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UNIQUENESS OF IDENTIFYING THE CONVECTION TERM

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ABSTRACT. The inverse boundary value problem for the steady state heat equation with covection term is considered in a simply connected bounded domain with smooth boundary. Taking the Dirichlet to Neumann map which maps the temperature on the boundary to the heat flux on the boundary as an observation data, the global uniqueness for identifying the convection term from the Dirichlet to Neumann map is proved.

1. Introduction

In this paper, the following inverse boundary value problem is considered. Assume that a bounded body with non–solid and inaccessible interior with a stationary velocity field is given. The problem is to estimate the velocity distribution based on stationary temperature measurements on the surface of the body. Mathematically, let $\Omega \subset \mathbb{R}^3$ be a simply connected bounded domain representing the body, and assume that boundary $\partial\Omega$ is smooth. Let $-2\vec{a}$ denote the (real valued) velocity field of the interior of the body. For simplicity, we assume that $\vec{a} \in C^\infty(\overline{\Omega})$. Assume further for simplicity that the diffusion coefficient of the temperature is constant (equal to unity) throughout the medium. Then the temperature distribution $u = u(x,t)$ satisfies the convection–diffusion equation

$$\partial_t u = \Delta u + 2\vec{a} \cdot \nabla u \text{ in } \Omega \times (0,T)$$

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where the factor two in front of $\vec{a}$ is included for notational convenience. The measurement setting can be described as follows. One applies a known stationary temperature distribution at the surface of the body and after stabilization measures the heat flux needed to maintain this distribution. This measurement procedure is repeated with different temperature distributions. Assuming that the convection field $\vec{a}$ is not affected by the warming of the body, the problem allows the following mathematical formulation: For any $f \in H^{1/2}(\Omega)$, let $u_f = u_f(x) \in H^1(\Omega)$ satisfy the Dirichlet problem

$$(\Delta + 2\vec{a}\cdot\nabla)u_f = 0 \text{ in } \Omega, \quad u_f\big|_{\partial\Omega} = f. \tag{1}$$

The problem considered here is to determine $\vec{a}$ from the knowledge of the pairs

$$\left(f, \frac{\partial u_f}{\partial \vec{n}} := (\vec{n}\cdot\nabla u_f)\big|_{\partial\Omega}\right),$$

where $\vec{n}$ is the exterior unit normal vector of $\partial\Omega$.

By applying Green's theorem, we can define the Dirichlet-to-Neumann map $\tilde{\Lambda}_{\vec{a}} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, $f \mapsto \frac{\partial u_f}{\partial \vec{n}}$ via the identity

$$\langle \tilde{\Lambda}_{\vec{a}} f, g\rangle_{\partial\Omega} = \int_\Omega \Big(\nabla u_f\cdot\nabla v - 2(\vec{a}\cdot\nabla u_f)v\Big)dx \tag{2}$$

for any $f, g \in H^{1/2}(\partial\Omega)$, where u_f is the solution to (1) and v is any $v \in H^1(\Omega)$ such that $v|_{\partial\Omega} = g$. As it is customary in the literature, the inverse problem can be formulated in terms of the Dirichlet-to-Neumann map as follows.

PROBLEM 1.1. *Identify the convection field $\vec{a} \in C^\infty(\overline{\Omega})^3$ from the knowledge of $\tilde{\Lambda}_{\vec{a}} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$.*

REMARK. Observe that the solution u_f to (1) exists and is unique due to the maximum principle (see Proposition 2.1 in Taylor's book [5], page 311 of Vol. 1).

We especially consider the uniqueness of the identification of $\vec{a}$ from $\tilde{\Lambda}_{\vec{a}}$. We prove the following global uniqueness result.

THEOREM 1.1. *Let $\vec{a}_j \in C^\infty(\overline{\Omega})^3$, $j = 1, 2$, be real valued convection fields that satisfy $\tilde{\Lambda}_{\vec{a}_1} = \tilde{\Lambda}_{\vec{a}_2}$. Then $\vec{a}_1 = \vec{a}_2$ in $\overline{\Omega}$.*

The corresponding two dimensional analogue of this result has been proved by J. Cheng and M. Yamamoto ([1]). Their proof is based on the use of the inverse scattering method that cannot be used in $\mathbb{R}^3$.

Note that the differential operator in (1) is not formally selfadjoint. One of the purpose of our study is to see what kind of new techniques are necessary to investigate the inverse problems for formally nonselfadjoint operators. It turns out that that the technique used here does not differ so much from that for the Scrödinger equation with magnetic potential.

2. Proof of the main result

In this section, we outline the proof of the main result, Theorem 1.1. The proof is based on several auxiliary results that will be proved in the next section.

In our proof of Theorem 1.1, we shall need an operators slightly more general than that of (1). Indeed, consider the Schrödinger type boundary value problem with a real valued convection term,

$$(3) \qquad (\Delta + 2\vec{a}\cdot\nabla + q)u_f = 0 \ \text{ in } \ \Omega, \quad u_f\big|_{\partial\Omega} = f,$$

where $\vec{a}$ is as before and $q \in L^\infty(\Omega)$ is a real-valued potential. Assuming that this Dirichlet problem has a unique solution, we may define the corresponding Dirichlet-to-Neumann map $\tilde{\Lambda}_{\vec{a},q} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, $f \mapsto \frac{\partial u_f}{\partial \vec{n}}$ as

$$(4) \qquad \langle \tilde{\Lambda}_{\vec{a},q} f, g\rangle_{\partial\Omega} = \int_\Omega \Big(\nabla u_f \cdot \nabla v - (2\vec{a}\cdot\nabla u_f + q u_f)v\Big)dx,$$

where v is as before. Associated with $\tilde{\Lambda}_{\vec{a},q}$, we also define an another Dirichlet-to-Neumann map $\Lambda_{\vec{a},q} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, $f \mapsto \vec{n}\cdot(\nabla u_f + \vec{a}u_f)|_{\partial\Omega}$ as

$$(5) \qquad \begin{aligned} &\langle \Lambda_{\vec{a},q} f, g\rangle_{\partial\Omega} \\ &= \int_\Omega \Big(\nabla u_f \cdot \nabla v + \vec{a}\cdot(u_f\nabla v - v\nabla u_f) + (\nabla\cdot\vec{a} - q)u_f v\Big)dx, \end{aligned}$$

where v as before and denote $\Lambda_{\vec{a},q}$ by $\Lambda_{\vec{a}}$ if $q = 0$. Later we will see that we know $\Lambda_{\vec{a}}$ if we know $\tilde{\Lambda}_{\vec{a}}$. $\Lambda_{\vec{a},q}$ plays an important role in our proof. It is well understood that the uniqueness of the inverse problem of recovering both $\vec{a}$ and q from $\Lambda_{\vec{a},q}$ fails. Indeed, as in the case of Schrödinger operator with a magnetic potential (see [4]), let us introduce the space of gauge functions,

$$C_0^\infty(\overline{\Omega}) = \left\{\varphi \in C^\infty(\overline{\Omega}) \mid \varphi|_{\partial\Omega} = 0\right\},$$

and the gauge transformation

$$(6)\; T_\varphi : C^\infty(\overline{\Omega})^3 \times L^\infty(\Omega) \;\to\; C^\infty(\overline{\Omega})^3 \times L^\infty(\Omega),$$
$$(\vec{a}, q) \;\mapsto\; (\vec{a} + \nabla\varphi, q + |\nabla\varphi|^2 + \Delta\varphi + 2\vec{a}\cdot\nabla\varphi),$$

where $\varphi \in C_0^\infty(\overline{\Omega})$. It is well known that the gauge transformations leave the Dirichlet-to-Neumann $\Lambda_{\vec{a},q}$ map intact. Indeed, if u_f satisfes the boundary value problem (3), with the potential $(\vec{a}, q)$, then $v_f = e^{-\varphi}u_f$ satisfies the corresponding equation with the potential $T_\varphi(\vec{a}, q) = (\vec{b}, p)$ and $v_f|_{\partial\Omega} = f$, $\vec{n}\cdot(\nabla v_f + \vec{b}v_f)|_{\partial\Omega} = \vec{n}\cdot(\nabla u_f + \vec{a}u_f)|_{\partial\Omega}$. However, in a restricted class of potentials, this is the only non-uniqueness that is related to the inverse problem of identifying $(\vec{a}, q)$ from $\Lambda_{\vec{a},q}$. More precisely, we shall show the following result.

THEOREM 2.1. *Let $(\vec{a}_j, q_j) \in C^\infty(\overline{\Omega})^3 \times L^\infty(\Omega)$, $j = 1, 2$ be two pairs of potentials, and assume that $\mathrm{supp}(\vec{a}_1 - \vec{a}_2) \subset \overline{\Omega}$. If $\Lambda_{\vec{a}_1,q_1} = \Lambda_{\vec{a}_2,q_2}$, then we have*

$$(\vec{a}_1, q_1) = T_\varphi(\vec{a}_2, q_2)$$

for some gauge function $\varphi \in C_0^\infty(\overline{\Omega})$.

Note that in particular $\nabla\times\vec{a}_1 = \nabla\times\vec{a}_2$. The proof of this result is outlined in the next section.

Before describing further the idea of the proof of the main result, let us formulate the property of the gauge transformation as a lemma.

LEMMA 2.1. *Let $\vec{a}_j \in C^\infty(\overline{\Omega})^3$, $j = 1, 2$, $q \in L^\infty(\Omega)$ and $\varphi \in C_0^\infty(\overline{\Omega})$ satisfy*

$$(\vec{a}_1, q) = T_\varphi(\vec{a}_2, q).$$

Then $\varphi = 0$ and consequently $\vec{a}_1 = \vec{a}_2$.

Evidently, for convection terms differing only in a simply connected bounded domain with smooth boundary, Theorem 2.1 and Lemma 2.1 immediately imply Theorem 1.1. Indeed, reminding $\tilde{\Lambda}_{\vec{a}_1} = \tilde{\Lambda}_{\vec{a}_2}$ implies $\Lambda_{\vec{a}_1} = \Lambda_{\vec{a}_2}$, we have $\Lambda_{\vec{a}_1,0} = \Lambda_{\vec{a}_2,0}$ if $\tilde{\Lambda}_{\vec{a}_1} = \tilde{\Lambda}_{\vec{a}_2}$, and hence $(\vec{a}_1, 0) = T_\varphi(\vec{a}_2, 0)$ for some $\varphi \in C_0^{(}\overline{\Omega})$ and so $\vec{a}_1 = \vec{a}_2$.

In the rest of this paper, we will show that it is possible to avoid the assumption that the convection terms differ only in a simply connected bounded domain with smooth boundary. This is achieved by appropriate extension of the potentials into a slightly larger set including $\overline{\Omega}$ and then argue as above. The extension is based on the following generalization of the result in V. Isakov [3].

THEOREM 2.2. *From $\tilde{\Lambda}_{\vec{a}}$, it is possible to compute $\vec{a}|_{\partial\Omega}$ and $\partial_x^\alpha \vec{a}|_{\partial\Omega}$ $(|\alpha| > 0)$. As by products, we have the followings:*

- (i) *We know $\Lambda_{\vec{a}}$ if we know $\tilde{\Lambda}_{\vec{a}}$.*
- (ii) *Assuming that $\vec{a}_j \in C^\infty(\overline{\Omega})^3$, $j = 1, 2$, be two convection terms for which $\tilde{\Lambda}_{\vec{a}_1} = \tilde{\Lambda}_{\vec{a}_2}$. Then, $\vec{a}_1 - \vec{a}_2$ vanishes at the boundary up to arbitrary order.*

We introduce the following notations. Let $\Omega' \subset \mathbb{R}^3$ be a bounded domain with smooth boundary such that $\overline{\Omega} \subset \Omega'$. Let $\vec{a}' \in C^\infty(\overline{\Omega'})$ and $q' \in L^\infty(\Omega')$ be extensions of the potentials $\vec{a}$ and q defined in Ω to the larger set Ω'. We denote by $\Lambda_{\vec{a}',q'}$ the Dirichlet-to-Neumann map on $\partial\Omega'$. We have the following result which can be easily proven by using the definition of the Dirichlet to Neumann map.

LEMMA 2.2. *Let $\vec{a}'_j \in C^\infty(\overline{\Omega'})^3$, $q'_j \in L^\infty(\Omega')$, $j = 1, 2$ be extensions of the potentials of $\vec{a}_j \in C^\infty(\overline{\Omega})^3$, $q_j \in L^\infty(\Omega)$, $j = 1, 2$ such that $\vec{a}'_1 = \vec{a}'_2$, $q'_1 = q'_2$ in $\Omega' \setminus \overline{\Omega}$. Then, we have that $\tilde{\Lambda}_{\vec{a}_1,q_1} = \tilde{\Lambda}_{\vec{a}_2,q_2}$ on $\partial\Omega$ implies $\tilde{\Lambda}_{\vec{a}'_1,q'_1} = \tilde{\Lambda}_{\vec{a}'_2,q'_2}$ on $\partial\Omega'$. In particular, $\tilde{\Lambda}_{\vec{a}_1} = \tilde{\Lambda}_{\vec{a}_2}$ on $\partial\Omega$ implies $\tilde{\Lambda}_{\vec{a}'_1} = \tilde{\Lambda}_{\vec{a}'_2}$ on $\partial\Omega'$.*

With these results, the proof of the main theorem is straightforward. Indeed, let $\vec{a}_j$, $j = 1, 2$, be two convection terms yielding the same boundary data. Let B be an open ball such that $\overline{\Omega} \subset B$. By Theorem 2.2, there are extensions $\vec{a}'_j \in C^\infty(\overline{B})$ $(j = 1, 2)$ of $\vec{a}_j$ $(j = 1, 2)$ such that $\vec{a}'_1 = \vec{a}'_2$ in $\overline{B} \setminus \Omega$. Then, we have from Lemma 2.2, $\tilde{\Lambda}_{\vec{a}'_1} = \tilde{\Lambda}_{\vec{a}'_2}$ on ∂B. Note that $\mathrm{supp}(\vec{a}'_1 - \vec{a}'_2) \subset \overline{B}$ and B is a simply connected bounded domain with smooth boundary. Hence, by the aforementioned global uniqueness result for convection terms differing only in a simply connected bounded domain with smooth boundary, $\vec{a}'_1 = \vec{a}'_2$ in $\overline{B}$. This immediately implies $\vec{a}_1 = \vec{a}_2$ in $\overline{\Omega}$.

3. Proofs

In this section, we work out the details that were skipped in the previous sections. We start by deriving an identity, sometimes referred to as Alessandrini's identity. To this end, consider the adjoint problem

$$\Delta v - 2\nabla \cdot (\vec{a}v) + qv = 0 \quad \text{in} \quad \Omega, \tag{7}$$

and the corresponding Dirichlet-to-Neumann map,

$$(\Lambda_{\vec{a},q})_* : v|_{\partial\Omega} \mapsto \vec{n} \cdot (\nabla v - \vec{a}v)|_{\partial\Omega}.$$

A straightforward integration by parts argument shows that if v satisfies the equation (7), then for any $u \in H^1(\Omega)$ we have

$$\begin{aligned} &\langle u, (\Lambda_{\vec{a},q})_* v\rangle_{\partial\Omega} \\ &= \int_\Omega \Big(\nabla u \cdot \nabla v + \vec{a} \cdot (u\nabla v - v\nabla u) + (\nabla \cdot \vec{a} - q)uv\Big)dx. \end{aligned} \tag{8}$$

By comparing with the definition of the mapping $\Lambda_{\vec{a},q}$, we observe that

$$\langle (\Lambda_{\vec{a},q})_* v, u\rangle_{\partial\Omega} = \langle v, \Lambda_{\vec{a},q} u\rangle_{\partial\Omega},$$

or

$$(\Lambda_{\vec{a},q})_* = \Lambda^*_{\vec{a},q}, \tag{9}$$

the adjoint of $\Lambda_{\vec{a},q}$. Assume now that we have two potentials $(\vec{a}_j, q_j)$, $j = 1, 2$, the corresponding boundary mappings being $\Lambda_{\vec{a}_j,q_j}$, respectively. Let u be any solution of the equation (3) with the potentials $(\vec{a}_1, q_1)$ and v any solution of the equation (7) with $(\vec{a}_2, q_2)$. By the identity (9) and the definitions (5) and (8), we get the identity

$$\begin{aligned} &\langle (\Lambda_{\vec{a}_1,q_1} - \Lambda_{\vec{a}_2,q_2})u, v\rangle \\ &= \langle \Lambda_{\vec{a}_1,q_1} u, v\rangle - \langle u, (\Lambda_{\vec{a}_2,q_2})_* v\rangle \\ &= \int_\Omega \Big((\vec{a}_1 - \vec{a}_2) \cdot (u\nabla v - v\nabla u) + (\nabla \cdot (\vec{a}_1 - \vec{a}_2) - (q_1 - q_2))uv\Big)dx. \end{aligned} \tag{10}$$

This identity has several implications. Let us start by proving Theorem 2.1.

Proof of Theorem 2.1. As in the article [4], we can prove the existence of exponentially growing solutions to the equations (3) and (7). By substituting the solutions in the identity (10) and following the reasoning of [2], we find first that $\nabla \times (\vec{a}_1 - \vec{a}_2) = 0$, and consequently $\vec{a}_1 -$

$\vec{a}_2 = \nabla\varphi$ with some $\varphi \in C_0^\infty(\overline{\Omega})$ reminding that Ω is simply connected. It follows, by the properties of the gauge transformation that

$$\Lambda_{\vec{a}_1,q_1} = \Lambda_{\vec{a}_2,q_2} = \Lambda_{T_\varphi(\vec{a}_2,q_2)} = \Lambda_{\vec{a}_1,p_2},$$

where

$$p_2 = q_2 + |\nabla\varphi|^2 + \Delta\varphi + 2\vec{a}_2 \cdot \nabla\varphi.$$

Applying the identity (10) to the pair $(\vec{a}_1, q_1)$, $(\vec{a}_1, p_2)$ and substituting the corresponding exponentially growing solutions to this identity, we find that $q_1 = p_2$, or $(\vec{a}_1, q_1) = T_\varphi(\vec{a}_2, q_2)$ as claimed. □

Observe also that the identity (10) immediately implies the claim of Lemma 2.2. Indeed, applying this identity in the extended domain Ω', the claim of Lemma 2.2 easily follows.

Proof of Lemma 2.1. Let $\vec{a}_j$, $j = 1, 2$ and q be such that for some $\varphi \in C_0^\infty(\overline{\Omega})$,

$$(\vec{a}_1, q) = T_\varphi(\vec{a}_2, q)$$

holds. Then, by the definition of the gauge transformation, φ satisfies a semilinear Dirichlet problem

$$\Delta\varphi + |\nabla\varphi|^2 + \vec{a}_2 \cdot \nabla\varphi = 0 \quad \text{in} \quad \Omega, \quad \varphi|_{\partial\Omega} = 0.$$

But by just checking the proof of the Proposition 2.1 in [5] one can deduce that $\varphi = 0$. □

Proof of Theorem 2.2. Let (x^1, x^2, x^3) be the boundary normal coordinates such that $(\partial x^3/\partial x_1, \partial x^3/\partial x_2, \partial x^3/\partial x_3) = \vec{n}$ on $x^3 = 0$. Also let $D_{x^j} = -\sqrt{-1}\partial/\partial x^j$ $(1 \le j \le 3)$, $x' = (x^1, x^2)$, $D_x = (D_{x^1}, D_{x^2}, D_{x^3}) = (D_{x'}, D_{x^3})$. Then, in terms of the boundary normal coordinates

$$\begin{aligned}(11)\quad P(x, D_x) &:= -(\Delta + 2\vec{a}\cdot\nabla) \\ &= D_{x^3}^2 + \sqrt{-1}E(x)D_{x^3} + Q(x, D_{x'}) - 2\sqrt{-1}\tilde{a}\cdot D_x,\end{aligned}$$

where $\tilde{a} = (\tilde{a}^1, \tilde{a}^2, \tilde{a}^3)$, $\tilde{a}^k = \sum_{j=1}^3 \partial x^k/\partial x_j \; a_j$ $(1 \le k \le 3)$, $E(x)$ is a scalar real valued C^∞ function, $Q(x, D_{x'}) = Q_2(x, D_{x'}) + Q_1(x, D_{x'})$ with C^∞ coefficient homogeneous partial differential operators $Q_j(x, D_{x'})$ $(j = 1, 2)$ of order j $(j = 1, 2)$. Moreover, $E(x)$, $Q(x, D_{x'})$ are independent of $\tilde{a}$ and $Q_2(x, \xi')$ is given by

$$(12)\qquad Q_2(x, \xi') = g(x, \xi') := \sum_{\alpha,\beta=1}^{2} g^{\alpha\beta}\xi_\alpha\xi_\beta$$

with $g^{\alpha\beta} = \sum_{\alpha,\beta=1}^{2} \partial x^\alpha/\partial x_j \; \partial x^\beta/\partial x_j$ $(1 \le \alpha, \beta \le 2)$.

It is not difficult to prove that modulo a smoothing operator, we can factorize $P(x, D_x)$ in the form

$$P(x, D_x) = (D_{x^3} + \sqrt{-1}E(x) - 2\sqrt{-1}\tilde{a}^3 - \sqrt{-1}B) \times (D_{x^3} + \sqrt{-1}B) \tag{13}$$

where $B(x, D_{x'})$ is a classical pseudodifferential operator of order 1 depending smoothly on x^3. Then, by a standard argument of the theory of pseudodifferential operators, we can prove that $\tilde{\Lambda}_{\vec{a}}$ is a classical pseudodifferential operator of order 1 and its full symbol $\tilde{\sigma}(\tilde{\Lambda}_{\vec{a}})(x', \xi')$ is given by

$$\tilde{\sigma}(\tilde{\Lambda}_{\vec{a}})(x', \xi') = b|_{x^3=0}, \tag{14}$$

where $b = b(x, \xi')$ is the full symbol of $B(x, D_{x'})$.

Let $\tilde{\sigma}(\tilde{\Lambda}_{\vec{a}})(x', \xi') = \sum_{j \le 1} \lambda_j(x', \xi')$ be the asymptotic expansion of $\tilde{\sigma}(\tilde{\Lambda}_{\vec{a}})(x', \xi')$ for $|\xi'| \ge 1$, where each $\lambda_j(x', \xi')$ is positive homogeneous of degree j in ξ'. Then, we can prove by induction

$$\lambda_{-r}(x', \xi') \equiv \left(\frac{1}{2\sqrt{g}}\right)^r \partial_{x^3}^r(\lambda_0)|_{x^3=0} \ (r \ge 1) \tag{15}$$

modulo terms which may depend on $\tilde{a}^j$ $(1 \le j \le 3)$ and their x^3 derivatives up to order $r - 1$ and

$$\lambda_0(x', \xi') \equiv \Big(-\sqrt{-1}\frac{1}{\sqrt{g}} \sum_{j=1}^{2} \tilde{a}^j \xi_j - \tilde{a}^3 \Big)\Big|_{x^3=0} \tag{16}$$

modulo terms which do not depend on $\vec{a}$. □

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