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MODIFIED HYERS-ULAM-RASSIAS STABILITY OF FUNCTIONAL EQUATIONS WITH SQUARE-SYMMETRIC OPERATION

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Abstract. In this paper, we obtain the modified Hyers-Ulam-Rassias stability for the family of the functional equation $f(x \circ y) = H(f(x)^{1/t}, f(y)^{1/t})$ $(x, y \in S)$, where H is a homogeneous function of degree t and $\circ$ is a square-symmetric operation on the set S.

1. Introduction

In 1940, S. M. Ulam [12] raised the following problem: Under what condition does there exist an additive mapping near an approximately additive mapping?

In 1941, this problem was solved by D. H. Hyers [3]. Thereafter we usually say that the equation $E_1(h) = E_2(h)$ has the Hyers-Ulam stability if for an approximate solution f of this equation, i.e., for a function f with $|E_1(f) - E_2(f)| \leq \delta$ there exists a function g such that $E_1(g) = E_2(g)$ and $|f(x) - g(x)| \leq \epsilon$. In 1978, the Hyers-Ulam stability for approximately linear mapping was generalized by Th. M. Rassias [8] who considers it for the bounded function by the sum of variable. It is called the Hyers-Ulam-Rassias stability. Thereafter P. Găvruta [2] generalized the stability of Rassias for the case of the bounded function as follows:

If for an approximate solution f of the equation $E_1(h) = E_2(h)$, i.e., for a function f such that $|E_1(f) - E_2(f)| \leq \phi$ holds with a given function ϕ, there exists a function g such that $E_1(g) = E_2(g)$ and $|g(x) -$

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$f(x)| \leq \Phi(x)$ for some fixed function Φ. We call it the modified Hyers-Ulam-Rassias stability(or stability in the spirit of Găvruta). Namely the result of Rassias is the case of special type of ϕ in this stability. One is referred to [1], [4], [5], [6], [7], [8], [9], [10], [11] for further generalizations and new open problems.

The aim of present paper is to investigate the modified Hyers-Ulam-Rassias stability for the following family of functional equation:

$$f(x \circ y) = H(f(x)^{\frac{1}{t}}, f(y)^{\frac{1}{t}}) \qquad (x, y \in S), \tag{1.1}$$

where S is a nonempty set, $\circ : S \times S \to S$ is a binary operation and $H : G \times G \to G$ is a homogeneous function of degree $t > 0$, that is, H satisfies

$$H(uv, uw) = u^t H(v, w) \qquad (u, v, w \in G; 0 < t; t \in R), \tag{1.2}$$

and G is a multiplicative subsemigroup of the real or complex field. A particular case of (1.1) is the Cauchy functional equation

$$f(x + y) = f(x) + f(y) \qquad (x, y \in S),$$

where S is a semigroup with the operation $+$ and $f : S \to \mathbb{C}$.

If the operation $\circ$ satisfies the following identity:

$$(x \circ y) \circ (x \circ y) = (x \circ x) \circ (y \circ y) \qquad (x, y \in S),$$

the operation $\circ$ will be called *square symmetric.*

2. Square-symmetric operation

Let S be a nonempty set and $\circ : S \times S \to S$ be a square symmetric operation. In addition, let G be a multiplicative subsemigroup of $\mathbb{C}$, and let $H : G \times G \to G$ satisfy (1.2).

In the following result we show that if the equation (1.1) has sufficiently many solutions, then $\circ$ is necessarily square symmetric.

THEOREM 1. *Assume that the set of solutions of the functional equation* (1.1) *separates the points of S, that is, for $u, v \in S$ with $u \neq v$,* *then there exists a solution $f : S \to G$ of* (1.1) *such that $f(u) \neq f(v)$. Then the operation $\circ$ is square symmetric.*

PROOF. Let $x, y \in S$, and let $f : S \to G$ be an arbitrary solution of (1.1). Then, using the homogeneity of H and (1.1) several times, we obtain

$$\begin{aligned} f((x \circ y) \circ (x \circ y)) &= H(f(x \circ y)^{\frac{1}{t}}, f(x \circ y)^{\frac{1}{t}}) \\ &= f(x \circ y) H(1,1) \\ &= H(f(x)^{\frac{1}{t}}, f(y)^{\frac{1}{t}}) H(1,1) \\ &= H((f(x)H(1,1))^{\frac{1}{t}}, (f(y)H(1,1))^{\frac{1}{t}}) \\ &= H(H(f(x)^{\frac{1}{t}}, f(x)^{\frac{1}{t}})^{\frac{1}{t}}, H(f(y)^{\frac{1}{t}}, f(y)^{\frac{1}{t}})^{\frac{1}{t}}) \\ &= H(f(x \circ x)^{\frac{1}{t}}, f(y \circ y)^{\frac{1}{t}}) \\ &= f((x \circ x) \circ (y \circ y)). \end{aligned}$$

By the assumption of separability, $\circ$ is square symmetric. □

The next result describes a set of square-symmetric operations.

COROLLARY 2. *Let G be a multiplicative subsemigroup of $\mathbb{C}$, let $H : G \times G \to G$ satisfy* (1.2), *and let $\phi : S \to G$ be an arbitrary bijective function. Then the binary operation $\circ : S \times S \to S$ defined by*

$$x \circ y := \phi^{-1}\Big(H(\phi(x), \phi(y))\Big) \qquad (x, y \in S) \tag{2.1}$$

is square symmetric.

PROOF. Clearly, ϕ is a solution of the functional equation (1.1) (with the operation $\circ$ defined in (2.1)). By its injectivity, it separates the points of S. Thus, due to Theorem 1, $\circ$ must be a square-symmetric operation. □

LEMMA 3. [7, Lemma 1] *Let $\circ$ be a square-symmetric operation on S. Define, for $x \in S$, the sequence $x[2^n]$ $(n = 0, 1, 2, \cdots)$ by*

$$x[1] = x[2^0] := x, \qquad x[2^{n+1}] := x[2^n] \circ x[2^n], \qquad n \in \mathbb{N} := \{1, 2, \cdots\}.$$

Then, for each $n \in \mathbb{N}$, the mapping $x \mapsto x[2^n]$ is an endomorphism of $(S, \circ)$, that is,

$$(x \circ y)[2^n] = x[2^n] \circ y[2^n] \qquad \textit{for all } x, y \in S.$$

3. The modified Hyers-Ulam-Rassias stability of (1.1)

In this section we shall investigate a more generalized modified Hyers-Ulam-Rassias stability than that in [7], that is, the Hyers-Ulam stability for the functional equation (1.1).

Let mappings φ and $\Phi_1, \Phi_2, \Phi_3, \Phi_4 : S\times S \to G$ satisfy the inequalities : For all $x, y \in S$,

$$\Phi_1(x,y) = \sum_{k=1}^{\infty} \frac{\varphi(x[2^{k-1}], y[2^{k-1}])}{|H(1,1)|^k} < \infty,$$

$$\Phi_2(x,y) = \sum_{k=1}^{\infty} \varphi(x[2^{-k}], y[2^{-k}])|H(1,1)|^{k-1} < \infty,$$

$$\Phi_3(x,y) = \sum_{k=1}^{\infty} \frac{\varphi(x[2^{k-1}], y[2^{k-1}])}{|a+b|^k} < \infty,$$

or

$$\Phi_4(x,y) = \sum_{k=1}^{\infty} \varphi(x[2^{-k}], y[2^{-k}])|a+b|^{k-1} < \infty,$$

where φ depend on that each $\Phi_i (i = 1, 2, 3, 4)$ converges to finite respectively. The definition of each bounded functions $\Phi_i (i = 1, 2, 3, 4)$ will be used at each of Theorem 4, 6, 10, 12, respectively.

By using an idea in P. Găvruta [2] and Z. Páles [6], we can obtain the following results:

THEOREM 4. *Let S be a nonempty set and $\circ$ be a square-symmetric operation on S. Let G be a closed multiplicative subsemigroup of $\mathbb{C}$ with $1 \in G$ and $H : G \times G \to G$ be a continuous homogeneous function of degree t such that $|H(1,1)| \neq 0$ and $\frac{1}{H(1,1)} \in G$. Assume that a function $g : S \to G$ satisfies the inequality*

$$|g(x \circ y) - H(g(x)^{\frac{1}{t}}, g(y)^{\frac{1}{t}})| \leq \varphi(x,y) \qquad (x, y \in S). \tag{3.1}$$

Then there exists a unique function $f : S \to G$ such that f is a solution of (1.1) and

$$|f(x) - g(x)| \leq \Phi_1(x,x) \qquad (x \in S). \tag{3.2}$$

PROOF. Substituting $x = y$ into (3.1) and using the t-homogeneity of H, we get

$$|g(x \circ x) - g(x)H(1,1)| \le \varphi(x,x) \qquad (x \in S). \tag{3.3}$$

Let $x \in S$ be fixed, and replace x by $x[2^{n-1}]$ in (3.3). Then we obtain

$$\left| \frac{g(x[2^n])}{H(1,1)^n} - \frac{g(x[2^{n-1}])}{H(1,1)^{n-1}} \right| \le \frac{\varphi(x[2^{n-1}], x[2^{n-1}])}{|H(1,1)|^n} \tag{3.4}$$

for all $x \in S$ and $n \in \mathbb{N}$. Let $g_0 := g$ and define, the function g_n $(n \in \mathbb{N})$ by

$$g_n(x) := \frac{g(x[2^n])}{H(1,1)^n} \qquad (x \in S).$$

Since $\dfrac{1}{H(1,1)} \in G$, $g_n : S \to G$ is a function and, in view of (3.4), we have

$$\begin{aligned} |g_n(x) - g_m(x)| &\le \sum_{j=m+1}^{n} |g_j(x) - g_{j-1}(x)| \\ &\le \sum_{j=m+1}^{n} \frac{\varphi(x[2^{j-1}], x[2^{j-1}])}{|H(1,1)|^j} \end{aligned}$$

for $n > m > 0$. Therefore, by letting $m \to \infty$ in the last inequality, the sequence $g_n(x)$ is a Cauchy sequence for all fixed $x \in S$ from the definition of Φ_1. Since the set G is closed, we can define a mapping $f : S \to G$ by

$$f(x) := \lim_{n \to \infty} g_n(x) \qquad (x \in S).$$

It follows from (3.4) that

$$\begin{aligned} |g_n(x) - g_0(x)| &\le \sum_{j=1}^{n} \frac{\varphi(x[2^{j-1}], x[2^{j-1}])}{|H(1,1)|^j} \\ &\le \sum_{j=1}^{\infty} \frac{\varphi(x[2^{j-1}], x[2^{j-1}])}{|H(1,1)|^j} \\ &= \Phi_1(x,x). \end{aligned}$$

Taking the limit of the last inequality as $n \to \infty$ produces the desired inequality (3.2).

To see that f satisfies (1.1), let $x, y \in S$ and replace x, y by $x[2^n]$, $y[2^n]$, respectively, in (3.1). Using Lemma 3, we get

$$|g((x \circ y)[2^n]) - H(g(x[2^n])^{\frac{1}{t}}, g(y[2^n])^{\frac{1}{t}})| \leq \varphi(x[2^n], y[2^n]).$$

Hence, by the t-homogeneity of H,

$$|g_n(x \circ y) - H(g_n(x)^{\frac{1}{t}}, g_n(y)^{\frac{1}{t}})| \leq \frac{\varphi(x[2^n], y[2^n])}{|H(1,1)|^n}$$

for all $x, y \in S$ and $n \in \mathbb{N}$. Taking the limit of the last inequality as $n \to \infty$, it follows from the continuity of H and the definition of Φ_1 that

$$|f(x \circ y) - H(f(x)^{\frac{1}{t}}, f(y)^{\frac{1}{t}})| = 0 \qquad (x, y \in S).$$

Thus (1.1) holds.

Assume that $h : S \to G$ is an another function which satisfies (1.1) and (3.2). Since $f(x[2^n]) = f(x)H(1,1)^n$ and $h(x[2^n]) = h(x)H(1,1)^n$, from (1.1), for all $x \in S$ and $n \in \mathbb{N}$, it follows from (3.2) that

$$\begin{aligned}
|h(x) - f(x)| &= \frac{1}{|H(1,1)|^n}|h(x[2^n]) - f(x[2^n])| \\
&\leq \frac{1}{|H(1,1)|^n}\left[|h(x[2^n]) - g(x[2^n])| + |g(x[2^n]) - f(x[2^n])|\right] \\
&\leq \frac{2}{|H(1,1)|^n}\Phi_1(x[2^n], x[2^n]) \\
&= \sum_{k=n+1}^{\infty} \frac{\varphi(x[2^k], y[2^k])}{|H(1,1)|^k}
\end{aligned}$$

for all $x \in S$ and $n \in \mathbb{N}$. By letting $n \to \infty$ in the last inequality, we immediately see the uniqueness of f from the definition of Φ_1. This finishes the proof of Theorem 4. □

COROLLARY 5. [7, Theorem 2] *Let S be a nonempty set and $\circ$ be a square-symmetric operation on S. Let G be a closed multiplicative subsemigroup of $\mathbb{C}$ with $1 \in G$ and $H : G \times G \to G$ a continuous homogeneous function of degree t such that $|H(1,1)| > 1$ and $\frac{1}{H(1,1)} \in G$. Assume that, for some $\varepsilon \geq 0$, a function $g : S \to G$ satisfies the stability inequality*

$$\left|g(x \circ y) - H(g(x)^{\frac{1}{t}}, g(y)^{\frac{1}{t}})\right| \leq \varepsilon \qquad (x, y \in S). \tag{3.5}$$

Then there exists a function $f : S \to G$ *such that f is a solution of* (1.1) *and*

$$|f(x) - g(x)| \le \frac{\varepsilon}{|H(1,1)| - 1} \qquad (x \in S).$$

PROOF. Consider $t = 1$ and $\varphi(x, y) = \varepsilon$ in Theorem 4. There exists a function $f : S \to G$ such that f is a solution of the equation $f(x \circ y) = H(f(x), f(y))$ and

$$|g(x) - f(x)| \le \frac{\varepsilon}{|H(1,1)| - 1} \qquad (x \in S).$$

We say that the operation $\circ$ has the *divisibility property* if, for each $x \in S$, there exists a unique element $y \in S$ such that $y \circ y = x$. □

THEOREM 6. *Let* S *be a nonempty set and* $\circ$ *be a square-symmetric operation with the divisibility property on* S. *Let* G *be a colsed multiplicative subsemigroup of* $\mathbb{C}$ *with* $1 \in G$ *and* $H : G \times G \to G$ *be a continuous homogeneous function of degree* t. *Assume that a function* $g : S \to G$ *satisfies the inequality* (3.1). *Then there exists a function* $f : S \to G$ *such that f is a solution of* (1.1) *and*

$$|f(x) - g(x)| \le \Phi_2(x, x) \qquad (x \in S). \tag{3.6}$$

PROOF. The proof of this theorem is analogous to that of Theorem 4.

Replacing x and y by $x[2^{-n}]$ in (3.1) and using the t-homogeneity of H, we obtain

$$|g(x[2^{1-n}]) - g(x[2^{-n}])H(1,1)| \le \varphi(x[2^{-n}], x[2^{-n}]) \qquad (x \in S; n \in \mathbb{N}).$$

Thus

$$\begin{aligned} \left|g(x[2^{1-n}])H(1,1)^{n-1} - g(x[2^{-n}])H(1,1)^{n}\right| \\ \le \varphi_2(x[2^{-n}], x[2^{-n}])\, |H(1,1)|^{n-1} \end{aligned} \tag{3.7}$$

for $x \in S$, and $n \in \mathbb{N}$. Let $g_0 := g$ and define the function g_n $(n \in \mathbb{N})$ by

$$g_n(x) := g(x[2^{-n}])H(1,1)^n \qquad (x \in S).$$

Then $g_n : S \to G$ and, by (3.7), exactly as in the proof of Theorem 4, we can deduce that the sequence $g_n(x)$ is a Cauchy sequence for all fixed

$x \in S$ from the definition of Φ_2. Define f as the pointwise limit function of the sequence g_n. It follows from (3.7) that

$$\begin{aligned} |g_n(x) - g_0(x)| &\leq \sum_{j=1}^{n} \varphi_2(x[2^{-j}], x[2^{-j}])|H(1,1)|^{j-1} \\ &\leq \sum_{j=1}^{\infty} \varphi_2(x[2^{-j}], x[2^{-j}])|H(1,1)|^{j-1} \\ &= \Phi_2(x,x), \end{aligned}$$

which, upon taking the limit as $n \to \infty$, yields (3.6).

To see that f satisfies (1.1), let $x, y \in S$ and replace x, y by $x[2^{-n}]$, $y[2^{-n}]$ in (3.1). Then we get

$$|g((x \circ y)[2^{-n}]) - H(g(x[2^{-n}])^{\frac{1}{t}}, g(y[2^{-n}])^{\frac{1}{t}})| \leq \varphi(x[2^{-n}], y[2^{-n}]).$$

It follows from the divisibility assumption that the equation $y[2^n] = x$ has a unique solution y for each fixed $x \in S$ and $n \in \mathbb{N}$. Denote this unique element y by $x[2^{-n}]$. Clearly, the mapping $x \to x[2^{-n}]$ is also an endomorphism of $(S, \circ)$.

Hence, by the t-homogeneity of H and an endomorphism of the above mapping,

$$|g_n(x \circ y) - H(g_n(x)^{\frac{1}{t}}, g_n(y)^{\frac{1}{t}})| \leq \varphi(x[2^{-n}], x[2^{-n}])|H(1,1)|^n$$

for all $x, y \in S$, and $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, by using the continuity of H and the definition of Φ_2, it follows that

$$|f(x \circ y) - H(f(x)^{\frac{1}{t}}, f(y)^{\frac{1}{t}})| = 0 \qquad (x, y \in S).$$

Therefore (1.1) holds and the uniqueness can be proved in a similar manner as in the proof of Theorem 4. Hence the proof of theorem is complete. □

COROLLARY 7. [7, Theorem 3] *Let S be a nonempty set and $\circ$ be a square symmetric operations on S. Assume that the operation $\circ$ has the divisible property. Let G be a closed multiplicative subsemigroup of $\mathbb{C}$ with $1 \in G$ and $H : G \times G \to G$ be a continuous homogeneous function of degree t such that $|H(1,1)| < 1$. Assume that, for some $\varepsilon \geq 0$, a function $g : S \to G$ satisfies the functional inequality*

$$|g(x,y) - H(g(x), g(y))| \leq \varepsilon \qquad (x, y \in S). \tag{3.8}$$

Then there exists a function $f : S \to G$ such that f is a solution of

$$f(x,y) = H(f(x), f(y)) \text{ and} \tag{3.9}$$

$$|g(x) - f(x)| \le \frac{\varepsilon}{1 - |H(1,1)|}.$$

COROLLARY 8. *Let G be a closed multiplicative subsemigroup of $\mathbb{C}$ with $1 \in G$ and $H : G \times G \to G$ be a continuous homogeneous of degree t function such that $|H(1,1)| \ne 0$ and $\dfrac{1}{H(1,1)} \in G$. Assume that a function $g : G \to G$ satisfies the inequality*

$$\left|g(H(x,y)) - H(g(x)^{\frac{1}{t}}, g(y)^{\frac{1}{t}})\right| \le \varphi(x,y) \qquad (x, y \in G).$$

Then there exists a function $f : G \to G$ such that f is a solution of

$$f(H(x,y)) = H(f(x)^{\frac{1}{t}}, f(y)^{\frac{1}{t}}) \qquad (x, y \in G)$$

and

$$|g(x) - f(x)| \le \varphi(x,x) \qquad (x, y \in G).$$

PROOF. Let $x \circ y = H(x,y)$. By Corollary 2, $\circ$ is a square-symmetric operation. In view of Theorems 4 and 6, we complete the proof. □

COROLLARY 9. [7, Corollary 2] *Let G be a closed multiplicative subsemigroup of C with $1 \in G$ and $H : G \times G \to G$ be a continuous homogeneous function of degree t such that $|H(1,1)| \notin \{0,1\}$ and $\frac{1}{H(1,1)} \in G$. Assume that, for some $\varepsilon \ge 0$, a function $g : G \to G$ satisfies the stability inequality* (3.8) *for all $x, y \in G$. Then there exists a function $f : G \to G$ such that f is a solution of* (3.9) *for all $x, y \in G$, and*

$$|g(x) - f(x)| \le \frac{\varepsilon}{||H(1,1)| - 1|} \qquad (x \in G).$$

4. Applications to the stability of homogeneous function $H(x,y) = ax + by$

In this section, we shall investigate the modified Hyers-Ulam-Rassias stability of the functional equation (4.2). In the case when $X = \mathbb{C}$ or $X = \mathbb{R}$ these results are also corollaries of Theorems 4 and 6 if one takes the homogeneous function $H(x,y) = ax + by$, respectively.

Since all proofs in this section are similar to those of Section 3, we shall skip most of the proofs.

THEOREM 10. *Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K}$ denotes the field of real or complex numbers. Let S be a nonempty set and $\circ$ be a square-symmetric operation on S. Let $a, b \in \mathbb{K}$ such that $|a+b| \neq 0$. Assume that a function $g : S \to X$ satisfies the inequality*

$$|g(x \circ y) - ag(x) - bg(y)| \leq \varphi(x, y) \qquad (x, y \in S). \tag{4.1}$$

Then there exists a unique function $f : S \to X$ such that f is a solution of

$$f(x \circ y) = af(x) + bf(y) \qquad (x, y \in S)\cdot \tag{4.2}$$

and

$$|f(x) - g(x)| \leq \Phi_3(x, x) \qquad (x \in S). \tag{4.3}$$

PROOF. We apply Theorem 4 with $H(1,1) = a + b$ and $t = 1$. By replacing y by x in (3.1), we have

$$|g(x \circ x) - ag(x) - bg(x)| \leq \varphi(x, x). \tag{4.4}$$

Replacing x by $x\left[2^{n-1}\right]$ in (4.4), we get

$$\left| \frac{g(x[2^n])}{|a+b|^n} - \frac{g(x[2^{n-1}])}{|a+b|^{n-1}} \right| \leq \frac{\varphi(x\left[2^{n-1}\right], x\left[2^{n-1}\right])}{|a+b|^n}.$$

Let $g_0 := g$ and define the function g_n $(n \in \mathbb{N})$ by

$$g_n(x) := \frac{g(x[2^n])}{|a+b|^n} \qquad (x \in S).$$

By employing analogous steps in the proof of Theorem 4, we can see that f is unique, satisfies (4.2), and estimates (4.3). □

COROLLARY 11. [7, Theorem 4] *Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K}$ denotes the field of real or complex numbers. Let S be a nonempty set and $\circ$ be a square-symmetric operation on S. Let $a, b \in \mathbb{K}$ such that $|a+b| > 1$. Assume that, for some $\varepsilon \geq 0$, a function $g : S \to X$ satisfies the inequality*

$$|g(x \circ y) - ag(x) - bg(y)| \leq \varepsilon \qquad (x, y \in S). \tag{4.5}$$

Then there exists a unique function $f : S \to X$ such that f is a solution of (4.2) and

$$|f(x) - g(x)| \leq \frac{\varepsilon}{|a+b| - 1} \qquad (x \in S).$$

THEOREM 12. *Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K}$ denotes the field of real or complex numbers. Let S be a non-empty set and $\circ$ a square-symmetric operation on S with the divisibility property. Assume that a function $g : S \to X$ satisfies the inequality* (4.1).

Then there exists a function $f : S \to X$ such that f is a solution of (4.2) *and*

$$|f(x) - g(x)| \leq \Phi_4(x, y) \qquad (x \in S).$$

PROOF. Setting $H(g(x), g(y)) = ag(x) + bg(y)$ and $t = 1$ in inequality (3.1), then it leads to (4.1). Replacing x and y by $x[2^{-n}]$ in (4.1), considering the multiplicativity of $\circ$, then we have

$$|g(x[2^{1-n}]) - (a+b)g(x[2^{-n}])| \leq \varphi(x[2^{-n}], x[2^{-n}]) \qquad (x \in S; n \in \mathbb{N}).$$

Thus we find that, for all $x \in S$ and $n \in \mathbb{N}$,

$$\begin{aligned} |g(x[2^{1-n}])(a+b)^{n-1} - g(x[2^{-n}])(a+b)^n| \\ \leq \varphi(x[2^{-n}], x[2^{-n}])|a+b|^{n-1}, \end{aligned}$$

which, upon employing the analogue of the proof of Theorem 6, completes the proof. □

COROLLARY 13. [7, Theorem 5] *Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K}$ denotes the field of real or complex numbers. Let S be a nonempty set and $\circ$ be a square-symmetric operation on S with the divisibility property. Let $a, b \in \mathbb{K}$ such that $|a+b| < 1$. Assume that, for some $\varepsilon \geq 0$, a function $g : S \to X$ satisfies the inequality* (4.5) *Then there exists a function $f : S \to X$ such that f is a solution of* (4.2) *and*

$$|f(x) - g(x)| \leq \frac{\varepsilon}{1 - |a+b|} \qquad (x \in S).$$

COROLLARY 14. *Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K}$ denotes the field of real or complex numbers. Let $H : X \times X \to X$ be a continuous X-homogeneous function. Let $a, b \in \mathbb{K}$ such that $|a+b| \neq 0$. Assume that a function $g : X \to X$ satisfies the inequality*

$$|g(H(x, y)) - ag(x) - bg(y)| \leq \varphi(x, y) \qquad (x, y \in X).$$

Then there exists a unique function $f : X \to X$ such that f is a solution of

$$f(H(x, y)) = af(x) + bf(y) \qquad (x, y \in X)$$

and

$$|f(x) - g(x)| \leq \Phi_i(x, x) \qquad (x, y \in X; i = 3 \text{ or } 4).$$

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