East Asian Math J 17(2001), No 1, pp 135-146

SOME FAMILIES OF INFINITE SUMS DERIVED BY MEANS OF FRACTIONAL CALCULUS

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ABSTRACT Several families of infinite series were summed recently by means of certain operators of fractional calculus (that is, calculus of derivatives and integrals of any real or complex order) In the present sequel to this recent work, it is shown that much more general classes of infinite sums can be evaluated *without* using fractional calculus Some other related results are also considered

1. Introduction, Definition and Motivation

The subject of fractional calculus (that is, calculus of derivatives and integrals of any real or complex order) has gained importance and popularity during the past three decades or so, due mainly to its demonstrated applications in many seemingly diverse fields of science and engineering (see, for details, [2] and [10]; see also [15]). Indeed one of the most frequently encountered tools in the theory and applications of fractional calculus is furnished by the Riemann-Liouville (*fractional differintegral*) operator D_z^μ defined by (*cf.*, *e.g.*, [10] and [11])

$$D_z^\mu\{f(z)\} := \begin{cases} \frac{1}{\Gamma(-\mu)} \int_0^z (z-\zeta)^{-\mu-1} f(\zeta)d\zeta & (\Re(\mu) < 0) \\ \frac{d^m}{dz^m} D_z^{\mu-m}\{f(z)\} & (m-1 \leq \Re(\mu) < m, m \in \mathbb{N}), \end{cases} \tag{1.1}$$

Received October 25, 2000 Revised May 26, 2001

2000 Mathematics Subject Classification Primary 26A33, 33C20, Secondary 33B15.

Key words and phrases· fractional calculus, infinite series, Riemann-Liouville operator, fractional derivative, fractional integral, generalized Leibniz rule, hypergeometric functions, expansion formula, reduction formula

provided that the integral in (1.1) exists, $\mathbb{N}$ being (as usual) the set of *positive* integers.

Recently, by applying the following (essentially equivalent) definition of a *fractional differintegral* (that is, *fractional derivative* and *fractional integral*) of order $\nu \in \mathbb{R}$, Nishimoto *et al.* [7] derived the sums of two interesting families of infinite series which are reproduced here, *in slightly modified forms*, as Theorem 1 and Theorem 2 below.

DEFINITION (*cf.* [5], [6], and [14]). If the function $f(z)$ is analytic and has no branch point inside and on $\mathcal{C}$, where

$$\mathcal{C} := \{\mathcal{C}^-, \mathcal{C}^+\}, \tag{1.2}$$

$\mathcal{C}^-$ is a contour along the cut joining the points z and $-\infty + i\mathfrak{J}(z)$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point at $-\infty$, $\mathcal{C}^+$ is a contour along the cut joining the points z and $\infty + i\mathfrak{J}(z)$, which starts from the point at ∞, encircles the point z once counter-clockwise, and returns to the point at ∞ ,

$$f_\nu(z) = {}_c f_\nu(z) := \frac{\Gamma(\nu+1)}{2\pi i}\int_{\mathcal{C}} \frac{f(\zeta)d\zeta}{(\zeta - z)^{\nu+1}}$$

$$(\nu \in \mathbb{R}\backslash\mathbb{Z}^-;\ \mathbb{Z}^- := \{-1, -2, -3, \dots\}), \tag{1.3}$$

and

$$f_{-n}(z) := \lim_{\nu \to -n} \{f_\nu(z)\} \qquad (n \in \mathbb{N}), \tag{1.4}$$

where $\zeta \neq z$,

$$-\pi \leq \arg(\zeta - z) \leq \pi \qquad \text{for} \quad \mathcal{C}^-, \tag{1.5}$$

and

$$0 \leq \arg(\zeta - z) \leq 2\pi \qquad \text{for} \quad \mathcal{C}^+, \tag{1.6}$$

then $f_\nu(z)$ $(\nu > 0)$ is said to be the *fractional derivative of* $f(z)$ *of order* ν and $f_\nu(z)$ $(\nu < 0)$ is said to be the *fractional integral of* $f(z)$ *of order* $-\nu$, provided that

$$|f_\nu(z)| < \infty \qquad (\nu \in \mathbb{R}). \tag{1.7}$$

THEOREM 1. *(cf Nishimoto et al. [7]).*
Let c and z be complex numbers.
Then

$$\sum_{k=2}^{\infty} \frac{(-c)^k}{k(k-1)} \cdot \frac{kz-c}{(z-c)^{k-1}} = c^2, \tag{1.8}$$

provided that

$$\left|\frac{c}{z-c}\right| < 1 \qquad (c, z \in \mathbb{C}). \tag{1.9}$$

THEOREM 2. *(cf Nishimoto et al. [7]).*
For complex parameters a, b, *and* c,

$$\begin{aligned}\sum_{k=1}^{\infty} &\frac{(k+c+a-1)\Gamma(k+a+b-1)}{k!}\left(-\frac{c}{b-c}\right)^k \\ &= \Gamma(a+b-1)\left\{(a-1)\left(\frac{b-c}{b}\right)^{a+b} - (c+a-1)\right\},\end{aligned} \tag{1.10}$$

provided that

$$\max\{|\Gamma(k+a)|, |\Gamma(k+a+b-1)|\} < \infty (a, b \in \mathbb{C}; k \in \mathbb{N}) \qquad (1.11),$$

and

$$\left|\frac{c}{b-c}\right| < 1 \qquad (b, c \in \mathbb{C}). \tag{1.12}$$

The proof of *each* of their results (Theorem 1 and Theorem 2 above) by Nishimoto *et al.* [7] is based rather heavily upon several lemmas involving the fractional differintegrals of logarithm and power functions (and, in the case of Theorem 2, also upon the generalized Leibniz rule for the differintegral of the product of two functions), which are defined by (1.3). The *main* object of the present sequel to the work of Nishimoto *et al.* [7] is to demonstrate that, not only *each* of the assertions of Theorems 1 and 2, but much more general families of infinite sums can also be evaluated *without* using the aforementioned fractional differintegral operator defined by (1.3). We also consider several other results relevant to our investigation here.

2. Alternative Derivations of Theorems 1 and 2

Our alternative derivation of the assertion (1.8) of Theorem 1 *without* using fractional calculus is based simply upon the familiar expansion formula:

$$\log(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k = z - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+1} z^{k+1} \qquad (|z| < 1), \tag{2.1}$$

which obviously holds true also when $z = 1$.

Denote, for convenience, the infinite series in (1.8) by $\mathcal{S}$.

Then, upon replacing the summation index k by $k+1$, it is easily seen that

$$\begin{aligned} \mathcal{S} :&= \sum_{k=2}^{\infty} \frac{(-c)^k}{k(k-1)} \cdot \frac{kz-c}{(z-c)^{k-1}} \\ &= \sum_{k=1}^{\infty} \frac{(-c)^{k+1}}{k(k+1)} \cdot \frac{(k+1)z-c}{(z-c)^k} \\ &= cz \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{c}{z-c}\right)^k - c^2 \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{k} - \frac{1}{k+1}\right) \left(\frac{c}{z-c}\right)^k \\ &= c(z-c) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{c}{z-c}\right)^k + c(z-c) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+1} \left(\frac{c}{z-c}\right)^{k+1}. \end{aligned} \tag{2.2}$$

Now, under the hypothesis (1.9) of Theorem 1, we can apply the expansion formula (2.1) to each of the infinite series in (2.2). We thus find that which evidently proves Theorem 1.

$$\begin{aligned}\mathcal{S} &= c(z-c)\left\{\log\left(1+\frac{c}{z-c}\right)+\left[\frac{c}{z-c}-\log\left(1+\frac{c}{z-c}\right)\right]\right\}\\ &= c^2 \quad \left(c, z \in \mathbb{C};\ \left|\frac{c}{z-c}\right| < 1\right),\end{aligned} \tag{2.3}$$

which evidently proves Theorem 1.

Alternatively (and *relatively* more simply), in view of the expansion formula (2.1) *and* the elementary identity:

$$kz - c = k(z-c) + (k-1)c,$$

the first member $\mathcal{S}$ of the assertion (1.8) of Theorem 1 can immediately be rewritten in the form:

$$\begin{aligned}\mathcal{S} :&= \sum_{k=2}^{\infty} \frac{(-c)^k}{k(k-1)} \cdot \frac{kz-c}{(z-c)^{k-1}}\\ &= \sum_{k=2}^{\infty} \frac{(-c)^k}{k(k-1)} \cdot \frac{k(z-c)+(k-1)c}{(z-c)^{k-1}}\\ &= \sum_{k=2}^{\infty} \frac{(-c)^k}{(k-1)(z-k)^{k-2}} + c\sum_{k=2}^{\infty} \frac{(-c)^k}{k(z-c)^{k-1}}\\ &= c(z-c)\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left(\frac{c}{z-c}\right)^k + c(z-c)\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+1}\left(\frac{c}{z-c}\right)^{k+1}\\ &= c(z-c)\log\left(1+\frac{c}{z-c}\right) + c(z-c)\left[\frac{c}{z-c}-\log\left(1+\frac{c}{z-c}\right)\right]\\ &= c^2 \quad \left(c, z \in \mathbb{C};\ \left|\frac{c}{z-c}\right| < 1\right),\end{aligned}$$

which is precisely the second member of the assertion (1.8) of Theorem 1.

Next we turn to our alternative derivation of the assertion (1.10) of Theorem 2 *without* using fractional calculus. First of all, for a generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters, defined by (*cf.* [17, p. 19 *et seq.*])

$$
\begin{aligned}
{}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z) &= {}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p; \\ \beta_1,\dots,\beta_q;\end{matrix}\; z\right] \\
&:= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k}\frac{z^k}{k!}
\end{aligned}
\tag{2.4}
$$

$$
(p,q\in\mathbb{N}_0:=\mathbb{N}\cup\{0\};\; p\leq q+1;\; p\leq q \textit{ and } |z|<\infty;
$$

$$
p=q+1 \textit{ and } |z|<1;\; p=q+1,\; |z|=1, \textit{ and } \Re(\omega)>0),
$$

where $(\lambda)_k$ denotes the Pochhammer symbol (or the *shifted* factorial, since $(1)_k=k!(k\in\mathbb{N}_0)$) given by

$$
(\lambda)_k := \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k=0) \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k\in\mathbb{N}), \end{cases} \tag{2.5}
$$

and

$$
\omega := \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j \quad (\beta_j\notin\mathbb{Z}_0^- := \mathbb{Z}^-\cup\{0\}; j=1,\dots,q), \tag{2.6}
$$

it is known that (*cf.*, *e.g.*, [3] *and* [12, p. 39, Equation (6)])

$$
\begin{aligned}
&{}_pF_q\left[\begin{matrix}\beta_1+m,\alpha_2,\dots,\alpha_p; \\ \beta_1,\dots,\beta_q;\end{matrix}\; z\right] \\
&= \sum_{j=0}^{m}\binom{m}{j}\frac{(\alpha_2)_j\cdots(\alpha_p)_j}{(\beta_1)_j\cdots(\beta_q)_j}\cdot z^j\; {}_{p-1}F_{q-1}\left[\begin{matrix}\alpha_2+j,\dots,\alpha_p+j; \\ \beta_2+j,\dots,\beta_q+j;\end{matrix}\; z\right],
\end{aligned}
\tag{2.7}
$$

provided that each member of (2.7) exists.

Since

$$
z = \frac{\Gamma(z+1)}{\Gamma(z)} \quad \left(z\in\mathbb{C};\; \left|\frac{\Gamma(z+1)}{\Gamma(z)}\right|<\infty\right), \tag{2.8}
$$

by appealing appropriately to the definitions (2.4) and (2.5), we readily find from the left-hand side of (1.10) that

$$\Omega(a,b,c):=\sum_{k=1}^{\infty}\frac{(k+c+a-1)\Gamma(k+a+b-1)}{k!}\left(-\frac{c}{b-c}\right)^{k}$$
$$=(c+a-1)\Gamma(a+b-1)\left({}_2F_1\left[\begin{matrix}c+a,a+b-1;\\ c+a-1;\end{matrix}\ -\frac{c}{b-c}\right]-1\right) \tag{2.9}$$

in terms of the Gauss hypergeometric function which corresponds to a special case of the definition (2.4) when

$$p=2\ \ and\ \ q=1.$$

Upon setting

$$p=2,\ q=1,\ m=1,\ \alpha_2=a+b-1,\ \beta_1=c+a-1,\ and\ z=-\frac{c}{b-c} \tag{2.10}$$

in the reduction formula (2.7), and recalling that [13, p. 20, Equation 1 2 (29)]

$${}_1F_0(\lambda;-;z)=(1-z)^{-\lambda}\quad(\lambda\in\mathbb{C};\ |z|<1), \tag{2.11}$$

we obtain

$${}_2F_1\left[\begin{matrix}c+a,a+b-1;\\ c+a-1;\end{matrix}\ -\frac{c}{b-c}\right]=\sum_{j=0}^{1}\binom{1}{j}\frac{(a+b-1)_j}{(c+a-1)_j}\left(-\frac{c}{b-c}\right)^j$$
$$\cdot\left(1+\frac{c}{b-c}\right)^{1-a-b-j}$$
$$=\left(\frac{b}{b-c}\right)^{1-a-b}-\frac{a+b-1}{c+a-1}$$
$$\cdot\frac{c}{b-c}\left(\frac{b}{b-c}\right)^{-a-b}$$
$$=\left(\frac{b-c}{b}\right)^{a+b}\left(\frac{b}{b-c}-\frac{c(a+b-1)}{(c+a-1)(b-c)}\right)$$
$$=\frac{a-1}{c+a-1}\left(\frac{b-c}{b}\right)^{a+b}, \tag{2.12}$$

which holds true under the constraints (1.11) and (1.12), exceptional parameter values (that would render any expression invalid or undefined) being tacitly excluded.

The assertion (1.10) of Theorem 2 would now follow immediately upon substituting from (2.12) into the last member of (2.9).

3. Generalizations of Theorems 1 and 2

There are at least two ways in which Theorem 1 can easily be stated in a more general setting. First of all, since [13, p. 20, Equation 1.2 (30)]

$$\log(1+z) = z\,{}_2F_1(1,1;2;-z) \quad (|z|<1), \tag{3.1}$$

it is not difficult to apply our alternative proof of Theorem 1 *mutatis mutandis* to show that

$$\sum_{k=2}^{\infty} \frac{(\alpha)_{k-2}(\beta)_{k-2}}{(\gamma)_{k-2}} \cdot \frac{(-c)^k}{(k-2)!(z-c)^{k-1}}$$

$$\cdot\left[z - c\left(1 - \frac{(\alpha+k-2)(\beta+k-2)}{(k-1)(\gamma+k-2)}\right)\right] = c^2 \tag{3.2}$$

$$\left(\gamma \notin \mathbb{Z}_0^-;\ \left|\frac{c}{z-c}\right| < 1;\ c, z \in \mathbb{C}\right)$$

or, more generally, that

$$\sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-2}\cdots(\alpha_p)_{k-2}}{(\beta_1)_{k-2}\cdots(\beta_q)_{k-2}} \cdot \frac{(-c)^k}{(k-2)!(z-c)^{k-1}}$$

$$\cdot\left[z - c\left(1 - \frac{(\alpha_1+k-2)\cdots(\alpha_p+k-2)}{(k-1)(\beta_1+k-2)\cdots(\beta_q+k-2)}\right)\right] = c^2 \tag{3.3}$$

$$\left(p \leq q;\ p = q+1 \text{ and } \left|\frac{c}{z-c}\right| < 1;\ c, z \in \mathbb{C}\right),$$

provided that each member of (3.2) and (3.3) exists.

The summation formula (3.2) corresponds to a special case of (3.3) when

$$p = 2,\ q = 1,\ \alpha_1 = \alpha,\ \alpha_2 = \beta,\ \textit{and}\ \beta_1 = \gamma.$$

Furthermore, in its special case when

$$\alpha = \beta = 1 \quad \textit{and} \quad \gamma = 2,$$

(3.2) would yield the assertion (1.8) of Theorem 1.

The aforementioned other way in which Thiorem 1 can easily be stated in a more general setting is based upon the following *obvious* variation of the expansion formula (2.1):

$$\log(1+z) = \sum_{k=1}^{m-1} \frac{(-1)^{k+1}}{k} z^k - \sum_{k=0}^{\infty} \frac{(-1)^{k+m}}{k+m} z^{k+m} \tag{3.4}$$

$$(|z| < 1; m \in \mathbb{N})$$

where (and in what follows) an empty sum is interpreted (*as usual*) to be zero. One of the *simplest* such generalizations of Theorem 1 has the *elegant* form;

$$\sum_{k=2}^{\infty} \frac{(-c)^k}{(k+m-1)(k+m-2)} \cdot \frac{(k+m-1)z - c}{(z-c)^{k-1}} = \frac{c^2}{m} \tag{3.5}$$

$$\left(m \in \mathbb{N};\ \left|\frac{c}{z-c}\right| < 1;\ c, z \in \mathbb{C}\right),$$

which, for m=1, reduces at once to the assertion (1.8) of Theorem 1. Slightly more generally, since

$$(k+m-1)z - (m-l)c = (k+m-1)(z-c) + (k+l-1)c, \tag{3.6}$$

by rewriting the expansion formula (3.4) in its *equivalent* form :

$$\sum_{k=0}^{\infty} \frac{z^{k+1}}{k+m+1} = -z^{-m}\log(1-z) - \sum_{k=0}^{m-1} \frac{z^{k-m+1}}{k+1} \tag{3.7}$$

$$(0 < |z| < 1; m \in \mathbb{N}_0),$$

it is readily observed that

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{(-c)^k}{(k+l-1)(k+m-1)} \cdot \frac{(k+m-1)z-(m-l)c}{(z-c)^{k-1}} \\
&\quad = c^2\left[\left(1-\frac{z}{c}\right)^{l+1} - \left(1-\frac{z}{c}\right)^{m}\right]\log\left(1-\frac{z}{c}\right) \\
&\quad + c(z-c)\sum_{k=0}^{l-1} \frac{\{c/(c-z)\}^{k-l+1}}{k+1} \\
&\quad + c^2 \sum_{k=0}^{m-1} \frac{\{c/(c-z)\}^{k-m+1}}{k+1}
\end{aligned} \tag{3.8}$$

$$\left(l \in \mathbb{N}_0; m \in \mathbb{N}; \left|\frac{c}{z-c}\right| < 1; c, z \in \mathbb{C}\right),$$

which, in the special case when $l = m - 1 (m \in \mathbb{N})$, yields (3.5), exceptional values of c *and* z (that would render either side of (3.8) invalid or undefined) being tacitly excluded.

A *closer* look at our alternative derivation of the assertion (1.10) of Theorem 2 *without* using fractional calculus would reveal the fact that Theorem 2 is essentially a special case of the known hypergeometric reduction formula (2.7).

This hypergeometric reduction formula (2.7) itself has already been extended as well as generalized in one and more variables in different several ways (see, for details, [1],[3],[4],[8],[9], and [12]).

The following yet another consequence of the hypergeometric reduction formula (2.7) does provide a generalization of Theorem 2 :

$$\begin{aligned}
&\sum_{k=1}^{\infty} (k+\beta)_m \Gamma(k+\alpha) \frac{z^k}{k!} \\
&\quad = \Gamma(\alpha)\left\{\sum_{j=0}^{m} \binom{m}{j} \frac{(\alpha)_j (\beta)_m}{(\beta)_j} \frac{z^j}{(1-z)^{\alpha+j}} - (\beta)_m\right\}
\end{aligned} \tag{3.9}$$

$$(\alpha \in \mathbb{C}; \beta \in \mathbb{C}\backslash\mathbb{Z}_0^-; |z| < 1).$$

Evidently, in its special case when

$$m = 1, \alpha = a + b - 1, \beta = c + a - 1, \quad \text{and} \quad z = -\frac{c}{b-c},$$

(3.9) would yield the assertion (1.10) of Theorem 2.

Acknowledgements

The present investigation was supported, in part, by th *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

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