Comm. Korean Math. Soc. **16** (2001), No. 2, pp. 287–290

ON A CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION BY CONDITIONAL EXPECTATIONS OF RECORD VALUES

MIN-YOUNG LEE

ABSTRACT. Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables with continuous cumulative distribution function $F(x)$. X_j is an upper record value of this sequence if $X_j > \max\{X_1, X_2, \cdots, X_{j-1}\}$. We define $u(n) = \min \{j | j > u(n-1),\ X_j > X_{u(n-1)},\ n \geq 2\}$ with $u(1) = 1$. Then $F(x) = 1 - e^{-\frac{x}{c}}$, $x > 0$ if and only if $E[X_{u(n+1)} - X_{u(n)} \mid X_{u(m)} = y] = c$ or $E[X_{u(n+2)} - X_{u(n)} \mid X_{u(m)} = y] = 2c, n \geq m+1$.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed(i.i.d) random variables with a common continuous distribution function $F(x)$ and probability density function $f(x)$. Suppose $Y_n = \max\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of this sequence if $Y_j > Y_{j-1}, j > 1$. By convention X_1 is an upper as well as a lower record value. We can transform from upper records to lower records by replacing the original sequence of random variables by $\{-X_j, j \geq 1\}$.

We define the record times $u(n)$ by $u(1) = 1$ and

$$u(n) = \min\{j | j > u(n-1), X_j > X_{u(n-1)}, n \geq 2\}.$$

The record times of the sequence $\{X_n, n \geq 1\}$ are random variables and are the same as those for the sequence $\{F(X_n); n \geq 1\}$. We know that

Received March 3, 2000.

2000 Mathematics Subject Classification: 60E15, 62E10.

Key words and phrases: absolutely continuous distribution, characterization, record value.

The research was conducted by the research fund of Dankook university in 2000.

the distribution of $u(n)$ does not depend on $F(x)$. Hence, the distribution of $u(n)$ can be determined by considering the uniform distribution $F(x) = x$. Also we will denote $L(n)$ as the indices where the lower record value occur. We will call the random variable $X \in \text{EXP}(\theta)$ if the corresponding probability density function $f(x)$ of x is of the form

$$f(x) = \begin{cases} \frac{1}{\theta}e^{-\frac{x}{\theta}}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Characterizations of the exponential distribution have been extensively studied in the literature. Tata(1969) characterized the exponential distribution by the independent X_1 and $X_{L_1} - X_{L_2}$. Ahsanullah(1982) showed that if F belongs to the class C_1 and $E(X_k), k \geq 1$ is finite then $X_k \in \text{EXP}(\sigma)$, if and only if for some fixed $n, n > 1, E(X_{u(n+1)} - X_{u(n)}) = E(X_k)$. Also he characterized for $F \in C_1$ with finite first moment $X_k \in \text{EXP}(\sigma)$ if and only if for some fixed $n, E(X_{L_n} - X_{L_{n-1}}|X_{L_{n+1}} = u)$ is independent of u.

In this paper we will give a characterization of the exponential distribution by considering conditional expectations of record values.

2. Results

THEOREM 1. *If $F(x)$ is absolutely continuous with $F(x) < 1$ for all x then*

$$(1) \qquad E[X_{u(n+1)} - X_{u(n)}|X_{u(m)} = y] = c, \;\; c > 0, \;\; n \geq m + 1$$

if and only if $F(x) = 1 - e^{-\frac{x}{c}}, \;\; x > 0$.

THEOREM 2. *If $F(x)$ is absolutely continuous with $F(x) < 1$ for all x then*

$$(2) \qquad E[X_{u(n+2)} - X_{u(n)}|X_{u(m)} = y] = 2c, \;\; c > 0, \;\; n \geq m + 1$$

if and only if $F(x) = 1 - e^{-\frac{x}{c}}, \;\; x > 0$.

3. Proof

PROOF OF THEOREM 1. If $X_k \in \text{EXP(c)}$, then $E[X_{u(n)}|X_{u(m)} = y] = y + (n-m)c$. Hence (1) holds.

Conversely, suppose (1) holds. Using Ahsanullah formula(1995) it follows the following equation.

$$\begin{aligned}&\frac{1}{1-F(y)}\int_y^\infty \frac{1}{(n-m)!}\left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m} xf(x)dx\\&-\frac{1}{1-F(y)}\int_y^\infty \frac{1}{(n-m-1)!}\left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m-1} xf(x)dx\\&= c, \text{ where } c>0, n\geq m+1\end{aligned}$$

i.e.

$$\begin{aligned}&\frac{1}{(n-m)!}\int_y^\infty \left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m} xf(x)dx\\&-\frac{1}{(n-m-1)!}\int_y^\infty \left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m-1} xf(x)dx\\&= c(1-F(y)).\end{aligned}\tag{4}$$

Since $F(x)$ is absolutely continuous, we can differentiate $(n-m+1)$ times both sides of (4) with respect to y and simplify, then we obtain the following equation

$$cf(y)+F(y)=1 \quad \text{i.e., } f(y)+\frac{1}{c}F(y)=\frac{1}{c}.\tag{5}$$

When we solve differential equation of (5), we get

$$F(y) = 1-e^{-\frac{y}{c}}.$$

This completes the proof. □

PROOF OF THEOREM 2. If $X_k \in \text{EXP(c)}$, then $E[X_{u(n)}|X_{u(m)} = y] = y + (n-m)c$. Hence (2) holds.

Conversely, suppose (2) holds. Using Ahsanullah formula(1995) if follows the following equation ; that is,

$$\begin{aligned}&\frac{1}{(n-m+1)!}\int_y^\infty \left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m+1} xf(x)dx\\&-\frac{1}{(n-m-1)!}\int_y^\infty \left(\ln\frac{1-F(y)}{1-F(x)}\right)^{n-m-1} xf(x)dx\\&= 2c(1-F(y)).\end{aligned}\tag{6}$$

Since $F(x)$ is absolutely continuous, we can differentiate $(n-m+2)$ times both sides of (6) with respect to y and simplify, then we obtain the following equation

$$3(1-F(y)) - 2cf(y) + \frac{(1-F(y))^2 f'(y)}{f(y)^2} = 0.$$

Let $y = F(y)$ (i.e., $y' = f$, $y'' = f'$). Then it expressed the following form

$$\frac{(1-y)^2 y''}{{y'}^2} + 3(1-y) - 2cy' = 0 \text{ i.e., } y'' = f(x, y, y'). \tag{7}$$

Therefore there exists a unique solution of the differential equation (7) that satisfies the prescribed initial conditions $y(0) = 0$, $y'(0) = \frac{1}{c}$.
By the existence and uniqueness Theorem, we get $F(x) = 1 - e^{-\frac{x}{c}}$.
This completes the proof. □

References

[1] M. Ahsanuallah, *Characterization of the exponential distibution by some properties of the record values*, 1982.

[2] ———, *Record Statistics*, Nova Science publishers, Inc, Commack NY, 1995.

[3] E. Boyce and C. Diprima, *Elementary differential equations and boundary value problems*, John Wiley & Sons, 1976.

[4] M. N. Tata, *An outstanding value in a sequence of random variables*, Zeit. Wahrscheinlichkeitsth **12** (1969), 9–20.

Department of Mathematics
Dankook University
Cheonan 330-714, Chungnam, Korea
E-mail: mylee@anseo.dankook.ac.kr