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ON THE JOINT WEYL AND BROWDER SPECTRA OF HYPONORMAL OPERATORS

YOUNG-YOON LEE

ABSTRACT. In this paper we study some properties of the joint Weyl and Browder spectra for the slightly larger classes containing doubly commuting n-tuples of hyponormal operators.

1. Introduction

Let H be a complex infinite dimensional Hilbert space and $\mathcal{B}(H)$ denote the Banach algebra of all bounded linear operators acting on H. Throughout this paper we let $\mathbf{T} = (T_1, \cdots, T_n)$ denote a commuting n-tuple of operators in $\mathcal{B}(H)$. Recall ([5], [7], [13]) that $\mathbf{T}$ is said to be (*Taylor*) *invertible* if the Koszul complex associated with $\mathbf{T}$ is exact at every stage and is said to be (*Taylor*) *Fredholm* if all cohomologies of the Koszul complex associated with $\mathbf{T}$ is finite dimensional. We let $\sigma_T(\mathbf{T})$ and $\sigma_{Te}(\mathbf{T})$ denote the *Taylor spectrum* and the *Taylor essential spectrum* of $\mathbf{T}$, respectively. If $\lambda \in \sigma_T(\mathbf{T}) \setminus \sigma_{Te}(\mathbf{T})$, then the *index* of $\mathbf{T} - \lambda$, denoted by $\text{ind}(\mathbf{T} - \lambda)$, is defined by the Euler characteristic of the Koszul complex associated with $\mathbf{T} - \lambda$ and let $\sigma_w^1(\mathbf{T})$ denote the *Taylor-Weyl spectrum.* We also recall ([10], [11], [12], [13]) that $\mathbf{T}$ is said to be (*Taylor*)*Browder* if $\mathbf{T}$ is Fredholm and there exists a deleted open neighborhood N_0 of $0 \in \mathbb{C}^n$ such that $\mathbf{T} - \lambda$ is invertible for all $\lambda \in N_0$. Then the *Taylor-Browder spectrum*, denoted by $\sigma_b^1(\mathbf{T})$, is defined by

$$\sigma_b^1(\mathbf{T}) = \sigma_{Te}(\mathbf{T}) \cup \operatorname{acc} \sigma_T(\mathbf{T}), \tag{1.1}$$

where $\operatorname{acc} \sigma_T(\mathbf{T})$ denotes the set of accumulation points of the Taylor spectrum of $\mathbf{T}$. Let $\mathcal{K}(H)$ denote the set of all compact operators acting

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on H and let $\mathbf{K} = (K_1, \cdots, K_n) \in \mathcal{K}(H)^n$ denote an n-tuple of compact operators. In [2] a *joint Weyl spectrum*, denoted by $\sigma_w^2(\mathbf{T})$, is defined by

$$(1.2) \qquad \sigma_w^2(\mathbf{T}) = \bigcap_{\mathbf{K} \in \mathcal{K}(H)^n} \{\sigma_T(\mathbf{T} + \mathbf{K})\}$$

and in [13] a *joint Browder spectrum*, denoted by $\sigma_b^2(\mathbf{T})$, is defined by

$$(1.3) \qquad \sigma_b^2(\mathbf{T}) = \bigcap_{\mathbf{K} \in \mathcal{K}(H)^n} \{\sigma_T(\mathbf{T} \uplus \mathbf{K})\},$$

where $\mathbf{T} \uplus \mathbf{K}$ means a commuting sum such that $\mathbf{T} + \mathbf{K}$ with $T_i K_j = K_i T_j$ for all i, j.

Also, if there exists a non-zero vector x such that

$$(T_i - \lambda_i)x = 0 \quad \text{for all } i = 1, \cdots n,$$

then $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$ is called a *joint eigenvalue* of $\mathbf{T}$. We denote the set of all joint eigenvalues by $\sigma_p(\mathbf{T})$ and the set of isolated eigenvalues of finite multiplicity by $\pi_{00}(\mathbf{T})$.

It is well known([13]) that in the case of an arbitrary single operator $T \in \mathcal{B}(H)$

$$(1.4) \qquad \sigma_w^1(T) = \sigma_w^2(T) \subset \sigma_b^1(T) = \sigma_b^2(T)$$

and in case of a normal operator $T \in \mathcal{B}(H)$

$$(1.5) \qquad \sigma_w^1(T) = \sigma_w^2(T) = \sigma_b^1(T) = \sigma_b^2(T).$$

However, the situation in case of an n-tuple of operators is different in general. It is well known ([13]) that for a commuting n-tuple $\mathbf{T}$ of arbitrary operators

$$(1.6) \qquad \sigma_w^1(\mathbf{T}) \subset \sigma_w^2(\mathbf{T}) \subset \sigma_b^2(\mathbf{T}) \subset \sigma_b^1(\mathbf{T})$$

and for a commuting n-tuple $\mathbf{T}$ of normal operators

$$(1.7) \qquad \sigma_w^1(\mathbf{T}) = \sigma_w^2(\mathbf{T}) = \sigma_b^1(\mathbf{T}) = \sigma_b^2(\mathbf{T}).$$

Recall([13], [15]) that an operator $T \in B(H)$ is said to be *M-hyponormal* if for every $\lambda \in \mathbb{C}$ there exists a positive number M such that

$$(T-\lambda)(T-\lambda)^* \leq M(T-\lambda)^*(T-\lambda).$$

We may note that if $M = 1$, then T is *hyponormal.* In [13], it is well known that for a doubly commuting n-tuple $\mathbf{T}$ of M-hyponormal operators

$$\sigma_b^1(\mathbf{T}) = \sigma_T(\mathbf{T}) \setminus \pi_{00}(\mathbf{T}). \tag{1.8}$$

Recall([1], [3], [4], [10]) that an operator $T \in \mathcal{B}(H)$ is said to be *p-hyponormal* if $(T^*T)^p - (TT^*)^p \geq 0$ for some $p \in (0,1]$. If $p = 1$, T is just hyponormal.

It is well known that the class of p-hyponormal operators properly contains the class of hyponormal operators and p-hyponormal operators have no general relations with M-hyponormal operators.

In this paper we give an extension of (1.7) and an analogue result of (1.8) for some doubly commuting n-tuples of p-hyponormal operators.

2. Main results

Following Frunză([9]) we say that a commuting n-tuple $\mathbf{T}$ has the *single valued extension property*, say *SVEP*, if, for any open polydisk $D \subset \mathbb{C}^n$, the Koszul complex $K(\mathbf{T}-\lambda, \mathcal{O}(D,H))$ has vanishing homology in positive degrees. Here $\mathcal{O}(D,H)$ denotes the Frechét space of H-valued analytic functions on D. There are many examples of n-tuples with the SVEP. As a typical example we mention all commuting n-tuples of analytic Toeplitz operators acting on the Bergman space of a bounded pseudoconvex domain in $\mathbb{C}^m$(c.f., [7]). We shall write $p_{00}(\mathbf{T}) :=$ iso$\sigma_T(\mathbf{T}) \setminus \sigma_{Te}(\mathbf{T})$ for the *(joint) Riesz points* of $\sigma_T(\mathbf{T})$. Then we can notice that the set $p_{00}(\mathbf{T})$ consists of all isolated points that the associated spectral space is finite dimensional.

THEOREM 1. *Let* $\mathbf{T}$ *be a doubly commuting* n*-tuple of* M*-hyponormal operators with the SVEP. Then*

$$\sigma_w^1(\mathbf{T}) = \sigma_w^2(\mathbf{T}) = \sigma_b^2(\mathbf{T}) = \sigma_b^1(\mathbf{T}) = \sigma_T(\mathbf{T}) \setminus \pi_{00}(\mathbf{T}). \tag{2.1}$$

PROOF. Since the fourth equality in (2.1) is just (1.8) and from (1.6)

$$\sigma_w^1(\mathbf{T}) \subset \sigma_w^2(\mathbf{T}) \subset \sigma_b^2(\mathbf{T}) \subset \sigma_b^1(\mathbf{T}),$$

it suffices to show that

$$\sigma_w^1(\mathbf{T}) = \sigma_b^1(\mathbf{T}).$$

We calim that

$$p_{00}(\mathbf{T}) = \text{iso}\sigma_T(\mathbf{T}) \setminus \sigma_w^1(\mathbf{T}) = \sigma_T(\mathbf{T}) \setminus \sigma_b^1(\mathbf{T}). \tag{2.2}$$

Indeed, the first equality in (2.1) follows from the continuity of the index ([5],[16]). And, for $\lambda \in \text{iso}\sigma_T(\mathbf{T})$, it is well known ([8, (2.4)]) that $\mathbf{T} - \lambda$ is Fredholm if and only if the spectral space corresponding to λ is finite dimensional, and so the second equality immediately follows. On the other hand, since $\mathbf{T}$ has the SVEP, from [14] we have

$$p_{00}(\mathbf{T}) = \sigma_T(\mathbf{T}) \setminus \sigma_w^1(\mathbf{T}). \tag{2.3}$$

Hence (2.2) and (2.3) complete the proof. □

As in [3] or [10], let T have the polar decomposition $T = U|T|$ and let $\widehat{T} = |T|^{1/2}U|T|^{1/2}$. Let $\widehat{T}$ have the polar decomposition $\widehat{T} = V|\widehat{T}|$. The operator $\widetilde{T}$ is then defined by $\widetilde{T} = |\widehat{T}|^{1/2}V|\widehat{T}|^{1/2}$. By the consequence of Löwner's inequality if T is p-hyponormal, then T is also q-hyponormal for every $0 < q \leq p$. Thus throughout this paper we can assume, without loss of generality, that $0 < p < 1/2$. If T is p-hyponormal, then $\widehat{T}$ is 1/2-hyponormal and $\widetilde{T}$ is hyponormal([1, Corollary 3]). We let $HU(p)$ denote the class of all p-hyponormal operators that the partial isometry U in the polar decomposition $T = U|T|$ is unitary.

Recall ([5], [6]) that the *left (right) joint spectrum*, denoted by $\sigma_\ell(\mathbf{T})$ ($\sigma_r(\mathbf{T})$), of $\mathbf{T}$ is defined by the set of all points $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$ such that $\{T_i - \lambda_i\}_{1\leq i\leq n}$ generates a proper left (right) ideal in the algebra $\mathcal{B}(H)$. Let $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ be the Calkin algebra with the canonical map $\pi : \mathcal{B}(H) \to \mathcal{C}(H)$. Then the *left (right) joint essential spectrum*, denoted by $\sigma_{\ell e}(\mathbf{T})$ ($\sigma_{re}(\mathbf{T})$), of $\mathbf{T}$ is defined by

$$\sigma_{\ell e}(\mathbf{T}) = \sigma_\ell(\pi(\mathbf{T}))\,(\sigma_{re}(\mathbf{T}) = \sigma_r(\pi(\mathbf{T}))),$$
$$\text{where } \pi(\mathbf{T}) = (\pi(T_1), \cdots, \pi(T_n)).$$

In the view of Theorem 6 in [3], the following theorem give an extension of Theorem 2.3 proved in [10] for a doubly commuting n-tuple of operators in $HU(p)$.

THEOREM 2. *Let* $\mathbf{T}$ *be a doubly commuting* n*-tuple of* p*-hyponormal operators. Then*

$$\sigma_b^1(\mathbf{T}) \setminus [0] = \{\sigma_T(\mathbf{T}) \setminus \pi_{00}(\mathbf{T})\} \setminus [0], \tag{2.4}$$

where $[0] = \{(\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n : \lambda_i = 0$ *for at least one* $i \in I = \{1, \cdots, n\}\}$.

PROOF. From Theorem 1 in [4] we have

$$\sigma_T(\mathbf{T}) = \sigma_T(\widetilde{\mathbf{T}}), \text{ where } \widetilde{\mathbf{T}} = (\widetilde{T_1}, \cdots, \widetilde{T_n}). \tag{2.4}$$

Since $\widetilde{\mathbf{T}}$ is a doubly commuting n-tuple of hyponormal operators, by [6, Theroem 2.8] we have

$$\sigma_T(\widetilde{\mathbf{T}}) = \overline{\sigma_\ell(\widetilde{\mathbf{T}^*})} = \sigma_r(\widetilde{\mathbf{T}}) \text{ and } \sigma_{Te}(\widetilde{\mathbf{T}}) = \sigma_{re}(\widetilde{\mathbf{T}}) \tag{2.5}$$

and by [6, Theorem 2.10]

$$\sigma_r(\widetilde{\mathbf{T}}) = \sigma_{re}(\widetilde{\mathbf{T}}) \cup \overline{\pi_0(\widetilde{\mathbf{T}^*})}, \tag{2.6}$$

where $\pi_0(\cdot)$ denotes the set of all joint eigenvalues of finite multiplicity. Thus applying Theorem 3 and Corollary 5 in [3] we have

$$\begin{aligned} \pi_0(\widetilde{\mathbf{T}}) \setminus [0] &= \{\sigma_{Te}(\widetilde{\mathbf{T}})^c \cap \sigma_p(\widetilde{\mathbf{T}})\} \setminus [0] \\ &= \{\sigma_{Te}(\mathbf{T})^c \cap \sigma_p(\mathbf{T})\} \setminus [0] \\ &= \pi_0(\mathbf{T}) \setminus [0] \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \sigma_b^1(\widetilde{\mathbf{T}}) \setminus [0] &= \{\sigma_{Te}(\widetilde{\mathbf{T}}) \cup \mathrm{acc}\,\sigma_T(\widetilde{\mathbf{T}})\} \setminus [0] \\ &= \{\sigma_{Te}(\mathbf{T}) \cup \mathrm{acc}\,\sigma_T(\mathbf{T})\} \setminus [0] \\ &= \sigma_b^1(\mathbf{T}) \setminus [0]. \end{aligned} \tag{2.8}$$

Also, from (2.7) we have

$$\begin{aligned} \pi_{00}(\widetilde{\mathbf{T}}) \setminus [0] &= \{\pi_0(\widetilde{\mathbf{T}}) \cap \mathrm{iso}\sigma_T(\widetilde{\mathbf{T}})\} \setminus [0] \\ &= \{\pi_0(\mathbf{T}) \cap \mathrm{iso}\sigma_t(\mathbf{T})\} \setminus [0] \\ &= \pi_{00}(\mathbf{T}) \setminus [0]. \end{aligned} \tag{2.9}$$

Since from Theorem 3 in [13]

$$\sigma_b^1(\mathbf{T}) = \sigma_T(\mathbf{T}) \setminus \pi_{00}(\mathbf{T})$$

applying (2.8) and (2.9) we have

$$\sigma_b^1(\mathbf{T}) \setminus [0] = \{\sigma_T(\mathbf{T}) \setminus \pi_{00}(\mathbf{T})\} \setminus [0].$$

Hence the proof completes □

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Department of Mathematics Education
Sangmyung University
Seoul 110-743, Korea
E-mail: yyoon@pine.sangmyung.ac.kr