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ON THE WEAK LAW FOR RANDOMLY INDEXED PARTIAL SUMS FOR ARRAYS

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Abstract. For randomly indexed sums of the form $\sum_{i=1}^{N_n}(X_{ni} - c_{ni})/b_n$, where $\{X_{ni}, i \geq 1, n \geq 1\}$ are random variables, $\{N_n, n \geq 1\}$ are positive integer-valued random variables, $\{c_{ni}, i \geq 1, n \geq 1\}$ are suitable conditional expectations and $\{b_n, n \geq 1\}$ are positive constants, we establish a general weak law of large numbers. Our result improves that of Hong [3].

Introduction

Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of random variables on the probability space $(\Omega, \mathcal{F}, P)$ and set $\mathcal{F}_{n,j} = \sigma\{X_{ni}, 1 \leq i \leq j\}, j \geq 1, n \geq 1$, and $\mathcal{F}_{n,0} = \{\emptyset, \Omega\}, n \geq 1$. Let $\{N_n, n \geq 1\}$ be a sequence of positive integer-valued random variables. In this paper, we establish a general weak law of large numbers(WLLN) of the form

$$(1) \qquad \frac{\sum_{i=1}^{N_n}(X_{ni} - c_{ni})}{b_n} \to 0 \quad \text{in probability}$$

as $n \to \infty$, where $\{c_{ni}, i \geq 1, n \geq 1\}$ is an array of random variables and $\{b_n, n \geq 1\}$ is a sequence of positive constants. Hong [3] obtained a WLLN of the following form under the assumption that $N_n/b_n \to c$ in probability as $n \to \infty$, where c ($c > 0$) is constant and

$$(2) \qquad \frac{\sum_{i=1}^{N_n}(X_{ni} - c_{ni})}{N_n} \to 0 \quad \text{in probability}$$

as $n \to \infty$. Note that (1) not only implies (2) under the condition that $N_n/b_n \to c$ ($c \neq 0$) in probability as $n \to \infty$ but also gives the following

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generalization

$$\frac{\sum_{i=1}^{N_n}(X_{ni}-c_{ni})}{\phi(N_n)} \to 0 \quad \text{in probability}$$

as $n \to \infty$ under the condition $\phi(N_n)/b_n \to c$ ($c \neq 0$) in probability as $n \to \infty$, where $\phi : N \to R$ is a real function. When $\{N_n, n \geq 1\}$ is a sequence of positive integers, the WLLNs of the form (1) have been established by Gut [2], Hong and Oh [4], Kowalski and Rychlik [5], and Sung [6]. Our result is a generalization and improvement of Hong's result [3]. The proof owes much to those of earlier articles.

Main results

To prove the main results, we need the following lemma.

LEMMA 1. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of random variables, and $\{b_n, n \geq 1\}$ be a sequence of constants satisfying*

$$b_n/n = O(1). \tag{3}$$

Suppose that

$$\frac{1}{m}\sum_{i=1}^{m} jP(|X_{ni}| > b_j) \to 0 \tag{4}$$

as $j \to \infty$ uniformly in n and m. Then

$$\frac{1}{m}\sum_{i=1}^{m} jP(|X_{ni}| > j) \to 0$$

as $j \to \infty$ uniformly in n and m.

PROOF. By (3), there exists a constant C such that $b_n \leq Cn$ for n sufficiently large. It follows that

$$\begin{aligned}\sum_{i=1}^{m} jP(|X_{ni}| > j) &= \sum_{i=1}^{m} jP(|X_{ni}| > C\frac{j}{C}) \\ &\leq \sum_{i=1}^{m} jP(|X_{ni}| > C[\frac{j}{C}]) \leq \sum_{i=1}^{m} jP(|X_{ni}| > b_{[j/C]}) \\ &= \sum_{i=1}^{m} \frac{j}{[j/C]}[\frac{j}{C}]P(|X_{ni}| > b_{[j/C]}),\end{aligned}$$

where $[a]$ denotes the integer part of a. If $j \geq 2C$, then

$$\frac{j}{[j/C]} \leq \frac{j}{j/C - 1} = \frac{jC}{j - C} \leq \frac{jC}{j/2} = 2C.$$

Thus we have for $j \geq 2C$

$$\frac{1}{m}\sum_{i=1}^{m} jP(|X_{ni}| > j) \leq \frac{2C}{m}\sum_{i=1}^{m}[\frac{j}{C}]P(|X_{ni}| > b_{[j/C]}),$$

and so the result is proved by (4). □

Now, we state and prove our main results.

THEOREM 2. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of random variables, and $\{b_n, n \geq 1\}$ be a sequence of constants satisfying (3) and $0 < b_n \to \infty$. Let $\{N_n, n \geq 1\}$ be a sequence of positive integer-valued random variables such that*

$$P(N_n > kb_n) = o(1) \tag{5}$$

for some positive integer k. Suppose that (4) holds. Then

$$\frac{\sum_{i=1}^{N_n}(X_{ni} - E(X_{ni}I(|X_{ni}| \leq b_n)|\mathcal{F}_{n,i-1}))}{b_n} \to 0 \quad \textit{in probability}$$

as $n \to \infty$.

PROOF. Let $X'_{ni} = X_{ni}I(|X_{ni}| \leq b_n)$ for $i \geq 1, n \geq 1$. Then we have by (3), (4), and (5) that

$$\begin{aligned} P(|\sum_{i=1}^{N_n} X_{ni} - \sum_{i=1}^{N_n} X'_{ni}| &> b_n\epsilon) \\ &\leq P(N_n > kb_n) + P(\cup_{i=1}^{kb_n}(X_{ni} \neq X'_{ni})) \\ &\leq o(1) + \frac{kb_n}{n}\frac{1}{kb_n}\sum_{i=1}^{kb_n} nP(|X_{ni}| > b_n) = o(1). \end{aligned}$$

Thus

$$\frac{\sum_{i=1}^{N_n} X_{ni} - \sum_{i=1}^{N_n} X'_{ni}}{b_n} \to 0 \quad \text{in probability}$$

as $n \to \infty$, and so it suffices to show that

$$\frac{\sum_{i=1}^{N_n}(X'_{ni} - E(X'_{ni}|\mathcal{F}_{n,i-1}))}{b_n} \to 0 \quad \text{in probability}$$

as $n \to \infty$. For $n \geq 1$ and $m \geq 1$, denote:

$$B_m^n = \{|\sum_{i=1}^{m}(X'_{ni} - E(X'_{ni}|\mathcal{F}_{n,i-1}))| > b_n\epsilon), \ D_n = \cup_{m=1}^{kb_n} B_m^n.$$

Then by (5)

$$P(B_{N_n}^n) \leq P(B_{N_n}^n, N_n \leq kb_n) + P(N_n > kb_n) \leq P(D_n) + o(1),$$

and hence it is sufficient to show that $P(D_n) = o(1)$. Since $\{X'_{ni} - E(X'_{ni}|\mathcal{F}_{n,i-1}),\ 1 \leq i \leq kb_n\}$ is a martingale difference sequence, the Hájek-Rényi inequality(see Theorem 7.4.8 in [1]) implies

$$\begin{aligned} P(D_n) &= P(\max_{1\leq m\leq kb_n} |\sum_{i=1}^{m}(X'_{ni} - E(X'_{ni}|\mathcal{F}_{n,i-1}))| > b_n\epsilon) \\ &\leq \frac{1}{b_n^2\epsilon^2}\sum_{i=1}^{kb_n} E(X'_{ni} - E(X'_{ni}|\mathcal{F}_{n,i-1}))^2 \leq \frac{1}{b_n^2\epsilon^2}\sum_{i=1}^{kb_n} E|X'_{ni}|^2. \end{aligned}$$

Moreover,

$$\begin{aligned} \sum_{i=1}^{kb_n} E|X'_{ni}|^2 &= \sum_{i=1}^{kb_n}\sum_{j=1}^{b_n} E|X_{ni}|^2 I(j-1 < |X_{ni}| \leq j) \\ &\leq \sum_{i=1}^{kb_n}\sum_{j=1}^{b_n} j^2 P(j-1 < |X_{ni}| \leq j) \\ &= \sum_{i=1}^{kb_n}\sum_{j=1}^{b_n} j^2[P(|X_{ni}| > j-1) - P(|X_{ni}| > j)] \\ &= \sum_{i=1}^{kb_n}[P(|X_{ni}| > 0) - b_n^2 P(|X_{ni}| > b_n) \\ &\quad + \sum_{j=1}^{b_n-1}((j+1)^2 - j^2)P(|X_{ni}| > j)] \\ &\leq kb_n + \sum_{i=1}^{kb_n}\sum_{j=1}^{b_n-1}(2j+1)P(|X_{ni}| > j). \end{aligned}$$

By Lemma 1, we have

$$\lim_{j\to\infty}\sup_{n\geq 1}\frac{1}{kb_n}\sum_{i=1}^{kb_n} jP(|X_{ni}| > j) = 0.$$

Thus by the Toeplitz lemma we have

$$\begin{aligned}
&\frac{1}{b_n^2}\sum_{i=1}^{kb_n}\sum_{j=1}^{b_n-1}(2j+1)P(|X_{ni}|>j)\\
&=\frac{k}{b_n}\sum_{j=1}^{b_n-1}\frac{2j+1}{j}\frac{1}{kb_n}\sum_{i=1}^{kb_n}jP(|X_{ni}|>j)\\
&=o(1),
\end{aligned}$$

which implies $P(D_n)=o(1)$, since $b_n\to\infty$. □

COROLLARY 3. *Let $\{X_{ni}\},\{b_n\}$, and $\{N_n\}$ be as in Theorem 2 except that condition (5) in Theorem 2 is replaced by*

$$N_n/b_n\to c \quad \textit{in probability} \tag{6}$$

as $n\to\infty$, where $c\neq 0$. Then

$$\frac{\sum_{i=1}^{N_n}(X_{ni}-E(X_{ni}I(|X_{ni}|\le b_n)|\mathcal{F}_{n,i-1}))}{N_n}\to 0 \quad \textit{in probability}$$

as $n\to\infty$.

PROOF. Take a positive integer k such that $c+\epsilon\le k$. It follows from (6) that

$$P(N_n>kb_n)\le P(|\frac{N_n}{b_n}-c|>\epsilon)=o(1).$$

Thus by Theorem 2 we have

$$\begin{aligned}
&\frac{\sum_{i=1}^{N_n}(X_{ni}-E(X_{ni}I(|X_{ni}|\le b_n)|\mathcal{F}_{n,i-1}))}{N_n}\\
&=\frac{b_n}{N_n}\frac{\sum_{i=1}^{N_n}(X_{ni}-E(X_{ni}I(|X_{ni}|\le b_n)|\mathcal{F}_{n,i-1}))}{b_n}\to 0 \quad \text{in probability}
\end{aligned}$$

as $n\to\infty$. □

REMARK. Hong [3] proved Corollary 3 under the stronger conditions that

$$\frac{b_n}{n}\downarrow,\quad \frac{b_n^2}{n}\uparrow\infty,\quad \text{and}\quad \sum_{i=1}^{n}\frac{b_i^2}{i^2}=O(\frac{b_n^2}{n}). \tag{7}$$

Note that the first condition of (7) implies (3) and the second condition implies $b_n\to\infty$.

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