

GAUSS SUMMATION THEOREM AND ITS APPLICATIONS

HYUNG JAE LEE, YOUNG JOON CHO AND JUNESANG CHOI

ABSTRACT The Gauss summation theorem plays a key role in the theory of (generalized) hypergeometric series. The authors study several proofs of the theorem and consider some applications of it.

1. Introduction and Preliminaries

The generalized hypergeometric function [9, p. 73] with p numerator and q denominator parameters is defined by

$$(1.1) \quad {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \\ = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n z^n}{(\beta_1)_n \dots (\beta_q)_n n!},$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined, for any complex number α , by

$$(1.2) \quad (\alpha)_n := \begin{cases} 1 & (n = 0) \\ \alpha(\alpha + 1) \dots (\alpha + n - 1) & (n \in \mathbb{N} := \{1, 2, 3, \dots\}), \end{cases}$$

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which can also be rewritten in the form:

$$(1.3) \quad (\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)},$$

where Γ is the well-known Gamma function whose Weierstrass canonical product form is

$$(1.4) \quad \Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} \right\},$$

γ being the Euler-Mascheroni constant defined by

$$(1.5) \quad \gamma := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right) \cong 0.577\,215\,664\,901\,532 \dots$$

With the notation (1.1), the Gaussian hypergeometric series is ${}_2F_1$, which is also denoted simply by F .

The Beta function $B(\alpha, \beta)$ is a function of two complex variables α and β , defined by

$$(1.6) \quad B(\alpha, \beta) := \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = B(\beta, \alpha) \quad (\Re(\alpha) > 0; \Re(\beta) > 0).$$

The Beta function is closely related to the Gamma function as follows:

$$(1.7) \quad B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad (\alpha, \beta \neq 0, -1, -2, \dots),$$

which not only confirms the symmetry property in (1.6), but also continues the Beta function analytically for all complex values of α and β , except when $\alpha, \beta = 0, -1, -2, \dots$.

Gauss proved the following useful theorem:

$$(1.8) \quad {}_2F_1 \left[\begin{matrix} a, & b; \\ & c; \end{matrix} 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$(\Re(c-a-b) > 0; c \neq 0, -1, -2, \dots),$$

which plays a vital role in the theory of the generalized hypergeometric series ${}_pF_q$ as well as in ${}_2F_1$ and is usually referred to as Gauss summation theorem.

In this note we aim at reviewing several methods of proofs of (1.8) including Gauss's original proof and considering some applications of it.

2. Proof of (1.8)

For Gauss's proof of (1.8), we begin by recalling Abel's theorem:

THEOREM 2.1 *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series, whose radius of convergence is unity, and let it be such that $\sum_{n=0}^{\infty} a_n$ converges, and let $0 \leq x \leq 1$. Then*

$$\lim_{x \rightarrow 1} \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} a_n.$$

Note also (see [6, p 47]) that for sufficiently large x ,

$$(2.1) \quad x^{b-a} \cdot \frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + O\left(\frac{1}{x}\right).$$

The hypergeometric series ${}_2F_1(a, b; c; x)$ converges absolutely for $\Re(c - a - b) > 0$ and $c \neq 0, -1, -2, \dots$, on the radius of convergence $|x| = 1$. It is a routine work to verify, by considering the coefficient of x^n in the variable, that if $0 \leq x \leq 1$, then

$$\begin{aligned} c\{c-1-(2c-a-b-1)x\}F(a, b; c, x) &+ (c-a)(c-b)x F(a, b; c+1; x) \\ &= c(c-1)(1-x)F(a, b; c-1; x) \\ &= c(c-1) \left\{ 1 + \sum_{n=1}^{\infty} (u_n - u_{n-1})x^n \right\}, \end{aligned}$$

where u_n is the coefficient of x^n in $F(a, b; c-1; x)$.

Now make $x \rightarrow 1$. By Theorem 2.1, the right-hand side tends to zero if $1 + \sum_{n=1}^{\infty} (u_n - u_{n-1})$ converges to zero, i.e., if $u_n \rightarrow 0$, which is the case when $\Re(c - a - b) > 0$.

Also, by Theorem 2.1, the left-hand side tends to

$$c(a + b - c)F(a, b ; c, 1) + (c - a)(c - b)F(a, b ; c + 1 ; 1)$$

under the same condition, and therefore

$$F(a, b ; c ; 1) = \frac{(c - a)(c - b)}{c(c - a - b)} F(a, b ; c + 1 ; 1).$$

Repeating this process, we see that

$$\begin{aligned} F(a, b ; c ; 1) &= \left\{ \prod_{n=0}^{m-1} \frac{(c - a + n)(c - b + n)}{(c + n)(c - a - b + n)} \right\} F(a, b ; c + m ; 1) \\ &= \left\{ \lim_{m \rightarrow \infty} \prod_{n=0}^{m-1} \frac{(c - a + n)(c - b + n)}{(c + n)(c - a - b + n)} \right\} \lim_{m \rightarrow \infty} F(a, b ; c + m \end{aligned}$$

if these two limits exist.

For the former limit, if c is not a negative integer, using (2.1), we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \prod_{n=0}^{m-1} \frac{(c - a + n)(c - b + n)}{(c + n)(c - a - b + n)} &= \lim_{m \rightarrow \infty} \frac{(c - a)_m (c - b)_m}{(c)_m (c - a - b)_m} \\ &= \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \lim_{m \rightarrow \infty} \frac{\Gamma(c - a + m)\Gamma(c - b + m)}{\Gamma(c + m)\Gamma(c - a - b + m)} \\ &= \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \lim_{m \rightarrow \infty} m^{-a} \left(1 + O\left(\frac{1}{m}\right) \right) \cdot m^a \left(1 + O\left(\frac{1}{m}\right) \right) \\ &= \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \end{aligned}$$

On the other hand, if $u_n(a, b, c)$ is the coefficient of x^n in $F(a, b ; c ; x)$, and $m > |c|$, we have

$$\begin{aligned} |F(a, b ; c + m ; 1) - 1| &\leq \sum_{n=1}^{\infty} |u_n(a, b, c + m)| \\ &\leq \sum_{n=1}^{\infty} u_n(|a|, |b|, m - |c|) \\ &< \frac{|ab|}{m - |c|} \sum_{n=0}^{\infty} u_n(|a| + 1, |b| + 1, m + 1 - |c|). \end{aligned}$$

Now the last series converges, when $m > |c| + |a| + |b| + 1$, and is a positive decreasing function of m ; therefore, since $\{m - |c|\}^{-1} \rightarrow 0$, we have

$$\lim_{m \rightarrow \infty} F(a, b ; c + m ; 1) = 1;$$

and therefore, finally, the Gauss's proof of (1.8) is complete.

Secondly, we have the following known integral representation for ${}_2F_1$ (see [6, p. 114]):

(2.2)

$${}_2F_1(a, b ; c ; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} (1-tz)^{-b} dt$$

$$(\Re(c) > \Re(a) > 0; |\arg(1-z)| < \pi)$$

which, upon setting $z = 1$ and considering (1.7), immediately yields (1.8) including its restrictions without any alteration.

The Riemann-Liouville fractional integral of f order ν is defined, for $\Re(\nu) > 0$, by

$$(2.3) \quad {}_0D_t^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} f(\xi) d\xi \quad (t > 0),$$

where f is piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval $[0, \infty)$ (see [8, p.45]).

Unless there is any possibility of ambiguity, the notation will be simplified by dropping the subscripts 0 and t on ${}_0D_t^{-\nu}$.

The Leibniz formula for fractional integrals is recalled here (see [8, p.75]):

(2.4)

$$D^{-\nu} [f(t)g(t)] = \sum_{k=0}^{\infty} \binom{-\nu}{k} [D^k g(t)] [D^{-\nu-k} f(t)] \quad (\nu > 0; 0 < t \leq X),$$

where f is continuous on $[0, X]$, and g is analytic at a , for all $a \in [0, X]$.

To prove (1.8), we start with the trivial identity

$$(2.5) \quad t^{\lambda+\mu} = t^\lambda t^\mu \quad (t > 0).$$

Now for $\nu > 0$ and $\lambda + \mu > -1$,

$$(2.6) \quad D^{-\nu} t^{\lambda+\mu} = \frac{\Gamma(\lambda + \mu + 1)}{\Gamma(\lambda + \mu + \nu + 1)} t^{\lambda+\mu+\nu}.$$

We shall show that if $\lambda, \mu \geq 0$, we may apply Leibniz's formula (2.4) to the product of $f(t) = t^\lambda$ and $g(t) = t^\mu$. This result may then be compared with (2.6) to establish (1.8).

We begin, in virtue of the binomial theorem (3.4), by expanding $g(\xi)$ in powers of $(\xi - t)$ as follows:

$$(2.7) \quad \begin{aligned} g(\xi) &= \xi^\mu = [t + (\xi - t)]^\mu \\ &= t^\mu \left(1 + \frac{\xi - t}{t}\right)^\mu \\ &= t^\mu \sum_{k=0}^{\infty} \binom{\mu}{k} \left(\frac{\xi - t}{t}\right)^k. \end{aligned}$$

Considered as a power series in $(\xi - t)/t$, the radius of convergence is 1. Using Raabe's test we see that the series converges absolutely for

$$\frac{\xi - t}{t} = \pm 1.$$

Furthermore, it converges to ξ^μ . Since

$$\left| \binom{\mu}{k} \left(\frac{\xi - t}{t}\right)^k \right| \leq \left| \binom{\mu}{k} \right|$$

for all $(\xi - t)/t$ in $[-1, 1]$, the Weierstrass M -test implies that the convergence is uniform in the closed interval $[-1, 1]$. Thus (2.7) converges uniformly for $\xi \in [0, t]$.

Therefore, it follows that

$$(2.8) \quad D^{-\nu} [t^\lambda t^\mu] = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu + k)}{k! \Gamma(\nu)} [D^k t^\lambda] [D^{-\nu-k} t^\mu]$$

is valid for $\nu > 0, t > 0, \lambda, \mu \geq 0$. Thus

$$\begin{aligned} D^{-\nu} [t^\lambda t^\mu] &= t^{\lambda+\mu+\nu} \frac{\Gamma(\mu+1)}{\Gamma(-\lambda)\Gamma(\nu)} \sum_{k=0}^{\infty} \frac{\Gamma(-\lambda+k)\Gamma(\nu+k)}{\Gamma(\mu+\nu+k+1)} \frac{1}{k!} \\ &= t^{\lambda+\mu+\nu} \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} {}_2F_1(-\lambda, \nu; \mu+\nu+1; 1). \end{aligned}$$

If we equate this result to (2.6), we obtain

$$(2.9) \quad \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+\mu+\nu+1)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} {}_2F_1(-\lambda, \nu; \mu+\nu+1, 1).$$

In more conventional notation let $a = -\lambda, b = \nu, c = \mu + \nu + 1$. Then (2.9) becomes (1.8) for

$$(2.10) \quad a \leq 0, \quad c - 1 \geq b > 0.$$

Thus they have established (1.8) only under the more restrictive conditions of (2.10) (see [8, pp.77-79]).

3. Applications

The Riemann Zeta function $\zeta(s)$ (see [10, pp.265-280]) is defined by

$$(3.1) \quad \zeta(s) := \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ (1-2^{1-s})^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0; s \neq 1). \end{cases}$$

Since the time of Euler, there have been many methods of proof of $\zeta(2n)$ ($n \in \mathbb{N}$) (cf. [3, 5]). We can also evaluate $\zeta(2)$ by using (1.8). Consider the following integral

$$(3.2) \quad \frac{1}{2}(\arcsin x)^2 = \int_0^x \frac{\arcsin t}{\sqrt{1-t^2}} dt.$$

We can expand $\arcsin t$ in powers of t as follows:

$$\begin{aligned}
 \arcsin t &= \int_0^t \frac{du}{\sqrt{1-u^2}} = \int_0^t (1-u^2)^{-\frac{1}{2}} du \\
 (3.3) \quad &= \int_0^t \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-u^2)^n du \\
 &= \sum_{n=0}^{\infty} \frac{(1/2)_n}{n!} \frac{t^{2n+1}}{2n+1} \quad (|t| < 1),
 \end{aligned}$$

where we have used termwise integration and the binomial theorem:

$$(3.4) \quad (1-z)^{-\alpha} = \sum_{n=0}^{\infty} \binom{-\alpha}{n} (-z)^n = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n \quad (|z| < 1).$$

Similarly as in (3.3), we have

$$\begin{aligned}
 \int_0^x \frac{t^{2n+1}}{\sqrt{1-t^2}} dt &= \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \int_0^x t^{2k+2n+1} dt \\
 &= \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \frac{x^{2k+2n+2}}{2k+2n+2} \\
 (3.5) \quad &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \frac{\Gamma(k+n+1)}{\Gamma(k+n+2)} x^{2k+2n+2} \\
 &= \frac{1}{2n+2} \sum_{k=0}^{\infty} \frac{(1/2)_k (n+1)_k}{(n+2)_k k!} x^{2k+2n+2} \\
 &= \frac{x^{2n+2}}{2n+2} {}_2F_1(1/2, n+1; n+2; x^2).
 \end{aligned}$$

Taking the limit as $x \rightarrow 1$ on (3.5) and using (1.8), we obtain

$$\begin{aligned}
 (3.6) \quad \int_0^1 \frac{t^{2n+1}}{\sqrt{1-t^2}} dt &= \frac{1}{2n+2} {}_2F_1(1/2, n+1; n+2; 1) \\
 &= \frac{1}{2n+2} \frac{\Gamma(n+2)\Gamma(1/2)}{\Gamma(n+3/2)}.
 \end{aligned}$$

Combining (3.3), (3.5), and (3.6) into (3.2), and setting $x = 1$ in the resulting identity, we obtain

$$(3.7) \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8},$$

which, in view of (3.1), yields

$$(3.8) \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

The Gauss summation theorem (1.8) may seem to have lots of applications in evaluations of some types of definite integrals. As an illustration, we consider the following integral (see [4]):

$$(3.9) \quad \int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{2\pi\sqrt{3}\sqrt{2}} \left\{ \Gamma\left(\frac{1}{3}\right) \right\}^3,$$

which was recorded in Gradshteyn and Ryzhik [7, p. 260, Entry 3.139]. Indeed,

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{1-x^3}} &= \int_0^1 \sum_{n=0}^{\infty} (-1)^n \binom{-1/2}{n} x^{3n} dx \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!} \frac{1}{3n+1} \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n}{\left(\frac{4}{3}\right)_n n!} = {}_2F_1\left(\frac{1}{2}, \frac{1}{3}; \frac{4}{3}; 1\right) \\ (3.10) \quad &= \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{6}\right)}. \end{aligned}$$

Now, recalling the well-known reflection and duplication formulae respectively:

$$(3.11) \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad \text{and} \quad \sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2),$$

we obtain

$$(3.12) \quad \Gamma\left(\frac{5}{6}\right) = \frac{2^{\frac{4}{3}}\pi^{\frac{3}{2}}}{\sqrt{3}} \left\{ \frac{1}{\Gamma\left(\frac{1}{3}\right)} \right\}^2.$$

Finally combining (3.10) and (3.12) leads at once to (3.9).

For another interesting application, consider the problem which was posed by Ananthanarayana Sastri [1, p.80, No.644]:

The length of the fourth positive pedal of a loop of the Lemniscate of Bernoulli is given by

$$18a \int_0^1 \frac{x^8 dx}{\sqrt{1-x^4}},$$

where a denotes one half of the distance between two fixed points in the definition of the Lemniscate of Bernoulli.

Similarly as above, we evaluate this integral

$$(3.13) \quad 18a \int_0^1 \frac{x^8 dx}{\sqrt{1-x^4}} = \frac{15\sqrt{2}}{28\sqrt{\pi}} a \left\{ \Gamma\left(\frac{1}{4}\right) \right\}^2,$$

the numerical value of $\Gamma(1/4)$ being 3.625 609 908 221 \dots .

Finally, among other things, we conclude this note by showing an applicability of the Gauss summation theorem (1.8) to evaluate some identities involving binomial coefficients. For example, an interesting identity was recorded in [8, p.294, Theorem A.7]:

$$(3.14) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\Gamma(\nu+k)}{\Gamma(\nu+k-m)} = 0 \quad (m=0, 1, \dots, n-1; n \in \mathbb{N}; \nu \in \mathbb{C}),$$

which was proved in an elementary way there. Indeed, the sum (3.14) is written in the following equivalent form and may be evaluated by using (1.8). Under the restrictions of (3.14),

$$\frac{\Gamma(\nu)}{\Gamma(\nu-m)} {}_2F_1(-n, \nu; \nu-m, 1) = \frac{\Gamma(\nu)\Gamma(n-m)}{\Gamma(\nu-m+n)} \cdot \frac{1}{\Gamma(-m)} = 0.$$

Note that some families of series involving binomial coefficients can easily be evaluated by employing the theory of the (generalized) hypergeometric series (see [2]).

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REFERENCES

- [1] K.V. Ananthanarayana Sastri, *Problem 644*, J Indian Math Soc -Madras **7** (1915), 80
- [2] Y J Cho, T Y Seo, and J Choi, *Sums of series associated with binomial coefficients*, Far East J Math Sci **2**.(2) (2000), 261 - 269.
- [3] J Choi and A K. Rathie, *An evaluation of $\zeta(2)$* , Far East J Math Sci **5** (1997), 393 - 398
- [4] J Choi and H M Srivastava, *Gamma function representation for some definite integrals*, Kyungpook Math J **37** (1997), 205 - 209.
- [5] J. Choi, A K. Rathie, and H M Srivastava, *Some hypergeometric and other evaluations of $\zeta(2)$ and allied series*, Appl Math Comput **104** (1999), 101 - 108
- [6] A. Erdélyi, W Magnus, F Oberhettinger, and F G Tricomi, *Higher Transcendental Functions*, McGraw-Hill Book Company, New York, Toronto, and London, 1953
- [7] I.S Gradshteyn and I M Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, San Diego, San Francisco, New York, Boston, London, Sydney, and Tokyo, 2000
- [8] K.S. Miller and B Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, Inc , New York, Chichester, Brisbane, Toronto, and Singapore, 1993.
- [9] E D Rainville, *Special Functions*, The Macmillan Company, New York, 1960
- [10] E T Whittaker and G N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, London, and New York, 1963

Young Joon Cho
 Department of Mathematics
 College of Natural Sciences
 Pusan National University
 Pusan 609-735, Korea

E-mail: choyj79@hanmail.net

Hyung Jae Lee and Junesang Choi

Department of Mathematics

College of Natural Sciences

Dongguk University

Kyongju 780-714, Korea

E-mail: junesang@mail.dongguk.ac.kr