(\overline{z},\overline{z}_{n+1}), j^{K+1}Q(z,\overline{z},\overline{z}_{n+1})\right) \\ &:= \Phi\left(z,\overline{z},\overline{z}_{n+1}, j^K\overline{F}(\overline{z},\overline{z}_{n+1})\right). \end{aligned} \tag{19}$$

Let $\overline{z} = \chi$ and $\overline{z}_{n+1} = \chi_{n+1}$. Then we can extend (19) as

$$F\left(z, Q(z,\chi,\chi_{n+1})\right) = \Phi\left(z,\chi,\chi_{n+1}, j^K\overline{F}(\chi,\chi_{n+1})\right). \tag{20}$$

Passing to the K-th jet and taking its complex conjugate, we have

$$J^K\overline{F}\left(\chi, \overline{Q}(\chi,z,z_{n+1})\right) = \Phi^K\left(\chi,z,z_{n+1}, j^{2K}F(z,z_{n+1})\right), \tag{21}$$

where Φ^K is holomorphic in its arguments.

Substituting for $J^K\overline{F}$ in (20), we have

$$F\left(w, Q(w,\chi,\overline{Q}(\chi,z,z_{n+1}))\right) = \Psi\left(z,z_{n+1},\chi,w, j^{2K}F(z,z_{n+1})\right), \tag{22}$$

where $w \in \mathbb{C}^n$ and Ψ is holomorphic in its arguments.

Also, we have

(23)
$$J^{2K}F\left(w, Q(w,\chi,\overline{Q}(\chi,z,z_{n+1}))\right) = \Psi^{2K}\left(z,z_{n+1},\chi,w, j^{4K}F(z,z_{n+1})\right),$$

where Ψ^{2K} is holomorphic in its arguments.

On the other hand, we have

$$F\left(u, Q(u,\tau,\overline{Q}(\tau,w,w_{n+1}))\right) = \Psi\left(w,w_{n+1},\tau,u, j^{2K}F(w,w_{n+1})\right), \tag{24}$$

where $u \in \mathbb{C}^n$.

LEMMA 8. *There exist $(p,p_{n+1}) \in \mathbb{C}^{n+1}$ sufficiently close to 0 and holomorphic functions $\chi = \chi(z,z_{n+1})$, $\tau = \tau(u,u_{n+1})$ defined on a neighborhood V of 0 such that*

$$p_{n+1} = Q\left(p,\chi,\overline{Q}(\chi,z,z_{n+1})\right) \tag{25}$$

and

$$u_{n+1} = Q\left(u,\tau,\overline{Q}(\tau,p,p_{n+1})\right) \tag{26}$$

on V.

Proof. It's enough to show that there exist $(p, p_{n+1}) \in \mathbb{C}^{n+1}$ and $\chi^0, \tau^0 \in \mathbb{C}^n$ which are sufficiently small such that

$$(27) \qquad \frac{\partial}{\partial \chi_j} \left[Q\left(p, \chi, \overline{Q}(\chi, z, z_{n+1})\right)\right] \Big|_{(\chi^0, 0)} = \frac{\partial Q}{\partial \chi_j}(p, \chi^0, 0) \neq 0$$

for some $j = 1, ..., n$ and

$$(28) \qquad \frac{\partial}{\partial \tau_j} \left[Q\left(u, \tau, \overline{Q}(\tau, p, p_{n+1})\right)\right] \Big|_{(0, \tau^0)} = \frac{\partial \overline{Q}}{\partial \tau_j}(\tau^0, p, p_{n+1}) \neq 0$$

for some $j = 1, ..., n$. Then by implicit function theorem we can prove the lemma.

But

$$\frac{\partial Q}{\partial \chi_j}(p, \chi^0, 0) = \frac{\partial R}{\partial \chi_j}(p, \chi^0)$$

and

$$\frac{\partial \overline{Q}}{\partial \tau_j}(\tau^0, p, p_{n+1}) = \frac{\partial \overline{R}}{\partial \tau_j}(\tau^0, p) + p_{n+1}\frac{\partial \overline{P}}{\partial \tau_j}(\tau^0, p, p_{n+1}).$$

Since M is of finite type in the sense of Bloom-Graham, $R \not\equiv 0$. Hence we can choose $(p, p_{n+1}) \in \mathbb{C}^{n+1}$ and $\chi^0, \tau^0 \in \mathbb{C}^n$ sufficiently close to 0 which satisfy the above conditions. □

Then substituting for $\chi = \chi(z, z_{n+1})$ and $\tau = \tau(u, u_{n+1})$ in (23) and (24), respectively, and substituting for $J^{2K}F(p, p_{n+1})$ in (24), we have

$$(29) \quad F(u, u_{n+1}) = H\left(J^{4K}F(z, z_{n+1}), z, z_{n+1}, \overline{z}, \overline{z}_{n+1}, u, u_{n+1}, \overline{u}, \overline{u}_{n+1}\right),$$

where H is holomorphic in its arguments.

Passing through $(4K+1)$-jet and taking $(u, u_{n+1}) = (z, z_{n+1}) \in M$, we have

$$(30) \qquad J^{4K+1}F(z, z_{n+1}) = H'\left(J^{4K}F(z, z_{n+1}), z, z_{n+1}, \overline{z}, \overline{z}_{n+1}\right),$$

where H' is holomorphic in its arguments.

Proof of Corollary 4

Let M and M' be as in Corollary 4. Suppose M is essentially finite at 0 and $F : M \to M'$ satisfies

$$(31) \qquad dF(\mathbb{C}T_0M) \not\subseteq H_0^{1,0}(M') \oplus H_0^{0,1}(M') \quad \text{(Hopf Lemma property)}.$$

In [3], Baouendi and Rothschild showed that F is of finite multiplicity at 0 and

$$(32) \qquad (\text{ess type } M_0) = (\text{mult } F_0) \cdot (\text{ess type } M_0').$$

If M' is k-nondegenerate at 0, then

$$\mathcal{O}[\zeta]/\left(a_\alpha(\zeta)\right) = \mathcal{O}[\zeta]/\left(\zeta_1, \cdots, \zeta_n\right). \tag{33}$$

Hence (ess type M'_0) = 1 and (mult F_0) = (ess type M_0).

Thus to prove Corollary 4, it's enough to show that if F is of finite multiplicity at 0, then M is of finite type and

$$\left\{\overline{L}^\gamma \left(r'_Z \circ F\right)(0) : |\gamma| \leq K\right\} \tag{34}$$

span $\mathbb{C}^{n+1}$, where $K = k \cdot$ (mult F_0).

Let $F = (f, g) = (f^1, \cdots, f^n, g)$ and (z) be the ideal of $\mathcal{O}[z]$ generated by z.

Lemma 9. *If F is of finite multiplicity at 0, then*

$$det \left(\frac{\partial}{\partial z_i} h^j(z,0)\right)_{i,j=1,\dots,n} \not\equiv 0, \tag{35}$$

where h^j, $j = 1, \dots, n$, are the (mult F_0)-th order Taylor series expansion of f^j.

Proof. Since we only deal with the Taylor series expansion of F, we may regard that F is smooth.

Since M and M' are in normal coordinates, $\dfrac{\partial^{|\alpha|} g}{\partial z^\alpha}(0) = 0$ for all α. Hence F is of finite multiplicity at 0 if and only if

$$dim_{\mathbb{C}} \mathcal{O}[z]/\left(f^1(z,0), \cdots, f^n(z,0)\right) = d < \infty, \tag{36}$$

where d = (mult F_0).

Now let $z^\alpha \in (z)^d$. We denote $\beta = (b_1, \cdots, b_n) < \alpha = (a_1, \cdots, a_n)$ if $b_j \leq a_j$ for all $j = 1, \dots, n$ and $\beta \neq \alpha$.

If $|\alpha| \geq d$, then we can choose β_l, $l = 1, \dots, d$, such that $0 < \beta_1 < \beta_2 \cdots < \beta_d = \alpha$. Suppose $z^\alpha \notin \left(f^1(z,0), \cdots, f^n(z,0)\right)$. Then

$$sp < \{1, z^{\beta_l} : l = 1, \dots, d\} > \cap \left(f^1(z,0), \cdots, f^n(z,0)\right) = \{0\}, \tag{37}$$

where $sp < \{1, z^{\beta_l} : l = 1, \dots, d\} >$ is the $\mathbb{C}$-vector space spanned by $\{1, z^{\beta_l} : l = 1, \dots, d\}$. Thus

$$\begin{aligned} d &= dim_{\mathbb{C}} \mathcal{O}[z]/\left(f^1(z,0), \cdots, f^n(z,0)\right) \\ &= dim_{\mathbb{C}} sp < \{1, z^{\beta_l} : l = 1, \dots, d\} > \\ &\quad + dim_{\mathbb{C}} sp < \{z^\gamma : \gamma \neq \beta_l, l = 1, \dots, d\} > / \left(f^1(z,0), \cdots, f^n(z,0)\right) \\ &\geq d + 1. \end{aligned}$$

Hence we conclude that

$$(z)^d \subset \left(f^1(z,0), \cdots, f^n(z,0)\right). \tag{38}$$

Then we have

$$\left(h^1(z,0),\cdots,h^n(z,0)\right)\subset\left(f^1(z,0),\cdots,f^n(z,0)\right)+(z)^{d+1}$$
$$\subset\left(f^1(z,0),\cdots,f^n(z,0)\right)$$

and

$$f^j(z,0)-h^j(z,0)\in(z)^{d+1}\subset(z)\cdot\left(f^1(z,0),\cdots,f^n(z,0)\right) \tag{39}$$

for all $j=1,...,n$. Thus by Nakayama's Lemma (see [3])

$$\left(h^1(z,0),\cdots,h^n(z,0)\right)=\left(f^1(z,0),\cdots,f^n(z,0)\right), \tag{40}$$

which implies

$$dim_{\mathbb{C}}\mathcal{O}[z]/\left(h^1(z,0),\cdots,h^n(z,0)\right)<\infty. \tag{41}$$

But in [3], it is proved that (35) holds if (41) holds. □

Let $h=(h^1,\cdots,h^n)$. By Lemma 9 we can show by following the same argument of the proof of Theorem 2 of [3] with h in place of F and with "$=$ modulo $(\xi)^{k(\text{mult } F_0)+1}\cdot(z)^{(\text{mult } F_0)+1}$" in place of "$=$" that M is essentially finite at 0 and hence of finite type at 0.

Now suppose there is a vector $s=(s_1,\cdots,s_n)\in\mathbb{C}^n$ such that

$$\sum_{j=1}^{n}s_ja_j(h)(z,0)\equiv 0. \tag{42}$$

By Lemma 9 there exists $z_0\in\mathbb{C}^n$ sufficiently close to 0 and a neighborhood U of z_0 such that $h(\cdot,0):U\to h(U,0)\subset\mathbb{C}^n$ is a biholomorphic map onto an open set $h(U,0)$ of $\mathbb{C}^n$. Thus

$$\sum_{j=1}^{n}s_ja_j(\zeta)\equiv 0 \tag{43}$$

for all $\zeta\in h(U,0)$. But $\sum_{j=1}^n s_ja_j(\zeta)$ is holomorphic in ζ, $\sum_{j=1}^n s_ja_j(\zeta)\equiv 0$ on $\mathbb{C}^n$.

Let

$$L'_j=\frac{\partial}{\partial\zeta_j}-\frac{r'_j}{r'_{n+1}}\frac{\partial}{\partial\zeta_{n+1}},\ j=1,...,n, \tag{44}$$

where $r'_j=\dfrac{\partial r'}{\partial\zeta_j}$, $j=1,...,n+1$. Since M' is in normal coordinates, we have

$$r'_Z(0)=(0,\cdots,0,1) \tag{45}$$

and

$$L'^{\gamma}(r'_{\overline{Z}})(0) = \left(\frac{\partial^{|\gamma|} a_1}{\partial \zeta^{\gamma}}(0), \cdots, \frac{\partial^{|\gamma|} a_n}{\partial \zeta^{\gamma}}(0), 0 \right) \tag{46}$$

for all $|\gamma| > 0$.

This implies that M' is k-nondegenerate at 0 if and only if

$$\sum_{j=1}^{n} \widetilde{s}_j a_j(\zeta) \not\equiv 0 \tag{47}$$

for all $\widetilde{s} = (\widetilde{s}_1, \cdots, \widetilde{s}_n) \neq 0$. Hence we conclude that

$$\sum_{j=1}^{n} s_j a_j(h)(z, 0) \equiv 0 \tag{48}$$

if and only if $s = 0$.

Now let

$$\begin{aligned} a_j(f)(z,0) &= \sum_{\alpha} c_\alpha z^\alpha \\ &= \sum_{|\alpha| = m_j} c_\alpha z^\alpha + \sum_{|\alpha| > m_j} c_\alpha z^\alpha, \end{aligned}$$

where $\sum_{|\alpha|=m_j} c_\alpha z^\alpha \not\equiv 0$. Then $a_j(f)(z,0) \equiv a_j(h)(z,0)$ modulo $\mathcal{I}^{m_j+1}$. Hence if $\sum_{j=1}^{n} s_j a_j(h)(z,0) \not\equiv 0$, then $\sum_{j=1}^{n} s_j a_j(f)(z,0) \not\equiv 0$ modulo $\mathcal{I}^{m+1}$, where $m = \max(m_1, \cdots, m_n) \leq k \cdot (\text{mult } F_0)$, which implies that the image

$$\left\{ \left(a_1^K(z), \cdots, a_n^K(z) \right) : z \in \mathbb{C}^n \right\} \tag{49}$$

is not contained in a hyperplane of $\mathbb{C}^n$ for $K = k \cdot (\text{mult } F_0)$ or equivalently

$$\left\{ \overline{L}^{\gamma} \left(r'_Z \circ F \right)(0) : |\gamma| \leq K \right\} \tag{50}$$

span $\mathbb{C}^{n+1}$, where $K = k \cdot (\text{mult } F_0)$.

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