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DETERMINATION OF TEMPERATURE FIELD FOR BACKWARD HEAT TRANSFER

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ABSTRACT. Consider an inverse problem of determining the 2-D temperature distribution from known temperature given at some time $T > 0$. Our aim is to find the temperature for $0 < t < T$. For this backward heat conduction problem, we give a stable method to determine the temperature from the measurement data with error pollution. Furthermore, the convergence rate of our approximate solution is also given. Numerical results illustrating our method are also presented.

1. Introduction

Consider a bounded domain $\Omega \subset R^2$ with piecewise smooth boundary $\partial\Omega$. We assume there do not exist any heat source and heat sink within the medium, and the temperature at boundary $\partial\Omega$ is prescribed. If the initial temperature at time $t = 0$ is also given, then the temperature field $T(x,t)$ in $t > 0$ is governed by the following system:

$$(1.1)\quad \begin{cases} \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_1}\left[k(x_1,x_2)\frac{\partial T}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[k(x_1,x_2)\frac{\partial T}{\partial x_2}\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \Omega \times (0,\infty) \\ T(x,t) = T_b(x,t) \qquad\qquad x \in \partial\Omega \\ T(x,0) = f_1(x) \qquad\qquad x \in \Omega, \end{cases}$$

where we set $x = (x_1, x_2) \in \Omega$, $k(x_1,x_2)$ is the medium parameter. The temperature $T(x,t)$ in $t > 0$ can be determined from this system by classical method.

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However, we can also meet the other situation in practice; for example the temperature is not known at the initial time $t = 0$, but at some final time $t = T_0 > 0$. We also want to determine the temperature $T(x, t)$ for $0 < t < T_0$. This problem is the so-called backward heat conduction problem. In this problem, we should solve $T(x, t)$ from

$$(1.2) \quad \begin{cases} \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_1}\left[k(x_1, x_2)\frac{\partial T}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[k(x_1, x_2)\frac{\partial T}{\partial x_2}\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \Omega \times (0, T_0) \\ T(x,t) = T_b(x,t) \qquad\qquad x \in \partial\Omega, t \in (0, T_0) \\ T(x, T_0) = h_1(x) \qquad\qquad x \in \Omega \end{cases}$$

to get $T(x, t)$ from known $h_1(x), T_b(x, t), k(x)$. It is well known that this problem is ill-posed in the following sense([5]):

(1) the solution to (1.2) may not exist for some given $h_1(x)$. From physics point of view, this implies that not all the function $h_1(x)$ can be considered as the final temperature distribution from the forward heat conduction system;

(2) even if there exists some $T(x, t)$ solving (1.2) for $h_1(x)$, then $T(x, t)$ does not depend on $h_1(x)$ continuously.

This property makes it very difficult for us to solve the temperature field from the final measured temperature data with error. That is, if there is a little error in our measurement for final temperature, which is impossible to avoid in practice, then solving (1.2) to get the temperature distribution by classical method may lead to a solution with error arbitrarily large. From the physics point of view, this solution is nonsense. For other inverse problems related to heat transfer problem, we refer as [3] which gives us some basic physics background for this inverse problem.

This problem can be simplified without any substantial change for its difficulties. Firstly, to illustrate our methodology for this inverse problem and to avoid complicated mathematical analysis, we assume that the thermal conductivity $k(x) = 1$, more discussions on the equation with variable coefficients may be found in [2]. Secondly, by letting

$$T(x, t) = w(x, t) + u(x, t)$$

and determining $w(x, t)$ from the following forward heat transfer problem

$$(1.3) \quad \begin{cases} \frac{\partial w}{\partial t} = \Delta w & (x,t) \in \Omega \times (0, \infty) \\ w(x,t) = T_b(x,t) & x \in \partial\Omega \\ w(x, 0) = w_0(x) & x \in \Omega, \end{cases}$$

for some initial temperature $w_0(x)$ given appropriately, we know the new temperature field $u(x,t)$ should satisfy

$$(1.4)\qquad \begin{cases} \frac{\partial u}{\partial t} = \Delta u & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0 & x \in \partial\Omega \\ u(x,0) = f_1(x) - w_0(x) = f(x) & x \in \Omega \end{cases}$$

for the direct problem and

$$(1.5)\qquad \begin{cases} \frac{\partial u}{\partial t} = \Delta u & (x,t) \in \Omega \times (0,T) \\ u(x,t) = 0, & x \in \partial\Omega \\ u(x,T) = h_1(x) - w_0(x,T_0) = g(x) & x \in \Omega \end{cases}$$

for the backward heat transfer problem. Here and henceforth, for simplicity of the notation, we denote by T the final time, rather than T_0 in (1.2).

In this paper, we consider (1.4) and (1.5) directly. Firstly, for initial temperature distribution $f(x) \in L^2(\Omega)$, (1.4) defines a map K

$$(1.6)\qquad K: \quad Kf(x) = u(x,T).$$

We denote by $u(g)(x,t)$ the solution to (1.5) if it exists, and we measure the temperature by

$$\|u(t)\| = \left(\int_\Omega |u(x,t)|^2 dx\right)^{1/2}, \qquad \|g\| = \left(\int_\Omega |g(x)|^2 dx\right)^{1/2}.$$

Now we can reveal the difficulty for the backward heat transfer problem stated above clearly from the following result and example for our simplified model.

THEOREM 1.1. *If $u(x,t)$ solves (1.5), then it satisfies*

$$(1.7)\qquad \|u(t)\| \geq \|g\| \exp\left(\frac{\int_\Omega |\nabla g(x)|^2 dx}{\int_\Omega |g(x)|^2 dx}(T-t)\right)$$

for any $0 \leq t \leq T$.

PROOF. Define $F(t) = \|u(t)\|^2$ for any $t \in [0,T]$. It is easy to show $(\ln F(t))'' \geq 0$ by simple calculation. In fact,

$$F'(t) = 2\int_\Omega |\nabla u|^2 dx, \qquad F''(t) = 4\left\|\frac{\partial u}{\partial t}\right\|^2,$$

which implies $F(t)F''(t)-(F'(t))^2 \geq 0$ from Schwarz's inequality. Therefore $\ln F(t)$ is a convex function in $(0,T)$. Now the Taylor's expansion for $\ln F(t)$ at $t=T$ says

$$\ln F(t) \geq \ln F(T) + \frac{F'(T)}{F(T)}(t-T)$$

from which we get (1.7) immediately since

$$F'(T) = -2\int_\Omega |\nabla g(x)|^2 dx$$

by simple calculation. □

Especially, let $\Omega = [0,\pi]\times[0,\pi]$ and $g_N(x) = \frac{1}{N}\sin(Nx_1)\sin(Nx_2)$. Then, by direct calculation, we know from the above theorem that

$$\|u_N(t)\| \geq \frac{1}{N}\exp\left(\frac{N^2}{2}(T-t)\right).$$

We can always choose N large enough such that $\|u_N(t)\|$ large arbitrarily and $\|g_N\| = \frac{1}{N}$ small enough.

This example shows that if we know some error data $g_\delta(x)$ of $g_0(x)$ satisfying

$$\|g_\delta - g_0\| \leq \delta, \tag{1.8}$$

then the exact solution $u(g_\delta)$ to (1.5) may be far away from $u(g_0)$ arbitrarily. Therefore an interesting problem is how to get the approximation of $u_0(g_0)$ from the error data g_δ satisfying (1.8), if we have known that the exact solution $u_0(g_0)$ to (1.5) corresponding to $g=g_0$ exists.

Essentially, this problem can be considered as solving some integral equation of first kind with smooth kernel. To get the approximate solution for such kind of equation from the error data, some regularization technique has been used both in theoretical analysis and in practical application. In this method, one takes the minimizer of some regularized functional as the approximate solution. However, there are two major difficulties in this method. On one hand, in order to guarantee the existence of exact minimizer, the error data $g_\delta(x)$ should lie in some range domain related to operator K, this condition is hard to verify in practice. On the other hand, the choice of regularization parameter needs some special technique to guarantee the convergence of the approximate solution. In this paper, by applying some conditional stability result, together with the new strategy for the choice of regularized parameter proposed in [1], we propose a new stable method to construct the temperature field from error data of $g_0(x)$. The convergence rate is also given. Our method relax the restriction on measured data. The key point is to give an approximate minimizer $f_\delta(x)$ of some regularized functional from the error data g_δ, then solve the corresponding forward heat transfer problem taking $f_\delta(x)$ as the initial value. Finally we prove that such an approximate solution approaches to $u(g_0)(x,t)$ for $0 < t < T$as $\delta \to 0$ and the convergence rate is Holder type. However, it is very interesting we still do not know whether or not $f_\delta(x) \to u_0(x,0)$ as $\delta \to 0$. A similar problem is discussed in [6] where the approximate solution is constructed in terms of the filtering method which trunced the large eigenvalues of the direct problem. For numerical solutions applying SOR method, we refer to [4].

2. Main result

For exact final temperature $g(x) = g_0(x)$ given at $t = T > 0$, we assume there exists a solution $u_0(x,t)$ to (1.5). This means the final temperature is meaningful. Now if we get the measured data g_δ of g_0 with the error level $\delta > 0$ in the sense of (1.8), we want to give a method to find the approximate temperature distribution $u_\delta(x,t)$ before time T from the error data g_δ. Of course, it should satisfy

$$u_\delta(x,t) \to u_0(x,t)$$

in some sense for all $t < T$ in case of $\delta \to 0$, if the approximate solution is reasonable.

Suppose we have an upper bound of the exact solution $u_0(x,t)$ at time $t = 0$, say, $\|u_0(0)\| \le m_0$ for some known constant $m_0 > 0$. We denote by $f_0(x)$ the initial temperature of $u_0(x,t)$, i.e., $f_0(x) = u_0(x,0)$. Furthermore, define a functional

$$F_\alpha^\delta(f) = \|Kf - g_\delta\|^2 + \alpha\,\|f\|^2 \tag{2.1}$$

over $L^2(\Omega)$, where $\|.\|$ is the L^2−norm in Ω.

THEOREM 2.1. *For any $C_0^2 > m_0^2 + 1$, there exists an approximate minimizer $f_\delta(x)$ for functional $F_{\delta^2}^\delta(f)$ over $L^2(\Omega)$ which satisfies*

$$F_{\delta^2}^\delta(f_\delta) \le C_0^2\delta^2, \tag{2.2}$$

$$\|Kf_\delta - Kf_0\| \le (C_0 + 1)\delta \tag{2.3}$$

and $f_\delta(x)$ can be solved from a known linear equation which contains δ and $g_\delta(x)$.

THEOREM 2.2. *For $f_\delta(x) \in L^2(\Omega)$ generated in the above theorem, we solve the forward heat conduction problem (1.4) with $f(x) = f_\delta(x)$ to get the approximate temperature $u_\delta(x,t)$. For such a approximate solution, it holds*

$$\|(u_\delta - u_0)(t)\| \leq 2(m_0+2)^2 \delta^{\frac{t}{T}} \tag{2.4}$$

for all $0 \leq t \leq T$.

REMARK 2.3. The two results stated above give a stable method to determine the approximation of $u_0(x,t)$ from the error data g_δ. Furthermore, the second result also gives convergence rate. Notice, the estimate for convergence is true but nonsense at $t = 0$, it gives no information about the convergence of $u_\delta(x,0)$ to $u_0(x,0)$. In other word, our estimate is valid only in $t > 0$. To get an estimate up to $t = 0$, we should modify the regularized term in functional F_α^δ, and give a more strong bound for $u_0(x,0)$. This problem will be discussed furthermore([2]).

REMARK 2.4. The only information in our method is the up bound of the exact solution $u_0(x,t)$ at $t = 0$. The constant m_0 is not difficult to get in many cases. Further, our estimate gives the error bound by m_0 and δ explicitly. From the convergence rate, we know that $u_\delta(x,t)$ converges to $u_0(x,t)$ fast near $t = T$ and slowly near $t = 0$. This is reasonable from the physics background.

3. Conditional stability and convergence rate

Firstly, we define function set

$$\begin{aligned}\mu_m = \{\phi(x,t):\quad & \phi(.,t) \in L^2(\Omega) \text{ for any fixed } t \in (0,T),\\ & \phi(x,.) \in L^2[0,T] \text{ for any fixed } x \in \Omega, \|\phi(0)\| \leq m\}\end{aligned}$$

for some known constant $m > 0$. Then it holds

THEOREM 3.1. *Assume that $u_i(x,t)$ solves (1.5) with $g = g_i$ for $i = 1,2$. Then*

$$\|(u_1 - u_2)(t)\| \leq (2m)^{1-t/T} \|g_1 - g_2\|^{t/T}. \tag{3.1}$$

PROOF. Firstly, we can show that if the solution $u(x,t)$ to (1.5) lies in μ_{2m}, then

$$\|u(t)\| \leq (2m)^{1-t/T} \|g\|^{t/T} \tag{3.2}$$

for all $t \in [0,T]$. In fact, since $\ln F(t)$ is convex, then we get

$$\ln F(\theta t_1 + (1-\theta)t_2) \leq \theta \ln F(t_1) + (1-\theta) \ln F(t_2)$$

for any $\theta \in [0,1]$ and $t_1, t_2 \in [0,T]$. Taking $\theta = t/T$ and $t_1 = T, t_2 = 0$ generates the result immediately from the definition of set μ_m.

Now take $u(x,t) = u_1(x,t) - u_2(x,t)$ which solves (1.5) with $g = g_1 - g_2$ and $\|u(0)\| \leq \|u_1(0)\| + \|u_2(0)\| \leq 2m$, the proof is complete from (3.2). □

Now Theorem 2.1 can be proved from the above theorem. Firstly, it is obvious that

$$\begin{aligned} F^{\delta}_{\delta^2}(f_0) &= \|Kf_0 - g_\delta\|^2 + \delta^2 \|f_0\|^2 \\ &= \|g_0 - g_\delta\|^2 + \delta^2 \|f_0\|^2 \\ &\leq \delta^2 + m_0^2 \delta^2 \\ &= (m_0^2 + 1)\delta^2, \end{aligned} \tag{3.3}$$

which implies $\{f : F^{\delta}_{\delta^2}(f) \leq C_0^2 \delta^2\} \neq \emptyset$ due to $C_0^2 > m_0^2 + 1$. Hence (2.2) is proven. From this inequality we also know

$$\|f_\delta\| \leq C_0, \tag{3.4}$$

$$\|Kf_\delta - g_\delta\| \leq C_0 \delta. \tag{3.5}$$

Therefore we get

$$\|Kf_\delta - Kf_0\| \leq \|Kf_\delta - g_\delta\| + \|g_\delta - Kf_0\| \leq (C_0 + 1)\delta.$$

So we get (2.3). Since the exact minimizer of functional $F^{\delta}_{\delta^2}(f)$ over $L^2(\Omega)$ satisfies

$$(\delta^2 I + K^* K) f = K^* g_\delta, \tag{3.6}$$

then $f_\delta(x)$ can be solved from this equation approximately, where K^* is the adjoint operator of K. In our problem, it is easy to see $K^* = K$. Hence Theorem 3.1 is proved.

REMARK 3.2. The estimate (2.2) gives an error level for us to solve (3.6) approximately. That is, it is enough for us to get the approximate solution to (3.6) such that it satisfies (2.2).

Now we prove Theorem 2.2. For certainty, we fix $C_0 = m_0 + 1$. Since $u_\delta(x,t)$ solves (1.5) with $f = f_\delta$, we know from (3.4) that $u_\delta(x,t) \in \mu_{m_0+1}$. On the other hand, it is obvious that $u_0(x,t) \in \mu_{m_0} \subset \mu_{m_0+1}$, therefore the conditional stability in Theorem 3.1 tells us

$$\begin{aligned} \|(u_\delta - u_0)(t)\| &\leq 2(m_0+1)\,\|Kf_\delta - Kf_0\|^{t/T} \\ &\leq 2(m_0+1)[(C_0+1)\delta]^{t/T} \\ &\leq 2(m_0+1)(C_0+1)\delta^{t/T} \\ &\leq 2(m_0+2)^2\delta^{t/T}, \end{aligned} \tag{3.7}$$

the proof is complete.

4. Derivation of operator K

In order to get the approximate minimizer $f_\delta(x)$, one possible way is to solve the linear equation (3.6). Here we derive the expressions of K. Of course, one can also get $f_\delta(x)$ by other method such as Newton iterative method. However, the expression for operator K is also necessary.

The basic idea to get operator K comes from the method of variable separation. Suppose that $\{\lambda_n, u_n(x)\}_{n=1}^\infty$ satisfies

$$\begin{cases} -\Delta u_n(x) = \lambda_n u_n(x) & x \in \Omega \\ u_n(x) = 0 & x \in \partial\Omega. \end{cases} \tag{4.1}$$

We know that $0 < \lambda_1 < \lambda_2 < \ldots.. < \lambda_n < \ldots \to +\infty$ and $\{u_n(x)\}_{n=1}^\infty$ is a basis of $L^2(\Omega)$. Without loss of generality, we assume $\|u_n\| = 1$. For given $f(x) \in L^2(\Omega)$, we know

$$u(x,t) = \sum_{n=1}^\infty c_n \exp\{-\lambda_n t\} u_n(x), \tag{4.2}$$

where the coefficient

$$c_n = \int_\Omega f(y) u_n(y) dy \tag{4.3}$$

for $n = 1, 2, \ldots.$ Now taking $t = T$ in (4.2) and inserting c_n into (4.2) get

$$Kf(x) = u(x,T) = \int_\Omega \left(\sum_{n=1}^\infty \exp\{-\lambda_n T\} u_n(x) u_n(y) \right) f(y) dy. \tag{4.4}$$

This is the expression for operator K. In some special cases such as Ω being a rectangle or a cycle, $\{\lambda_n, u_n(x)\}$ can be written out explicitly.

5. Numerical implementations

To test the validity of our method we consider two examples. Firstly, we set $\Omega = [0, \pi] \times [0, \pi]$ and

$$f_0(x) = \sin x_1 \sin x_2 \tag{5.1}$$

for $x = (x_1, x_2) \in \Omega$. Then the exact solution to the direct problem (1.4) is

$$u_0(x, t) = e^{-2t} \sin x_1 \sin x_2 \tag{5.2}$$

for $0 < t \leq T$ and the final value of temperature at time $t = T$ for (1.5) is

$$g_0(x) = u_0(x, T) = e^{-2T} \sin x_1 \sin x_2. \tag{5.3}$$

We generate the noisy inversion data by

$$g^\delta(x) = g_0(x) + \eta(x) \frac{\delta}{\pi}, \tag{5.4}$$

where $-1 \leq \eta(x) \leq 1$ is some random function. We will solve the approximate temperature $u^\delta(x, t)$ from noisy data $g^\delta(x)$ applying the method proposed in this paper and verify the validity of our method, by comparing our numerical solution with the exact solution.

According to the argument in the last section, we can express the solution to (1.4) in terms of the eigenvalues and eigenfunctions from which we get

$$Kf(x) = \int_\Omega H(x, z; T) f(z) dz, \tag{5.5}$$

$$\begin{aligned} K^2 f(x) &= \int_\Omega \left(\int_\Omega H(x, y; T) H(y, z; T) dy \right) f(z) dz \\ &= \int_\Omega H(x, z, 2T) f(z) dz \end{aligned} \tag{5.6}$$

with the kernel function

$$\begin{aligned} &H(x, y; T) \\ &= \frac{4}{\pi^2} \sum_{n,m=1}^{\infty} e^{-(n^2+m^2)T} (\sin nx_1 \sin mx_2)(\sin ny_1 \sin my_2). \end{aligned} \tag{5.7}$$

Noticing $K^* = K$, we apply (5.5)-(5.7) in solving the Euller equation (3.6). Then for known $f^\delta(x)$, we apply the series

$$u^\delta(x,t) = \sum_{n,m=1}^{\infty} C_{n,m} e^{-(n^2+m^2)t} (\sin nx_1 \sin mx_2) \tag{5.8}$$

with the coefficients

$$C_{n,m} = \frac{4}{\pi^2} \int_\Omega f^\delta(y)(\sin ny_1 \sin my_2) dy \tag{5.9}$$

to get $u^\delta(x,t)$. For the infinite series, we choose the first 10 terms as an approximation.

Now we divide the interval $[0,\pi]$ into N subintervals with the nodal points $t^j = (j-1) * \pi/N$ for $j = 1,2,...,N+1$, and treat the integral terms by

$$\int_\Omega f(x_1,x_2)dx_1dx_2 = \left(\frac{\pi}{N}\right)^2 \sum_{j,k=1}^{N} f(x_1^j, x_2^k)$$

to get a linear algebra equations with the unknowns $f^\delta(x_1^j, x_2^k)$.

Under the above configurations, the discrete form of (3.6) is

$$\delta^2 f^\delta(l,m) + \left(\frac{\pi}{N}\right)^2 \sum_{j,k=1}^{N} H(l,m;j,k;2T) f^\delta(j,k) \tag{5.10}$$

$$= \left(\frac{\pi}{N}\right)^2 \sum_{j,k=1}^{N} H(l,m;j,k;T) g^\delta(j,k),$$

which is a linear algebra equations with respect to $f^\delta(l,m)$ for $l,m = 1,2,...N+1$, where we set

$$f^\delta(l,m) = f^\delta(x_1^l, x_2^m); \quad g^\delta(j,k) = g^\delta(z_1^j, z_2^k),$$
$$H(l,m;j,k;T) = H(x_1^l, x_2^m; z_1^j, z_2^k; T).$$

The unknowns $f^\delta(j,k)$ in this linear system have double subscripts which are not convenient in solving this equations. We establish a 1-1 correspondence between double subscripts (j,k) and single subscript mm by following rules:

$$mm = j + (k-1)(N+1),$$

$$(j,k) = \begin{cases} (N+1, \frac{mm}{N+1}) & \text{if } \frac{mm}{N+1} \text{ is an integral} \\ \left(mm - \left[\frac{mm}{N+1}\right](N+1), \left[\frac{mm}{N+1}\right] + 1\right) & \text{if } \frac{mm}{N+1} \text{ is not an integral,} \end{cases}$$

where [.] denotes the integral part. Now the linear equations read as

$$\delta^2 f^\delta(mm) + \sum_{nn=1}^{N^2} H2(mm;nn) f^\delta(nn) = \sum_{nn=1}^{N^2} H1(mm;nn) g^\delta(nn) \tag{5.11}$$

for $mm = 1, 2, ..., (N+1)^2$, where

$$H1(mm;nn) = \left(\frac{\pi}{N}\right)^2 H(l,m;j,k;T),$$
$$H2(mm;nn) = \left(\frac{\pi}{N}\right)^2 H(l,m;j,k;2T).$$

When $f^\delta(nn)$ are solved from this linear equations, $u^\delta(x,t)$ can be obtained from (5.8) and (5.9) approximately.

More precisely, (5.11) can be rewritten as

$$\sum_{n=1}^{(N+1)^2} C(m,n) f^\delta(n) = \sum_{n=1}^{N^2} H1(m;n) g^\delta(n) \tag{5.12}$$

for $m = 1, 2, ..., (N+1)^2$, where the coefficients

$$C(m,n) = \begin{cases} 0 & N^2+1 \le n \le (N+1)^2, \\ H2(m,n) & 1 \le n \le N^2,\ m \ne n, \\ H2(m,n) + \delta^2 & 1 \le n \le N^2,\ m = n \end{cases} \tag{5.13}$$

for $1 \le m \le N^2$, and

$$C(m,n) = \begin{cases} H2(m,n) & 1 \le n \le N^2, \\ 0 & N^2+1 \le n \le (N+1)^2, m \ne n, \\ \delta^2 & N^2+1 \le n \le (N+1)^2, m = n \end{cases} \tag{5.14}$$

for $N^2 + 1 \le m \le (N+1)^2$.

In the first example, we take $f(x) = \sin(x_1)\sin(x_2)$ and $N = 15, \delta = 0.00001, T = 1$. The error date that the iteration procedure for solving linear equations stoped is $ee = .0001$. By 249 iterations, the numerical results at some nodal points and the error at some time layers $t = kt * T/N$ are listed in Tab.1. Notice, the initial value, as well as the corresponding solution $u_0(x,t)$, is small in this example.

Tab. 1 Recovery of $u_0(x,t) = e^{-2t} \sin x_1 \sin x_2$

$(k1, k2, kt)$	real $u_0(k1, k2, kt)$	inve $u_0(k1, k2, kt)$	kt	$err(kt)$
2,6,1	.1629221	.1623722	1	.01786
2,6,16	.0363528	.0359883	3	.01378
2,10,8	.0888484	.0877896	5	.01085
2,14,1	.0765179	.0748262	7	.00868
2,14,16	.0170734	.0168690	9	.00702
6,10,8	.3700849	.3656209	11	.00570
6,14,1	.3187239	.3116328	13	.00465
14,14,1	.1496916	.1462723	19	.00254

In order to illustrate the validity of our method for large initial value, we also consider another function $f(x) = 10\sin(x_1)\sin(x_2)$. The numerical results after 286 itertions for linear equations are given in Tab.2 where we also set $N = 15$ and $\delta = 0.001$. Notice, for the initial temperature distribution large enough, our numerical inversion results are quite satisfactory. On the other hand, if we choose N large, the results are expected to be more exact. In the above examples, we divide time interval $[0, T]$ into 20 subintervals and set $\eta(x_1^j, x_2^k) = (-1)^{j+k}$ for simplicity.

Tab. 2 Recovery of $u_0(x,t) = 10e^{-2t} \sin x_1 \sin x_2$

$(k1, k2, kt)$	real $u_0(k1, k2, kt)$	inve $u_0(k1, k2, kt)$	kt	$err(kt)$
2,6,1	1.6292210	1.6327300	1	.12600
2,6,16	.3635284	.3617875	3	.08807
2,10,8	.8884841	.8824583	5	.06434
2,14,1	.7651787	.7517892	7	.04883
2,14,16	.1707344	.1695681	9	.03814
6,10,8	3.7008490	3.6752100	11	.03037
6,14,1	3.1872390	3.1310210	13	.02447
14,14,1	1.4969160	1.4696230	19	.01321

From the numerical results $err(kt)$, we know our inversion error $\|(u_0 - u_\delta)(t)\|$ is monotone with respect to t, which concides with our theoretical result (2.4).

Finally, we point out that the method proposed in this paper can be applied to treat with the backward heat transfer problem for more general mathematical models.

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