East Asian Math J 17(2001), No 1, pp. 159-165

CHARACTERIZATIONS OF HOLOMORPHIC FUNCTIONS IN INFINITE DIMENSIONAL COMPLEX SPACES

CHUL JOONG KANG, SU MI KWON AND KWANG HO SHON

1. Introduction

We state classical definition of envelopes of holomorphy and pseudoconvex domains and prove various results on such domains. We begin by looking at a set of conditions [2, 3] on a domain in $\mathbb{C}^n$ and locally convex spaces [1, 8]. Many efforts have been devoted to the study of these conditions in infinite dimensional spaces. The technique for studying the Levi problem in infinite dimensional spaces has been to assume a suitable approximation property in the space and to use the known finite dimensional results [4].

2. Notations and preliminaries

Let E be a linear space over the field K. A topology on E is said to be *a linear topology* if addition and scalar multiplication are continuous mappings from $E \times E$ to E and $K \times E$ to E, respectively. We call a topological linear space any linear space E equipped with a linear topology (see [6, 7]).

Received November 9, 2000 Revised June 2, 2001

2000 Mathematics Subject Classification 32D05, 32E10, 32F15

Key words and phrases domains of holomorphy, pseudo convex domains, Stein spaces, infinite dimensional complex spaces, Riemann spaces, Hartogs theorem

DEFINITION 2 1 A topological linear space E is called a *locally convex space* if it is Hausdorff and any 0-neighborhood contains a convex 0-neighborhood.

DEFINITION 2 2 Let E and F be complex locally convex spaces, and Ω be an open subset of E. A mapping f from Ω to F is said to be *holomorphic* in Ω if it is G-analytic and continuous in Ω.

DEFINITION 2.3 ([5]) Let (Ω, φ) be a Riemann domain over E and let $F \subset \mathcal{H}(X)$. A morphism $\tau : X \to Y$ is said to be an F- *envelope of holomorphy* of X if :

(a) τ is an F-extension of X.

(b) If $\mu : X \to Z$ is an F-extension of X then there is a morphism of X $\rho : Z \to Y$ such that $\rho \circ \mu = \tau$.

A morphism $\tau : X \to Y$ is said to be an *envelope of holomorphy* of X if τ is an $\mathcal{H}(X)$-envelope of holomorphy of X.

DEFINITION 2 4 Let I be an index set. For each point $a \in E$ consider the collection of all pairs (U, φ) such that U is an open neighborhood of a and $\varphi = (\varphi_i)_{i \in I} \subset \mathcal{H}(U)$. Two such pairs (U, φ) and (V, ψ) are said to be equivalent if there is an open neighborhood W of a with $W \subset U \cap V$ such that $\varphi_i = \psi_i$ on W for every $i \in I$. We shall denote by $\mathcal{H}_a^I$ the collection of all equivalent classes. The members of $\mathcal{H}_a^I$ are called *germs of holomorphic I-families* at the point a. The germ of (U, φ) at a will be denoted by φ_a. Clearly $\mathcal{H}_a^I$ is an algebra. Next consider the collection

$$\mathcal{H}_E^I = \bigcup_{a \in E} \mathcal{H}_a^I,$$

where the algebras $\mathcal{H}_a^I$ are regarded as disjoint sets. For each $\varphi_a \in \mathcal{H}_E^I$ let $N(\varphi_a)$ denote the collection of all sets of the form

$$N(U, \varphi) = \{\varphi_b : b \in U\},$$

where (U, φ) varies over all representatives of the germ φ_a. The set $\mathcal{H}_E^I$ will be endowed with the unique topology such that $N(\varphi_a)$ is a neighborhood base at φ_a for each $\varphi_a \in \mathcal{H}_E^I$.

3. Properties of Riemann domains

PROPOSITION 3.1 *Let* $\pi : \mathcal{H}_E^I \to E$ *be defined by* $\pi(\varphi_a) = a$ *for each* $\varphi_a \in \mathcal{H}_E^I$. *Then* $(\mathcal{H}_E^I, \pi)$ *is a Riemann domain over* E.

PROOF. Let φ_a and ψ_b be two distinct points of $\mathcal{H}_E^I$. If $a \neq b$ then we can find a representative (U, φ) of φ_a and a representative (V, ψ) of ψ_b such that U and V are disjoint. Then the sets $N(U, \varphi)$ and $N(V, \psi)$ are also disjoint. Next suppose $a = b$. Let (U, φ) be a representative of φ_a, and let (V, ψ) be a representative of ψ_a and let W be a connected open neighborhood of a such that $W \subset U \cap V$. Then the sets (W, φ) and (W, ψ) are necessarily disjoint, for otherwise the Identity Principle would imply that $\varphi_i = \psi_i$ on W for every $i \in I$, and therefore $\varphi_a = \psi_a$, a contradiction. Thus $\mathcal{H}_E^I$ is a Hausdorff space. Since the mapping π is clearly a local homeomorphism, the proof is complete.

THEOREM 3.2 *Let* (X, ξ) *be a Riemann domain over* E *and let* $F \subset \mathcal{H}(X)$. *Then the* F-*envelope of holomorphy of* X *always exists.*

PROOF. Let $F = (f_i)_{i \in I} \in \mathcal{H}(X)$. Given $x \in X$ let U be a chart in X containing x, let $\varphi_i = f_i \circ (\xi|U)^{-1}$ for each $i \in I$, let $\varphi = (\varphi_i)_{i \in I} \subset \mathcal{H}(\xi(U))$ and let $\varphi_{\xi(x)} \in \mathcal{H}_{\xi(x)}^I$ be the germ of $(\xi(U), \varphi)$ at $\xi(x)$. Then the mapping

$$\tau : x \in X \to \varphi_{\xi(x)} \in \mathcal{H}_E^I$$

is clearly well defined and a morphism. Given $\varphi_a \in \mathcal{H}_E^I$ let (V, φ) be a representative of φ_a and define

$$g_i(\varphi_a) = \varphi_i(a)$$

for each $i \in I$. Clearly each g_i is well defined. Since $g_i = \varphi_i \circ \pi$ on a neighborhood of φ_a we see that each g_i is holomorphic on $\mathcal{H}_E^I$. For each $i \in I$ and $x \in X$ we have that

$$g_i(\tau(x)) = g_i(\varphi_{\xi(x)}) = \varphi_i(\xi(x)) = f_i(x).$$

If Y is the union of those connected components of $\mathcal{H}_E^I$ which intersect $\tau(X)$ then it is clear that $\tau : X \to Y$ is a F-extension of X. Let (Z, ζ) be another Riemann domain over E and suppose that $\mu : X \to Z$

is an F-extension of X too. Then for each $i \in I$ there is a unique function $h_i \in \mathcal{H}(Z)$ such that $h_i \circ \mu = f_i$. Given $z \in Z$ let W be a chart in Z containing z, let $\psi_i = h_i \circ (\zeta|W)^{-1}$ for each $i \in I$, let $\psi = (\psi_i)_{i\in I} \subset \mathcal{H}(\zeta(W))$ and let $\psi_{\zeta(z)} \in \mathcal{H}^I_{\zeta(z)}$ be the germ of $(\zeta(W), \psi)$ at $\zeta(z)$. Then the mapping

$$\nu : z \in Z \to \psi_{\zeta(z)} \in \mathcal{H}^I_E$$

is clearly well defined and morphism. Given $x \in X$ let U be a chart in X containing x and let W be a chart in Z containing $\mu(x)$ such that $W = \mu(U)$. Then

$$\varphi_i = f_i \circ (\xi|U)^{-1} = h_i \circ \mu \circ (\xi|U)^{-1} = h_i \circ (\zeta|W)^{-1} = \psi_i$$

for every $i \in I$. Hence

$$\nu \circ \mu(x) = \psi_{\zeta\circ\mu(x)} = \varphi_{\xi(x)} = \tau(x)$$

and in particular $\nu(\mu(x)) = \tau(x) \subset Y$. Since each connected component of Z intersect $\mu(X)$ we see that $\nu(Z) \subset Y$. This completes the proof.

DEFINITION 3.3 Let E be a linear space over $\mathbb{C}$, $\mathcal{C}$ a Haudorff topology on E and Ω an open set for $(E, \mathcal{C})$. Let v be a function defined on Ω and with range in $[-\infty, +\infty[$, with $v \neq -\infty$. The function v is called *plurisubharmonic* if

(a) v is upper semi-continuous (i.e. the set $\{z \in \Omega : v(z) < c\}$ is open for any $c \in \mathbb{R}$)

(b) if $(a, b) \in \Omega \times (E - \{0\})$ the function $\xi \longmapsto v(a + \epsilon b), \xi \in \mathbb{C}$ is subharmonic or identical to $-\infty$ on each connected component of $\mathbb{C}$ where it is defined.

DEFINITION 3.4. Let E be a complex l.c.s. and Ω an open subset of E. We denote by d_Ω the function :

$$\Omega \times (E - \{0\}) \to]0, +\infty]$$
$$(z, z') \longmapsto d_\Omega(z, z') = \inf_{z+\lambda z' \notin \Omega} |\lambda|.$$

We say that Ω is *pseudo-convex* if the function $-logd_\Omega$ is plurisubharmonic on $\Omega \times (E - \{0\})$, for every fixed $z' \in E - \{0\}$, the function d_Ω is the distance from z to the complement of Ω in the direction z'.

The following lemma is on Noverraz [9, 10].

LEMMA 3 5 *For a Riemann domain (Ω, φ) over a locally convex space E, the following conditions are equivalent:*

(a) *Ω is pseudoconvex.*
(b) *$-\log d_\Omega^\alpha$ is plurisubharmonic on Ω for any $\alpha \in cs(E)$.*
(c) *$\varphi^{-1}(F)$ is a Stein Manifold for each finite dimensional linear subspace F of E.*
(d) *For every $x \in \Omega$, there exists an open neighborhood U of x in E such that $\left(\varphi^{-1}(U),\ \varphi|_{\varphi^{-1}(U)}\right)$ is a pseudoconvex Riemann domain over E.*

THEORM 3 6 *Let (Ω, φ) be a Schlicht Riemann domain over $\mathbf{C}^{\mathbf{N}}$. If Ω is a pseudoconvex domain, then there exists a number $n \in \mathbf{N}$ and a psedoconvex Riemann domain $(V,\ \varphi|_V)$ over $\mathbf{C}^n$ such that*

$$\Omega = \mathbf{C}^{\mathbf{N}-\{1,2,\ \ ,n\}} \times V.$$

PROOF For $x \in \Omega$, there is $\alpha \in \mathrm{cs}(\mathbf{C}^{\mathbf{N}})$ with $d_\Omega^\alpha(x) \geqq 1$. Thus for $z = (z_i)$, there exist $n \in \mathbf{N}$ and $c > 0$ such that

$$c\left(\sup_{1 \leqq i \leqq n} |z_i| \right) \geqq \alpha(z).$$

Hence there exists a section

$$s : B_{\mathbf{C}^{\mathbf{N}}}^{\alpha}(\varphi(x),\ 1) \longrightarrow \Omega$$

satisfying $\mathfrak{s} \circ \varphi(x) = x$. In fact, we have

$$
\begin{aligned}
& B^{\alpha}_{\mathbf{C}^{\mathbf{N}}}(\varphi(x), 1) \\
=& \{\varphi(x) + \zeta \in \mathbf{C}^{\mathbf{N}} : \alpha(\zeta) < 1\} \\
\supset& \{\varphi(x) + \zeta \in \mathbf{C}^{\mathbf{N}} : c(\sup_{1 \leqq i \leqq n} |\zeta_i|) < 1\} \\
=& \{\varphi(x) + (\zeta_1, \zeta_2, \cdots, \zeta_n, \zeta_{n+1}, \cdots) \in \mathbf{C}^{\mathbf{N}} : |\zeta_i| < \frac{1}{c}, \\
& \quad i = 1, 2, \cdots, n,\ \zeta_j \in \mathbf{C},\ j = n+1, n+2, \cdots\} \\
=& \varphi(x) + \{(\zeta_1, \zeta_2, \cdots, \zeta_n, 0, 0, \cdots) \in \mathbf{C}^{\mathbf{N}} : |\zeta_i| < \frac{1}{c},\ i = 1, 2, \cdots, n\} \\
& \quad + \{(0, 0, \cdots, 0, \zeta_{n+1}, \zeta_{n+2}, \cdots) \in \mathbf{C}^{\mathbf{N}} : \zeta_j \in \mathbf{C}, j = n+1, n+2, \cdots\} \\
=& (p_1 \circ \varphi(x), p_2 \circ \varphi(x), \cdots, p_n \circ \varphi(x), 0, 0, \cdots, 0, \cdots) \\
& \quad + (0, 0, \cdots, 0, p_{n+1} \circ \varphi(x), p_{n+2} \circ \varphi(x), \cdots) \\
& \quad + \{(\zeta_1, \zeta_2, \cdots, \zeta_n, 0, 0, \cdots, 0, \cdots) \in \mathbf{C}^{\mathbf{N}} : |\zeta_i| < \frac{1}{c}, i = 1, 2, \cdots, n\} \\
& \quad + \{(0, 0, \cdots, 0, \zeta_{n+1}, \zeta_{n+2}, \cdots) \in \mathbf{C}^{\mathbf{N}} : \zeta_j \in \mathbf{C},\ j = n+1, n+2, \cdots\} \\
=& \mathbf{C}^{\mathbf{N} - \{1, 2, \cdots, n\}} \times \prod_{j=1}^{n} D(p_j \circ \varphi(x), \frac{1}{c}).
\end{aligned}
$$

That is,

$$
\mathfrak{s}|_{\mathbf{C}^{\mathbf{N}-\{1,2,\cdots,n\}} \times \prod_{j=1}^{n} D(p_j \circ \varphi(x), \frac{1}{c})} : B^{\alpha}_{\mathbf{C}^{\mathbf{N}}}(\varphi(x), 1) \longrightarrow \Omega
$$

is a section satisfying $\mathfrak{s}|_{\mathbf{C}^{\mathbf{N}-\{1,2,\cdots,n\}} \times \prod_{j=1}^{n} D(p_j \circ \varphi(x), \frac{1}{c})} \circ \varphi(x) = x$. From Lemma 3.5, we have the result.

References

[1] S Dineen, *Complex analysis in locally convex spaces, North-Holland Math. Studies 57*, North-Holland Pub Co., Amsterdam, New York, Oxford, 1981

[2] H Grauert and K Fritzsche, *Several Complex Variables. Graduate Texts in Math. 38*, Springer-Verlag, New York, 1976.

[3] H Grauert and R Remmert, *Theory of Stein Spaces Grundlehren Math. Wiss. 236, Translated from : Theorie der Steinschen Räume. Grundlehren Math. Wiss. 227*, Springer-Verlag, Heidelberg, 1979

[4] L. Hormander, *An Introduction to Complex Analysis in Several Variables*, Van Nostrand Princeton, N T , 1966.

[5] M Matos, *The envelope of holomorphy of Riemann domains over a countable product of complex planes*, Tran Amer Math Soc **167** (1972), 379-387

[6] J Mujica, *Holomorphic approximation in infinite-dimensional Riemann domains*, Studia Mathematica T **82** (1985), 107-134

[7] J Mujica, *Complex analysis in Banach spaces, North - Holland Math. Studies 120*, Elsevier Science Publishers B V , North - Halland, New York, Oxford, 1986

[8] L. Nachbin, *Holomorphic Functions, Domains of Holomorphy and Local Properties*, North Holland Publ Comp , Amsterdam, 1970

[9] P Noverraz, *Fonctions Plurisousfarmoniques et Analytiues dans les Espaces Vectoriels Topologiques Complexes*, Ann Inst Fourie, Grenoble **19-2** (1969), 419-493

[10] P Noverraz. *Pseudo-Convexité, Convexité Polynomiale et Domaines d'Holomorphie en Dimension Infinite, North-Holland Math. Studies 3*, North-Holland Publ. Co , New York, 1973.

Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail: khshon@hyowon.pusan.ac.kr