Journal of the Korean Association for Spatial Structures
Vol. 1, No. 1, pp. 109~115, Sep. 2001

Nonlinear Random Vibration Analysis of Thin Laminated Plates

얇은 적층 평판의 비선형 불규칙 진동해석

Kang, Joo-won*

강 주 원

Abstract

Composite materials also known as fiber reinforced plastics have been developed and used in many engineering applications due to their outstanding mechanical properties. Laminated plates as structural components that are made of in composite material are widely used. Therefore, nonlinear response of laminated composite plates modeled with finite elements and excited by stochastic loading is studied. The classical laminated plate theory is used to account for the variation of strains through the thickness for modeling laminated thin plates. Approximate nonlinear random vibration analysis is performed using the method of equivalent linearization to account for material non-linearity.

***Keywords** : thick plate, orthotropic plate, perforated plate, transverse shear deformation*

1. INTRODUCTION

Laminated plates are finding an increasing use in many engineering applications. Structural components made of laminated composite materials have a great potential for their use in civil engineering structures. Their most attractive properties are high strength to weight ratio, excellent corrosion resistance, very good fatigue strength, ease of formability, low coefficient of thermal expansion, high damping characteristics, etc. The composite components used in many of these areas are noticeably exposed to stochastic dynamic loads. One of the most important characteristics of composite materials is their strong anisotropic properties and significant non-linearity in the shear stress-strain law. The classical laminated plate theory based on the Kirchhoff's hypothesis is used for modeling laminated composite plates with thin thickness. The method of equivalent linearization to account for material non-linearity in inplane direction is used for the nonlinear random vibration analysis. The solutions are obtained using an iterative approach, where linear random vibration analysis is performed in each iteration. A computer program is developed and implemented for the nonlinear random vibration analysis. Cantilevered composite laminated plates consisting of three-ply plates with rectangular geometry and Boron/ Epoxy material are considered in numerical examples.

* 정회원 · Professor of Youngnam University

2. NONLINEAR CONSTITUTIVE EQUATIONS

A uni-directional composite lamina exhibits significant non-linearity in its shear stress-strain relations. The in-plane strain-stress law in material coordinates proposed by Hahn and Tsai (1973) is adopted in this study. Based on experimental results, they proposed the following stress-strain law for in plane stress problems:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} + S^*{}_{66}\tau_{12}^2 \begin{Bmatrix} 0 \\ 0 \\ \tau_{12} \end{Bmatrix} \quad (1)$$

where ε_1 and σ_1 are normal strain ε_2 and σ_2 stress in the fiber direction, γ_{12} and τ_{12} are the normal strain and stress in the perpendicular direction to the fiber direction. and are the engineering shear strain and shear stress corresponding to the material coordinates. is the linear compliance matrix, and the last term on the right-hand side of (1) represents shear non-linearities. By inverting Eq. (1), the shear stress-strain law is:

$$\{\sigma'\} = [Q]\{\varepsilon'\} + f(\gamma_{12})[\mathrm{diag}(0,0,1)]\{\varepsilon'\} \quad (2)$$

where $[Q]=[S]^{-1}$ is a matrix of elastic constants and $[\mathrm{diag}(0,0,1)]$ is the diagonal matrix with elements 0, 0, 1 diagonally and others are 0 in the matrix. The function, $f(\gamma_{12})$is the root of cubic polynomials. Since the exact forms of $f(\gamma_{12})$ is complicated and difficult to apply in the finite element method, it is expedient to approximate it by

$$f(\gamma_{12}) = a_1\gamma_{12}^2 + a_2\gamma_{12}^4 + \cdots + a_n\gamma_{12}^{2n} = \sum_{i=1}^{n} a_i\gamma_{12}^{2i} \quad (3)$$

By suitable choice of the parameters, a_i, the stress-strain law can be made to approximate the law in Eq. (1) for $f(\gamma_{12})$. Figure 1 shows the shear stress-strain law given by Eq. (1) for Boron/Epoxy composite material. An approximate fit using Eqs. (2) and (3) with $n=2$ is shown in Fig. 1 as a dotted line. The material parameters used were: $S_{66}=1.81\times10^{-10}\mathrm{m}^2/\mathrm{N}$, $S^*{}_{66}=2.08\times10^{-25}\mathrm{m}^6/\mathrm{N}^2$, $a_1=-1.558\times10^{13}\mathrm{Pa}$, $a_2=2.417\times10^{16}\mathrm{Pa}$. The fifth-order approximation is satisfactory for strains up to 0.02. The constitutive equation in terms of global coordinates can be expressed as

$$\{\sigma\} = [\overline{Q}]\{\varepsilon\} + f([T_3]\{\varepsilon\})[T^*]\{\varepsilon\} \quad (4)$$

where $[\overline{Q}]=[T]^{-1}[Q][T]^{-T}$ and $[T]$ is a rotational transformation matrix. $[T_3]$ is row matrix consisting of the third row of $[T]^{-1}$ and $[T^*]=[T]^{-1}[diag(0,0,1)][T]^{-T}$ and $f([T_3]\{\varepsilon\}) = f(\gamma_{12})$ in Eq. (4).

3. FINITE ELEMENT FORMULATION

A four-noded isoparametric element with five degrees-of-freedom (DOFs) at each node is used together with the Kirchhoff's assumed displacement field. The in-plane and out-of-plane displacements are interpolated through $[u_0, v_0]^T=[N_1]\{u_e\}$ and $w_0=[N_2]\{u_e\}$, in which $\{u_e\}$ are the element nodal DOFs. The shape functions $[N_1]$ are defined in the natural $\{\xi,\eta\}$ coordinate system as bilinear functions, $[N_2]$ and are non-conforming shape functions developed by Zienkiewicz and Cheung (1964). Using the energy approach, the stiffness equations for a plate element consisting N layers can be derived in the form:

$$\{P_e\} = [K_e]\{u_e\} + \{\phi_e\} \quad (5)$$

where $[K_e]$ is the linear stiffness matrix, and

$$\{\phi_e\} = \sum_{k=1}^{N} \int_{-1}^{1} \int_{-1}^{1} \int_{z_{k-1}}^{z_k} f(\gamma_{12})[B]^T [T^*][B]\{u_e\} \mid J \mid d\xi d\eta dz \tag{6}$$

is a nonlinear load vector. The dynamic equation of motion of the complete system is

$$[M]\{\ddot{u}\}+[C]\{\dot{u}\}+[K]\{u\}+\{\phi\}=\{P\} \tag{7}$$

where $[M]$ and $[K]$ are the system mass and stiffness matrices assembled from element matrices, $[C]$ is the damping matrix, $\{\phi\}$ is a vector of nonlinear terms, and $\{P\}$ is the external force vector.

4. EQUIVALENT LINEARIZATION

Approximate nonlinear random vibration analysis may be performed using the method of equivalent linearization. In this method, the nonlinear equation of motion given by Eq. (7) is replaced by an equivalent linear one

$$[M]\{\ddot{u}\}+[C]\{\dot{u}\}+([K]+[K^*])\{u\}=\{P\} \tag{8}$$

where $[K^*]$ is the equivalent stiffness matrix determined by minimizing the magnitude of the error vector $\{e\}=\{\phi\}-[K^*]\{u\}$. Assuming that $\{P\}$ and $\{u\}$ are zero-mean Gaussian random vectors, it can be shown that the equivalent stiffness matrix as:

$$[K^*]=E\left[\frac{\partial\{\phi\}}{\partial\{u\}}\right] \tag{9}$$

The matrix $[K^*]$ may be assembled through equivalent element stiffness matrices given by

$$[K^*]=\sum_{k=1}^{N}\sum_{i=1}^{2} a_i \int_{-1}^{1}\int_{-1}^{1}\int_{z_{k-1}}^{z_k} [B]^T[T^*][B]E\left[\frac{\partial}{\partial\{u_e\}}\gamma_{12}^{2i}\{u_e\}\right] d\xi\, d\eta\, dz \tag{10}$$

5. RANDOM VIBRATION ANALYSIS

Random vibration analysis to solve Eq. (8) is performed using an iterative approach, with each iteration consisting of a linear analysis. The matrix $[K^*]$ in any given iteration is computed using the nodal displacement covariances from the previous iteration, and the iterations are terminated when the covariances converge. The steps required in a frequency domain analysis are as follows:

1. Using the stiffness matrix $[K]+[K^*]$ with $[K^*]=[0]$ in the very first iteration and mass matrix $[M]$, determine the frequencies ω_j, and mode shapes $\{\Psi_j\}$ for a chosen number of modes.
2. Perform a linear random $\{u\}$ vibration analysis to determine the covariance matrix of the nodal displacements . The rsth element of the covariance matrix is given by

$$E[u_r u_s]=\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\Psi_{rj}\Psi_{sk}}{M_j M_k}\sum_{l=1}^{n}\sum_{m=1}^{n}\Psi_{lj}\Psi_{mk}\int_{-\infty}^{\infty}H_j(-\omega)H_k(\omega)S_{lm}(\omega)d\omega \tag{11}$$

where Ψ_{rj} are elements of the mode shape matrix, M_j is the jth modal mass, $H_j(\omega)$ is the jth modal frequency response function and S_{lm} is the cross-spectral density function for the excitation P_l and P_m. Note that for synchronous loading only the auto spectra are non-zero. For certain classes of excitation spectra, closed-form solutions can be used to rapidly compute the integrals in Eq. (11), while for more general excitations numerical integration will need to be used.

3. Compute the equivalent element stiffness matrices $[K_e^*]$ and assemble the global equivalent matrix $[K^*]$. The three steps

outlined above are repeated until convergence is obtained in covariance of the nodal displacements. It is convenient to check for convergence by using the nodal displacement variances and mth iteration is assumed to have converged if

$$\frac{\sqrt{\sum_i(\sigma_{u_i,m}-\sigma_{u_i,m-1})^2}}{\sqrt{\sum_1\sigma^2_{u_i,m}}}\leq tolerance \qquad (12)$$

where, $\sigma_{u_i}=\sqrt{E[u_iu_i]}$

6. NUMERICAL EXAMPLE

The cantilevered three-ply laminated plate is assumed to be made of Boron Epoxy with elastic moduli of 205 GPa and 19 GPa in the fiber and perpendicular to the fiber directions, respectively; shearing modulus of 6 GPa; mass density of 2000 kg/m^3; n = 2 in Eq. (3); and the first two coefficients on the RHS of Eq. (3) being $a_1=-1.558\times10^{13}$ Pa and $a_2=2.417\times10^{16}$ Pa. A ply arrangement with fiber orientations of α, 0° and -α in the top, middle and bottom layers respectively, is used. Two values of α, 30° and 60° are used for comparative purposes. Each layer is modeled with nine elements of equal size as indicated by the dashed lines and the element numbers. The four nodes at the free end of the cantilever are loaded with identical zero-mean white noise excitations. The level of the excitation spectrum S_0, is increased and the root-mean-square (RMS) displacement, strain and stress responses are computed as the level of the excitation spectrum, S_0, is raised. Two types of loading conditions are examined. For the comparable purpose, the responses are normalized by dividing by the linear response, which is obtained at the end of the first iteration.

6.1. Three-ply laminated plate loaded in shear direction

The cantilevered three-ply laminated plate shown in Fig. 2 is considered. The loads are applied along the in-plane DOF in the y-direction. The level of the excitation spectrum S_0, was increased from 5000 to 30000 N^{2}sec. Table 1 shows the first five undamped natural frequencies from linear and nonlinear analysis of three-ply laminated plate for the load spectrum level S_0=30000 N^{2}sec. Due to the softening effect of shear non-linearity, the natural frequencies show about 3 to 7% decrease. The plate with α=30° is stiffer than the plate with α=60° as indicated by the higher natural frequencies. Fig. 4 shows the variation of the absolute RMS shear strain in material coordinates at the center of element 2 with excitation level. The responses are nonlinear. The effect of non-linearity is comparable for both the [30° /0° /-30°] and the [60° /0° /-60°] laminated plates. The variation of the normalized RMS displacement at free corner nodes with excitation intensity is shown in Fig. 5. There is a steady increase in displacement with excitation spectrum level due to the effect of the non-linearity. The fifth order approximation is not applicable for the [60° /0° /-60°] laminated plate at levels above 20000 N^{2}sec.

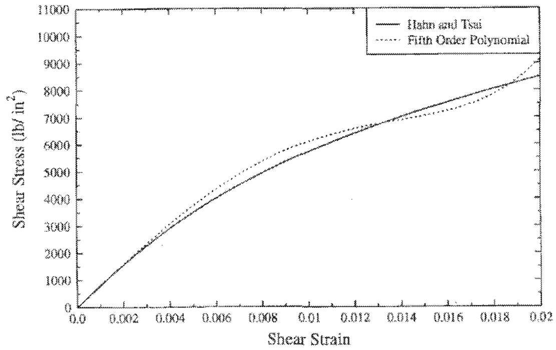
6.2. Three-ply laminated plate loaded in flexure

The same cantilever plate used in the in-plane shear loading is considered. However the out-of-plane DOF in the z-direction are loaded. The excitation intensity is increased from 0 to 80 N^{2}sec. The variation of the absolute shear strain in material coordinates at the center of element 2 with excitation intensity is shown in Fig. 6 and the responses are clearly nonlinear. For any given excitation level, the shear strain for the [60° /0°

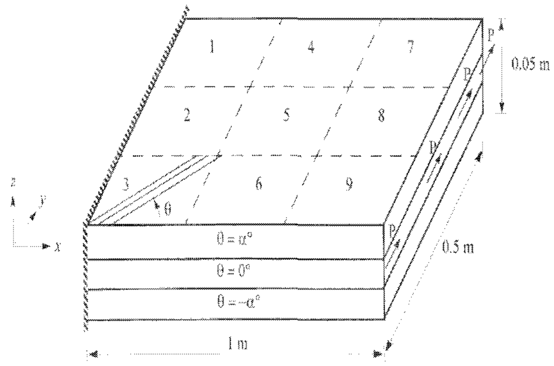
/-60°] laminated plate is significantly less than that for the [30° /0° /-30°] laminated plate. The variation of the normalized RMS z-displacement of the corner nodes with excitation spectrum level is shown in Fig. 7. The effect of non-linearity is less pronounced for the [60° /0° /-60°] laminated plate than for the [30° /0° /-30°] laminated plate. This is because the smaller shear strains in the former case reduce the overall level of non-linearity. The variation of normalized RMS shear and normal stresses in material coordinates at the corner of element 2 with excitation intensity is shown in Fig. 8. The non-linearity in the constitutive law results in a significant increase in the shear and normal stresses for the [30° /0° /-30°] laminated plate compared with the results from the [60° /0° /-60°] laminated plate.

7. CONCLUSION

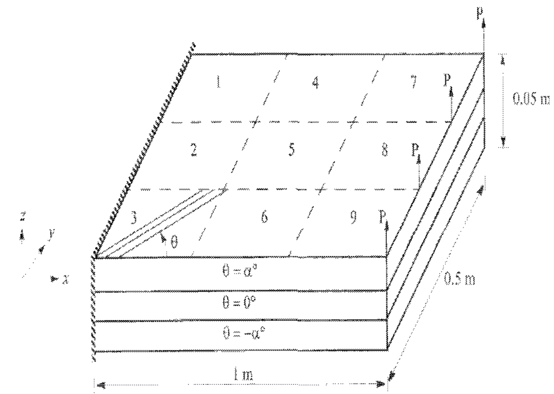
A general formulation for the nonlinear random vibration analysis of laminate plates modeled using finite elements and Kirchhoff plate theory is presented. An approximate representation of the nonlinear shear stress-strain law in terms of a fifth order polynomial results in a tractable formulation that is sufficiently accurate for practical purposes. The solution is performed iteratively using linear random vibration analysis during each iteration. The numerical examples presented indicate that the effect of non-linearity on the responses for any give excitation intensity depends on the ply-arrangement and as expected becomes more significant for higher excitation intensities.



⟨Fig. 1⟩ Fit of approximate shear stress-strain law



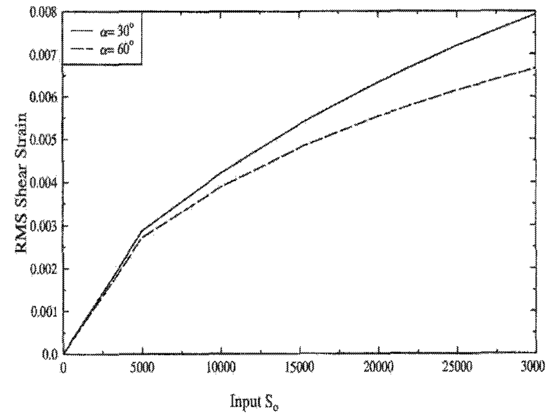
⟨Fig. 2⟩ Three-ply laminated plate loaded in shear



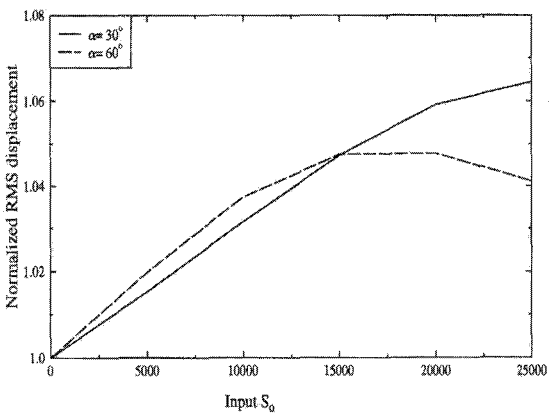
⟨Fig. 3⟩ Three-ply laminated plate loaded in flexure

〈Table 1〉 First five natural frequencies from linear and nonlinear analysis for the load level S_0=30000 N^2sec

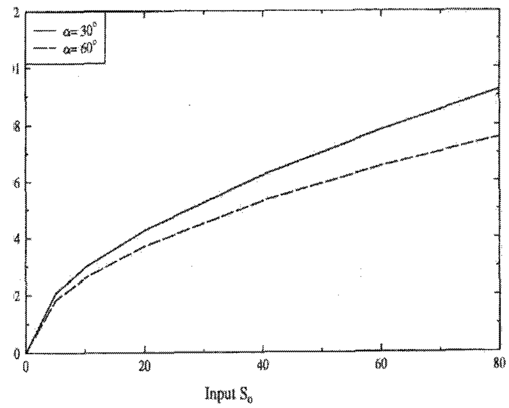
Mode	[30° /0° /-30°]		[60° /0° /-60°]	
	First Iteration	Last Iteration	First Iteration	Last Iteration
1	23.4 Hz	21.8 Hz	18.0 Hz	17.7 Hz
2	81.6 Hz	79.7 Hz	65.3 Hz	64.0 Hz
3	101.6 Hz	100.4 Hz	80.7 Hz	78.1 Hz
4	147.9 Hz	137.7 Hz	110.0 Hz	106.6 Hz
5	261.3 Hz	253.6 Hz	215.3 Hz	213.3 Hz



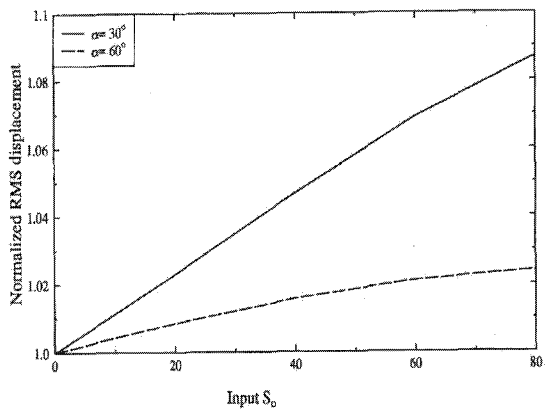
〈Fig. 4〉 Variation of absolute RMS shear strain with excitation intensity



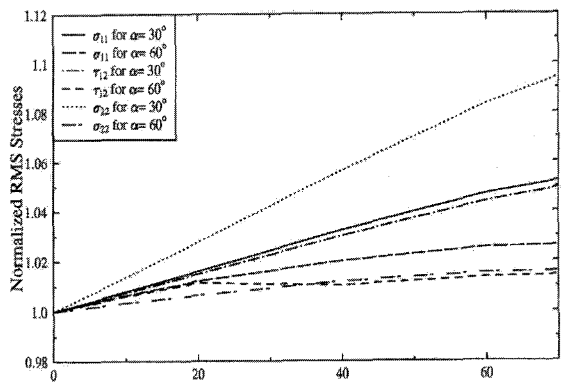
〈Fig. 5〉 Variation of normalized RMS displacement with excitation intensity



〈Fig. 6〉 Variation of absolute RMS shear strain with excitation intensity



〈Fig. 7〉 Variation of normalized RMS displacement with excitation intensity



〈Fig. 8〉 Variation of normalized RMS stresses with excitation intensity

REFERENCES

1) Bathe, K. J. and Cimento, A. P. (1980), "Some practical procedures for the solution of non-linear finite element equations," Computer Methods in Applied Mechanics and Engineering, 22, 59-85

2) Hahn, H. T. and Tsai, S. W. (1973), "Nonlinear elastic behavior of unidirectional composite laminae," *Journal of Composite Materials*, 7, 102-118

3) Kang, J. and Harichandran, R. S.(1997), "Nonlinear random vibration of laminated composite plates using high-order shear theory," *Proceedings of the 7th International Conference on Computing in Civil and Building Engineering*, Seoul, Korea.

4) Harichandran, R. S. and Naja, M. K. (1992), "Non-linear random vibration of filamentary composites," *Computing Systems in Engineering*, 3, 469-475

5) Harichandran , R. S. (1992) "Random vibration under propagating excitation: closed-form solutions," *Journal of Engineering Mechanics*, ASCE, 118(3), 575-586

6) Zienkiewicz, O. C. and Cheung, Y. K. (1964), "The finite element method for analysis of elastic isotropic and orthotropic slabs," *Proceedings of the Institute of Civil Engineers*, 28, 471-488