Comm. Korean Math. Soc. **16** (2001), No. 3, pp. 399–404

PROPERTIES OF ELASTIC SYMBOLS AND CONSTRUCTION OF SOLUTIONS OF THE DIRICHLĖT PROBLEM

MISHIO KAWASHITA* AND HIDEO SOGA**

ABSTRACT. We examine plane waves of the elastic reduced wave equation in the half-space, and show that linear combinations of them can cover all plane waves on the boundary. The proof is based on the complex analysis for the symbol in the (dual) variable in the normal direction to the boundary.

1. Introduction

In this note we consider the following elastic (reduced wave) equation (1.1) in the half-space $\mathbb{R}^n_+ = \{x = (x', x_n) : x_n > 0\}$ with the Dirichlet boundary condition, and study the plane waves of this equation:

$$(1.1) \qquad (\sigma^2 I + \sum_{i,j=1}^{n} a_{ij}\partial_{x_i}\partial_{x_j})u(x) = 0 \qquad \text{in} \quad \mathbb{R}^n_+.$$

Plane waves play an important role on various problems, e.g., inverse problems analyzed by means of the Fourier analysis, etc.

We assume that the coefficients a_{ij} $(i, j = 1, \cdots, n)$ are constant real $n\times n$-matrices satisfying

$$(\text{A.1}) \qquad a_{ij} = {}^t a_{ji}, \quad i, j = 1, 2, \cdots, n,$$

$$(\text{A.2}) \qquad L(\xi) \equiv \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \ \text{ is positive definite for any } \ \xi \neq 0.$$

Received February 23, 2001.
2000 Mathematics Subject Classification: Primary 74J05; Secondary 35C99.
Key words and phrases: elastic equations, plane waves.
The paper has been presented at the ip2001 workshop.
*Partly supported by Grant-in-Aid for Encouragement of Young Scientists A-09740085 from Ministry of Educ., Japan.
**Partly supported by Grant-in-Aid for Sci. Res. (c) 11640098, Ministry of Educ., Japan.

Then all the eigen-values of $L(\xi)$ become positive. We assume that

(A.3) every eigen-value of $L(\xi)$ is of constant multiplicity for all $\xi \neq 0$,

and denote the eigen-values by $\lambda_j(\xi)$ $(j = 1, \cdots, d;\ 0 < \lambda_1 < \cdots < \lambda_d)$. We set the following assumption on the eigen-values $\{\lambda_j\}$:

(A.4) every slowness surface $\Sigma_i = \{\xi : \lambda_j(\xi) = 1\}$ is strictly convex, and its Gaussian curvature does not vanish.

We say that η' is non-glancing if for every eigen-value λ_j

$$\partial_{\xi_n}\lambda_j(\eta', z) \neq 0 \ \text{ whenever real } z \text{ satisfies } \lambda_j(\eta', z) = 1.$$

In the whole space $\mathbb{R}^n$, the plane waves are of the form

$$e^{i\sigma\eta x}v, \ \text{ where } \lambda_j(\eta) = 1 \text{ and } v \in \text{Ker}\ [I - L(\eta)]$$

for some eigen-value λ_j. In the half-space, we need to add other waves to $e^{i\sigma\eta x}v$ so that those sum satisfies the boundary condition. Our purpose is to examine those added waves. Namely we construct the bounded solutions satisfying the equation (1.1) and

$$u|_{x_n=0} = e^{i\sigma\eta' x'}v \quad \text{on} \quad \mathbb{R}^{n-1}, \tag{1.2}$$

where v is any given vector in $\mathbb{C}^n$. Taking a root $\tilde{z}$ of the equation (in z)

$$\det\ (I - L(\eta', z)) = 0 \tag{1.3}$$

and a vector $\tilde{v}$ belonging to $\text{Ker}(I - L(\eta', \tilde{z}))$, we can make a solution of the form $e^{i\sigma\eta' x'}e^{i\sigma\tilde{z}x_n}\tilde{v}$, but it is not guaranteed that this solution can satisfies (1.2) (i.e. $\tilde{v} = v$).

Obviously, the non-real roots of (1.3) are complex conjugate each other. The roots $\{z^j_\pm(\eta')\}_{j=1,\cdots,d'}$ of (1.3) are classified in the following way if η' is non-glancing:

(i) $z^j_\pm(\eta')$ $(j = 1, \cdots, k)$ are real and satisfy $\pm\partial_{\xi_n}\lambda_j(\eta', z^j_\pm(\eta')) > 0$,
(ii) $z^j_\pm(\eta')$ $(j = k+1, \cdots, d')$ are non-real and satisfy $\pm\text{Im}\ z^j_\pm(\eta') > 0$.

Furthermore, the multiplicities of the real roots $z^j_\pm$ $(j = 1, \cdots, k)$ coincide with those of the eigen-values $\lambda_j(\eta', z^j_\pm(\eta'))$ (cf. Lemma 2.1 in Soga [2]).

One of our main assertions is that under some assumption on the non-real roots $z_{\pm}^{j}(\eta')$ $(j = k+1, \cdots, d')$, linear combinations of the solutions

$$E_{+}^{0} = \{e^{i\sigma\eta' x'} e^{i\sigma z_{+}^{j}(\eta')x_n} v^j : v^j \in \mathrm{Ker}\,[I - L(\eta', z_{+}^{j}(\eta'))],\ j = 1, \cdots, d'\}$$

can cover the boundary data $e^{i\sigma\eta' x'} v$ for any $v \in \mathbb{C}^n$ (cf. Theorem 2.1). In general systems, this is not necessarily obtained, and is set as an assumption.

As another main assertion, even if the assumption on the non-real roots is not satisfied, adding the solutions represented of the same form as for the Poisson operators, we can make linear combinations of the solutions in the extended class cover the boundary data (cf. Theorem 2.4).

In Soga [2] we have already obtained the first assertion in more general situations. In this note, however, we give a new idea of the proof due to a suggestion by Professor J. Ralston, which is a modified method of Kostyuchenko and Shakalikov [1] for operator pencils. We only explain the main results together with outlines of the proof. And in a forthcoming paper, we will describe the precise proofs, and moreover intend to improve the present results.

2. Main results

As is easily seen, the first assertion stated in §1 follows from

THEOREM 2.1. *Let η' be non-glancing, and assume that*

$$\dim \mathrm{Ker}\,[I - L(\eta', z_{+}^{j}(\eta'))] = \text{multiplicity of } z_{+}^{j}(\eta') \tag{2.1}$$

for the non-real roots $z_{+}^{j}(\eta')$ $(j = k+1, \cdots, d')$. Then we have

$$\sum_{j=1}^{d'} \mathrm{Ker}\,[I - L(\eta', z_{+}^{j}(\eta'))] = \mathbb{C}^n.$$

In the isotropic case the condition (2.1) is satisfied, but in general cases it is not necessarily satisfied. Let us note

REMARK 2.2. Let η' be non-glancing. Then, for any root $\tilde{z}$ of (1.3) we have

$$\dim \mathrm{Ker}\,[I - L(\eta', \tilde{z})] \leq \text{multiplicity of } \tilde{z}.$$

For proof of this remark, see Remark 2.4 in Soga [2]. The proof of Theorem 2.1 is based on

PROPOSITION 2.3. *Let the assumptions in Theorem 2.1 be satisfied. Then the function* $(I-L(\eta',z))^{-1}$ *has only the simple pole at every of the roots* $\{z_{\pm}^{j}(\eta')\}_{j=1,\cdots,d'}$.

For proof of this proposition, see the proof of Lemma 2.3 in Soga [2]. When (2.1) is not guaranteed, we cannot expect the conclusion of Theorem 2.1. Let c_+ be a closed path in $\mathbb{C}$ surrounding only the non-real roots $\{z_{+}^{j}(\eta')\}_{j=k+1,\cdots,d'}$, and put

$$Q_+(x_n;\eta') = \frac{1}{2\pi i}\int_{c_+} e^{i\sigma z x_n}(I-L(\eta',z))^{-1}\,dz. \tag{2.2}$$

Then, for any $v\in\mathbb{C}^n$ the function $e^{i\sigma\eta' x'}Q_+(x_n;\eta')v$ becomes a bounded solution of (1.1). We employ the class

$$\begin{aligned} E_+^1 \;&=\; \{e^{i\sigma\eta' x'}e^{i\sigma z_+^j(\eta')x_n}v^j : v^j\in \mathrm{Ker}\;[I-L(\eta',z_+^j(\eta'))],\; j=1,\cdots,k\} \\ &\quad \bigcup\{e^{\sigma\eta' x'}Q_+(x_n;\eta')v :\; v\in\mathbb{C}^n\}. \end{aligned}$$

Linear combinations of the solutions in E_+^1 can cover any boundary data. Namely we obtain

THEOREM 2.4. *Let* η' *be non-glancing. Then we have*

$$\sum_{j=1}^{k}\mathrm{Ker}\;[I-L(\eta',z_+^j(\eta'))] + Q_+(0;\eta')\mathbb{C}^n = \mathbb{C}^n.$$

The solution $e^{i\sigma\eta' x'}Q_+(x_n;\eta')v$ is represented by sum of the solutions in E_+^0 if the roots surrounded by c_+ are all simple. Namely, we have

PROPOSITION 2.5. *Let* $\tilde z$ *be a root of* (1.3) *and let* $\tilde c$ *be a small path surrounding* $\tilde z$. *If* $\tilde z$ *is the simple pole of* $(I-L(\eta',z))^{-1}$, *we have*

(i) $\mathrm{Ker}\;[I-L(\eta',\tilde z)] = \int_{\tilde c}(I-L(\eta',z))^{-1}dz\,\mathbb{C}^n\,,$

(ii) $\dfrac{1}{2\pi i}\int_{\tilde c} e^{i\sigma z x_n}(I-L(\eta',z))^{-1}dz v = e^{i\sigma\tilde z x_n}\mathrm{Res}_{\,z=\tilde z}(I-L(\eta',z))^{-1}v\,.$

For the proof, see Lemma 2.5 in Soga [2].

3. Outline of the proofs

Let $(v, w) = \sum_{i=1}^{n} v_i \overline{w_i}$, and for $v \in \mathbb{C}^n$ put

$$F_v(z) = ((I - L(\eta', z))^{-1} v, v). \tag{3.1}$$

Then $F_v(z)$ becomes a meromorphic function and may have poles at $\{z_{\pm}^j(\eta'), \infty\}_{j=1,\cdots,d'}$ at most. We obtain

LEMMA 3.1. *Let $\tau \in \{z_{\pm}^j(\eta')\}_{j=1,\cdots,d'}$, and assume that $(I - L(\eta', z))^{-1}$ has a simple pole at τ or is analytic at τ. Then, if v is orthogonal to* $\mathrm{Ker}[I - L(\eta', \overline{\tau})]$, *$F_v(z)$ becomes analytic at τ.*

If the root τ of (1.3) is real, we can know a precise form of $(I - L(\eta', z))^{-1}$ for real z near τ:

LEMMA 3.2. *Let η' be non-glancing, and let the real root τ of* (1.3) *satisfy $\lambda_l(\eta', \tau) = 1$. Then we have a C^∞ function $a(z)$ defined for real z near τ such that*

$$\begin{aligned} (I - L(\eta', z))^{-1} &= \sum_{j=1;\neq l}^{d} \frac{1}{1 - \lambda_j(\eta', z)} P_j(\eta', z) + \frac{a(z)}{z - \tau} P_l(\eta', z), \\ a(\tau) &= -\{\partial_{\xi_n} \lambda_l(\eta', \tau)\}^{-1} \end{aligned}$$

where $P_j(\xi', z)$ is the projection to the eigen-space of $\lambda_j(\xi', z)$.

In order to prove Theorem 2.1, assuming that $v \in \mathbb{C}^n$ is orthogonal to $\sum_{j=1}^{d'} \mathrm{Ker}\ [I - L(\eta', z_+^j(\eta'))]$, we have only to show that v must be equal to 0. We insert the v into $F_v(z)$ in (3.1). Using Lemma 3.1 and the equality $\overline{F_v(z)} = F_v(\overline{z})$, we see that $\{z_+^j(\eta')\}_{j=1,\cdots,d'}$ and $\{z_-^j(\eta')\}_{j=k+1,\cdots,d'}$ are not the poles, and also that ∞ is not the pole, which follows from

$$(I - L(\eta', z))^{-1} \quad \sim \quad -z^{-2} a_{nn}^{-1} \qquad \text{as} \qquad |z| \to \infty. \tag{3.2}$$

Let c_-^j be a small circle surrounding only $z_-^j(\eta')$. Then, (3.2) yields that

$$\begin{aligned} &\lim_{r \to +\infty} (2\pi i)^{-1} \int_{|z|=r} (I - L(\eta', z))^{-1} dz \\ &= \sum_{j=1}^{k} (2\pi i)^{-1} \int_{c_-^j} (I - L(\eta', z))^{-1} dz \\ &= 0. \end{aligned}$$

Therefore, by Proposition 2.3 and Lemma 3.2, we obtain

$$\sum_{j=1}^{k}\{\partial_{\xi_n}\lambda_j(\eta', z_-^j(\eta'))\}^{-1}(P_j(\eta', z_-^j(\eta'))v, v) = 0. \tag{3.3}$$

From the non-glancingness of η' we see that $(P_j(\eta', z_-^j(\eta'))v, v) = 0$ for every $j = 1, \cdots, k$, and therefore, by (3.2) we have $z^2F_v(z) = -(a_{nn}^{-1}v, v) = 0$, which proves that $v = 0$. Thus Theorem 2.1 is proved.

Assume that $v \in \mathbb{C}^n$ is orthogonal to the subspace $\sum_{j=1}^{k} \mathrm{Ker}\ [I - L(\eta', z_+^j(\eta'))] + Q_+(0;\eta')\mathbb{C}^n$, and insert this v into $F_v(z)$ in (3.1). In the same way as for Theorem 2.1, we see that $F_v(z)$ is analytic at the real roots $z_+^j(\eta')$ $(j = 1, \cdots, k)$.

Let c_+ (c_-) be a closed path in $\mathbb{C}$ surrounding only the non-real roots $\{z_+^j(\eta')\}_{j=k+1,\cdots,d'}$ $(\{z_-^j(\eta')\}_{j=k+1,\cdots,d'})$. Then, in the same way as for (3.3), we obtain

$$0 = \sum_{j=1}^{k}\partial_{\xi_n}\lambda_j(\eta', z_-^j(\eta'))^{-1}(P_j(\eta', z_-^j(\eta'))v, v) + (Q_+v, v) + (Q_-v, v),$$

where $Q_\pm = (2\pi i)^{-1}\int_{c_\pm}(I - L(\eta', z))^{-1}dz$. From the above equality, using $Q_- = {}^t\overline{Q_+}$, we have $v = 0$ in the same way as for Theorem 2.1. Thus Theorem 2.4 is obtained.

References

[1] A. G. Kostyuchenko and A. A. Shkalikov, *Self-adjoint quadratic operator pencils and elliptic problems*, Func. Anal. Appl. **17** (1983), 109–128.
[2] H. Soga, *Asymptotic solutions of the elastic wave equation in the case of total reflection*, To appear in Comm. PDE.

Faculty of Education
Ibaraki University
Mito Ibaraki 310-8512, Japan
E-mail: kawasita@mito.ipc.ibaraki.ac.jp
E-mail: soga@mito.ipc.ibaraki.ac.jp