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PICARD VALUES AND NORMALITY CRITERION

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ABSTRACT. In this paper, we study the value distribution of meromorphic functions and prove the following theorem: Let $f(z)$ be a transcendental meromorphic function. If f and f' have the same zeros, then $f'(z)$ takes any non-zero value b infinitely many times.

1. Introduction

Let $f(z)$ be a non-constant meromorphic function in the whole complex plane. We use the following standard notations of value distribution theory,

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \cdots$$

(see Hayman [1], Yang [2]). We denote by $S(r,f)$ any function satisfying

$$S(r,f) = o\{T(r,f)\},$$

as $r \to +\infty$, possibly outside of a set with finite measure. We define Ahlfors-Shimizu characteristic function $T_0(r,f)$:

$$T_0(r,f) = \int_0^r \frac{A(t,f)}{t} dt, \tag{1.1}$$

where

$$A(t,f) = \frac{1}{\pi} \int\int_{|z|\le t} [f^{\#}(z)]^2 dxdy, \quad f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2}. \tag{1.2}$$

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The relation between $T(r,f)$ and $T_0(r,f)$ is given by

$$T(r,f) = T_0(r,f) + O(1). \tag{1.3}$$

The order λ of the function $f(z)$ is defined as

$$\lambda = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log r}.$$

Let $f(z)$ be a meromorphic function. If $f(z) = 0$ if and only if $f'(z) = 0$, then it is called that $f(z)$ and $f'(z)$ have the same zeros.

In 1959, Hayman [3] proved the following result.

THEOREM A. *Let $f(z)$ be a transcendental meromorphic function. If $f \neq 0$, then f' takes any non-zero value b infinitely many times.*

Bergweiler and Eremenko [4] proved

THEOREM B. *Let $f(z)$ be a transcendental meromorphic function with finite order. If the zeros of f are of multiplicity ≥ 2, then f' takes any non-zero value b infinitely many times. Moreover, the assumption that f is of finite order is necessary.*

Naturally, we ask that under what condition f' takes any non-zero value b infinitely many times for any transcendental meromorphic function with infinite order. In this paper, we prove

THEOREM 1. *Let $f(z)$ be a transcendental meromorphic function with infinite order. If f and f' have the same zeros, then f' takes any non-zero value b infinitely many times.*

In fact, we have proved

THEOREM 2. *Let $f(z)$ be a transcendental meromorphic function with infinite order. If f and f' have the same zeros, then $f'(z) - b(z)$ has infinitely many zeros for any $b(z) \in S$. Here $S = \{az^n : a \neq 0, n = 0, 1, 2, \cdots\}$.*

In order to prove Theorem 2, we shall first prove the following result.

THEOREM 3. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D and let $a(z)$ be a non-vanish analytic function in D. If, for every function $f \in \mathcal{F}$, f and f' have the same zeros, and $f(z) = a(z)$ whenever $f'(z) = a(z)$, then $\mathcal{F}$ is normal in D.*

Theorem 3 implies the following result obtained by Xu [5] and Pang [6].

THEOREM C. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D and b be a non-zero value. If, for every function $f \in \mathcal{F}$, f and f' have the same zeros, and $f(z) = b$ if and only if $f'(z) = b$, then $\mathcal{F}$ is normal in D.*

For a transcendental function with finite order, the following result is proved by using the method of Bergweiler [7].

THEOREM 4. *Let $f(z)$ be a transcendental meromorphic function with finite order. If the zeros of f are of multiplicity ≥ 2, then $f'(z) - p(z)$ has infinitely many zeros for any polynomial $p(z) \not\equiv 0$.*

2. Proof of Theorem 3

For the proof of Theorem 3, we need the following lemmas.

LEMMA 1 ([8, 9]). *Let $\mathcal{F}$ possesses the property that every function $f \in \mathcal{F}$ has only zeros of multiple at least k. If $\mathcal{F}$ is not normal at a point z_0, then for $0 \leq \alpha < k$, there exist a sequence of functions $f_j \in \mathcal{F}$, a sequence of complex numbers $z_j \to z_0$ and a sequence of positive numbers $\rho_j \to 0$, such that $\rho_j^{-\alpha} f_j(z_j + \rho_j \zeta)$ converges locally uniformly to a non-constant meromorphic function $g(\zeta)$ on $\mathbb{C}$. Moreover, g has only zeros of multiple at least k.*

LEMMA 2 ([8]). *Let $R(z)$ be a non-constant rational function, k a positive integer and let b be a non-zero value. If the zeros of $R(z)$ are of multiplicity at least $k+1$, and $R^{(k)}(z) \neq b$, then $R(z) = \frac{(\gamma z + \delta)^{k+1}}{\alpha z + \beta}$, where α, β, γ, δ are constants such that $\alpha\gamma \neq 0$, $|\beta| + |\delta| \neq 0$.*

LEMMA 3. *Let $f(z)$ be a meromorphic function of finite order and let b be a non-zero complex number. If f and f' have the same zeros, $f' \neq b$, then $f(z)$ is a constant.*

Proof. Obviously, the zeros of $f(z)$ are of multiplicity at least 2 by the assumption on f, and f can not be a polynomial of degree 2. If $f(z)$ is a transcendental meromorphic function with finite order, then by Theorem B we get that $f' = b$ has infinitely many solutions, a contradiction. Hence $f(z)$ is a rational function. Suppose that $f(z)$ is a non-constant rational function. Then by Lemma 2 we know that $f(z) = \frac{(\gamma z+\delta)^2}{\alpha z+\beta}$, where α, β, γ, δ are constants such that $\alpha\gamma \neq 0$, $|\beta| + |\delta| \neq 0$. Thus we have $f'(z) = b + \frac{A}{(\alpha z+\beta)^2}$, where A is a non-zero constant. Hence we deduce that $f'(z) = 0$ if and only if $z \in \{z : \ b + \frac{A}{(\alpha z+\beta)^2} = 0\}$ and $f(z) = 0$ if and only if $z = \frac{\delta}{\gamma}$. Thus f and f' do not have the same zeros, a contradiction. Hence f is a constant. This completes the proof of the lemma. □

Proof of Theorem 3. Suppose that $\mathcal{F}$ is not normal at a point $z_0 \in D$. Then by Lemma 1, for $\alpha = 1$, there exist a sequence of functions $f_j \in \mathcal{F}$, a sequence of complex numbers $z_j \to z_0$, and a sequence of positive numbers $\rho_j \to 0$, such that $g_j(\zeta) = \rho_j^{-1} f_j(z_j + \rho_j\zeta)$ converges locally uniformly to a non-constant meromorphic function $g(\zeta)$. Moreover, g has only zeros of multiple at least 2.

Suppose that $g'(\zeta_0) = 0$. Then there exist ζ_j, $\zeta_j \to \zeta_0$, such that

$$g_j'(\zeta_j) = f_j'(z_j + \rho_j\zeta_j) = 0, \quad j = 1, 2, \cdots .$$

Hence $f_j(z_j + \rho_j\zeta_j) = 0$ and $g_j(\zeta_j) = 0$ for $j = 1, 2, \cdots$, since f_j and f_j' have the same zeros. Thus we get $g(\zeta_0) = \lim_{j\to\infty} g_j(\zeta_j) = 0$. Hence we prove that $g(\zeta)$ and $g'(\zeta)$ have the same zeros, since the zerso of $g(\zeta)$ are of multiplicity ≥ 2. Obviously, $a(z_0) \neq 0, \infty$. From Lemma 3, there exists ζ_0 such that $g'(\zeta_0) = a(z_0)$. Hence there exists $\delta > 0$ such that $g(\zeta)$ is analytic on $D_{2\delta} = \{\zeta : \ |\zeta - \zeta_0| < 2\delta\}$. Thus $g_j'(\zeta)$ are analytic on $D_\delta = \{\zeta : \ |\zeta - \zeta_0| < \delta\}$ for sufficiently large j and $g_j'(\zeta)$ converges uniformly to $g'(\zeta)$ on D_δ. Next consider two cases.

Case 1. There exist ϵ $(0 < \epsilon < \delta)$ and infinitely many j such that

$$g_j'(\zeta) - a(z_j + \rho_j\zeta) = f_j'(z_j + \rho_j\zeta) - a(z_j + \rho_j\zeta) \neq 0,$$

on $D_\epsilon = \{\zeta : |\zeta - \zeta_0| < \epsilon\}$. Since $g_j'(\zeta) - a(z_j + \rho_j\zeta)$ converges uniformly to $g'(\zeta) - a(z_0)$ on D_ϵ. Hence by Hurwitz's theorem we deduce that $g'(\zeta) - a(z_0) \equiv 0$ on D_ϵ, thus we have

$$g'(\zeta) - a(z_0) \equiv 0, \quad \text{for all } \zeta \in \mathbb{C}.$$

Next we can easily obtain that $g(\zeta)$ is a constant, a contradiction.

Case 2. There exist infinitely many j such that $\zeta_j \to \zeta_0$ and $f_j'(z_j + \rho_j\zeta_j) = a(z_j + \rho_j\zeta_j)$. Without loss of generality we assume that

$$g_j'(\zeta_j) - a(z_j + \rho_j\zeta_j) = f_j'(z_j + \rho_j\zeta_j) - a(z_j + \rho_j\zeta_j) = 0,$$

for $j = 1, 2, 3, \cdots$. Since $f_j(z) = a(z)$ whenever $f_j'(z) = a(z)$, we have $f_j(z_j + \rho_j\zeta_j) = a(z_j + \rho_j\zeta_j)$ and $g_j(\zeta_j) = \rho_j^{-1} f_j(z_j + \rho_j\zeta_j) = \rho_j^{-1} a(z_j + \rho_j\zeta_j) \to \infty$. This contradicts that $\lim_{j\to\infty} g_j(\zeta_j) = g(\zeta_0) \neq \infty$.

The proof of the theorem is complete. □

3. Proof of Theorem 2

Since $f(z)$ is of infinite order, we have

$$\overline{\lim_{r\to\infty}} \frac{T(r,f)}{(\log r)^2} = \infty. \tag{3.1}$$

Hence we obtain

$$\overline{\lim_{r\to\infty}} \frac{T(r, \frac{f(z)}{z^{n+1}})}{(\log r)^2} = \infty. \tag{3.2}$$

Thus by (1.1)-(1.3) and (3.2) we have

$$\overline{\lim_{r\to\infty}} \frac{A(r, \frac{f(z)}{z^{n+1}})}{\log r} = \infty. \tag{3.3}$$

Set

$$\mathcal{F} = \{g_j(z) = \frac{f(2^j z)}{2^{(n+1)j} z^{n+1}}, j = 1, 2, 3, \cdots, \frac{1}{2} < |z| < \frac{5}{2}\}.$$

Claim: $\mathcal{F}$ is not a normal family.

Suppose that $\mathcal{F}$ is a normal family. Then by Marty's criterion, there exists $M > 0$ satisfying

$$g_j^{\#}(z) \le M, \quad \text{for } j = 1, 2, 3, \cdots, 1 \le |z| \le 2.$$

Hence

$$\begin{aligned} A\left(2^j, \frac{f(z)}{z^{n+1}}\right) &= \frac{1}{\pi} \iint_{|z| \le 2^n} \left(\left(\frac{f(z)}{z^{n+1}} \right)^{\#} \right)^2 dxdy \quad (z = x + iy) \\ &= \frac{1}{\pi} \sum_{m=0}^{j-1} \iint_{2^m \le |z| \le 2^{m+1}} \left(\left(\frac{f(z)}{z^{n+1}} \right)^{\#} \right)^2 dxdy \\ &= \frac{1}{\pi} \sum_{m=0}^{j-1} \iint_{1 \le |w| \le 2} (g_m^{\#}(w))^2 d\xi d\eta \quad (w = \xi + i\eta) \\ &\le 3M^2 j = M_1 j, \quad (M_1 = 3M^2). \end{aligned}$$

Thus, for any $r > 0$, $2^{j-1} \le r < 2^j$, we have

$$A\left(r, \frac{f(z)}{z^{n+1}}\right) \le A\left(2^j, \frac{f(z)}{z^{n+1}}\right) \le M_1 j \le M_1 \left(\frac{\log r}{\log 2} + 1\right),$$

which contradicts (3.3). Therefore $\mathcal{F}$ is not normal. Hence the family

$$\mathcal{F}_1 = \{h_j(z) = \frac{f(2^j z)}{2^{(n+1)j}}, j = 1, 2, 3, \cdots, \frac{1}{2} < |z| < \frac{5}{2}\}$$

is not normal. Thus by using Theorem 3 for $a(z) = az^n$, we know that there exist infinitely many j and z_j such that $h_j'(z_j) = az_j^n$, that is $f'(2^j z_j) = a(2^j z_j)^n$. Hence we deduce $f'(z) - az^n$ has infinitely many zeros. The proof of the theorem is complete.

4. Proof of Theorem 4

For the proof of Theorem 4, we need the following lemmas.

LEMMA 4 ([10]). *Let $f(z)$ be a transcendental meromorphic function. Then for each positive number ϵ and each positive integer k, we have*

$$k\overline{N}(r,f) \le N\left(r,\frac{1}{f^{(k)}}\right) + N(r,f) + \epsilon T(r,f) + S(r,f). \tag{4.1}$$

LEMMA 5 ([4]). *Let $g(z)$ be a transcendental meromorphic function with finite order. If $g(z)$ has only finitely many critical values, then $g(z)$ has only finitely many asymptotic values.*

LEMMA 6 ([11]). *Let $g(z)$ be a transcendental meromorphic function and suppose that $g(0) \neq \infty$ and the set of finite critical and asymptotic values of $g(z)$ is bounded. Then there exists $R > 0$ such that*

$$|g'(z)| \ge \frac{|g(z)|}{2\pi|z|}\log\frac{|g(z)|}{R},$$

for all $z \in \mathbb{C}\backslash\{0\}$ which are not poles of $g(z)$.

Proof of Theorem 4. Let $p(z) = az^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$, $a \neq 0$. In the following, we consider two cases.

Case 1. $f(z)$ has only finitely many zeros. In this case , we have

$$N\left(r,\frac{1}{f}\right) = O(\log r) = S(r,f). \tag{4.2}$$

Obviously

$$\begin{aligned}
&m\left(r,\frac{1}{f}\right) + m\left(r,\frac{1}{f'-p}\right)\\
&\le m\left(r,\frac{1}{f^{(n+1)}}\right) + m\left(r,\frac{1}{f^{(n+1)}-n!a}\right) + S(r,f)\\
&\le m\left(r,\frac{1}{f^{(n+2)}}\right) + S(r,f)\\
&= T(r,f^{(n+2)}) - N\left(r,\frac{1}{f^{(n+2}}\right) + S(r,f)\\
&\le T(r,f') + (n+1)\overline{N}(r,f) - N\left(r,\frac{1}{f^{(n+2)}}\right) + S(r,f),
\end{aligned}$$

thus by (4.1) and (4.2) we have

$$
\begin{aligned}
T(r,f) &\le (n+1)\overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f'-p}\right) \\
&\quad - N\left(r,\frac{1}{f^{(n+2)}}\right) + S(r,f) \\
&\le \frac{n+1}{n+2}N(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f'-p}\right) \\
&\quad + \frac{1}{2n+4}T(r,f) + S(r,f) \\
&\le \frac{2n+3}{2n+4}T(r,f) + N\left(r,\frac{1}{f'-p}\right) + S(r,f).
\end{aligned}
$$

Hence we obtain

$$T(r,f) \le (2n+4)N\left(r,\tfrac{1}{f'-p}\right) + S(r,f).$$

Therefore, $f'(z)-p(z)$ has infinitely many zeros and the conclusion of the theorem is valid in this case.

Case 2. $f(z)$ has infinitely many zeros $z_1, z_2, \cdots$. Define

$$g(z) = f(z) - \left(\tfrac{a}{n+1}z^{n+1} + \tfrac{a_1}{n}z^n + \cdots + a_n z\right).$$

Then $g'(z) = f'(z)-p(z)$. We has to show that $g'(z)$ has infinitely many zeros. Suppose that $g'(z)$ has finitely many zeros, then $g(z)$ has finitely many critical values. Hence by Lemma 5 we know that $g(z)$ has only finitely many asmyptotic values. Without loss of generality we assume that $f(0)\neq\infty$, thus by Lemma 6 we deduce that

$$\frac{|z_j g'(z_j)|}{|g(z_j)|} \ge \frac{1}{2\pi}\log\frac{|g(z_j)|}{R}.$$

In particular, $\frac{|z_j g'(z_j)|}{|g(z_j)|} \to \infty$ as $j\to\infty$, since $\frac{1}{2\pi}\log\frac{|g(z_j)|}{R} \to \infty$ as $j\to\infty$. On the other hand, $\frac{|z_j g'(z_j)|}{g(z_j)} \to n+1$ as $j\to\infty$, a contradiction. Hence we deduce that $f'(z)-p(z)$ has infinitely many zeros. The theorem is proved. □

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