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OTHER PROOFS OF KUMMER'S SECOND THEOREM

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ABSTRACT The aim of this research note is to derive the well known Kummer's second theorem by transforming the integrals which represent some generalized hypergeometric functions This theorem can also be shown by combining two known Bailey's and Preece's identities for the product of generalized hypergeometric series

1. Introduction

The generalized hypergeometric function [4] with p numerator and q denominator parameters is defined by

(1.1)
$$
{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\,;\\ \\ \beta_1,\ldots,\beta_q\,;\end{matrix}\;z\right] = {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)
$$
$$
= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}\frac{z^n}{n!},
$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined, for any complex number α, by

$$
(1.2) \qquad (\alpha)_n := \begin{cases} 1 & (n=0) \\ \alpha(\alpha+1)\ldots(\alpha+n-1) & (n=1,\,2,\,3,\ldots), \end{cases}
$$

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which can also be rewritten in the form:

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \tag{1.3}$$

where Γ is the well-known Gamma function whose Weierstrass canonical product form is

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left\{ \left(1+\frac{z}{n}\right)^{-1} e^{z/n} \right\}, \tag{1.4}$$

γ being the Euler-Mascheroni constant defined by

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) \cong 0.577\,215\,664\,901\,532 \cdots . \tag{1.5}$$

With the notation (1.1), the Gaussian hypergeometric series is ${}_2F_1$, which is also denoted simply by F.

From the theory of differential equations, Kummer [2] established the following very interesting and useful result, which is, in the literature, referred to as the Kummer's second theorem, *via*.

$$e^{-x}\,{}_1F_1(\alpha;\, 2\alpha;\, 2x) = {}_0F_1\left(-;\, \alpha+\frac{1}{2};\, \frac{x^2}{4}\right). \tag{1.6}$$

Later on, Bailey [1] established the result (1.6) in the form

$$e^{-\frac{x}{2}}\,{}_1F_1(\alpha;\, 2\alpha;\, x) = {}_0F_1\left(-;\, \alpha+\frac{1}{2};\, \frac{x^2}{16}\right) \tag{1.7}$$

by making use of classical Gauss's second summation theorem:

$$ {}_2F_1\left[\begin{matrix} a, \quad b\,; \\ \frac{1}{2}(a+b+1)\,; \end{matrix} \; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \tag{1.8}$$

provided $a+b \neq -1,\, -3,\, -5, \ldots$.

Very recently Rathie and Choi [5] derived the result (1.6) by using classical Gauss's summation theorem:

$$ {}_2F_1\left[\begin{matrix} a, & b\,; \\ & c\,; \end{matrix} 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \tag{1.9} $$

provided $\Re(c-a-b)>0$.

The following results will be required in our present investigations.

Preece's identity [3]:

$$ \{{}_1F_1(\alpha;\,2\alpha;\,x)\}^2 = e^x\,{}_1F_2\left(\alpha,\,\alpha+\frac{1}{2},\,2\alpha\,;\,\frac{x^2}{4}\right); \tag{1.10} $$

Bailey's result [1]:

$$ \{{}_0F_1(-;\,\rho;\,x)\}^2 = {}_1F_2\left(\rho-\frac{1}{2};\,\rho,\,2\rho-1;\,4x\right); \tag{1.11} $$

Integral representation for ${}_1F_1$ [4]:

$$ {}_1F_1(\alpha;\,\rho,\,z) = \frac{\Gamma(\rho)}{\Gamma(\alpha)\,\Gamma(\rho-\alpha)}\int_0^1 e^{zt}\,t^{\alpha-1}\,(1-t)^{\rho-\alpha-1}\,dt; \tag{1.12} $$

Finite integral:

$$ \int_{-1}^{1} e^{zx}\left(1-x^2\right)^{\alpha-1}dx = \frac{\Gamma(\alpha)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)}\,{}_0F_1\left(-;\,\alpha+\frac{1}{2};\,\frac{z^2}{4}\right) \tag{1.13} $$

provided $\Re(\alpha)>0$.

We are aiming at deriving Kummer's second theorem (1.6) by transforming the integrals (1.12) and (1.13). The equivalent form (1.7) of (1.6) can also be shown to be obtained by combining (1.10) and (1.11).

2. Derivation of Kummer's Second Theorem

Write (1.6) in the form

$$ {}_1F_1(\alpha,\,2\alpha;\,2x) = e^x\,{}_0F_1\left(-;\,\alpha+\frac{1}{2};\,\frac{x^2}{4}\right). \tag{2.1} $$

By using (1.13), the right-hand side of (2.1) becomes

$$e^{x}\,\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)\,\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} e^{xt}\,\left(1-t^{2}\right)^{\alpha-1}\,dt$$
$$=\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)\,\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} e^{(1+t)x}\,\left(1-t^{2}\right)^{\alpha-1}\,dt,$$

which, upon putting $1+t=2\lambda$ and using duplication formula for the Gamma function (see [4, p. 24]), leads to

$$\frac{\Gamma(2\alpha)}{\Gamma(\alpha)\,\Gamma(\alpha)}\int_{0}^{1} e^{2x\lambda}\,\lambda^{\alpha-1}\,(1-\lambda)^{\alpha-1}d\lambda. \tag{2.2}$$

By applying (1.12) to the integral part of (2.2), we find that (2.2) becomes ${}_1F_1(\alpha;\,2\alpha;\,2x)$, which completes the proof of (2.1).

We conclude this note by remarking that the identity (1.7), which is equivalent to (1.6), can also be obtained from Preece's identity (1.10) and Bailey's identity (1.11) as follows: Combining (1.10) and (1.11), we obtain

$$e^{-x}\,\left\{{}_1F_1(\alpha;\,2\alpha;\,x)\right\}^{2}={}_1F_2\left(\alpha;\,\alpha+\frac{1}{2},\,2\alpha\,;\,\frac{x^{2}}{4}\right)$$

or

$$\left\{e^{-\frac{x}{2}}\,{}_1F_1(\alpha;\,2\alpha;\,x)\right\}^{2}=\left\{{}_0F_1\left(-;\,\alpha+\frac{1}{2};\,\frac{x^{2}}{16}\right)\right\}^{2},$$

which, upon taking square root and considering the value when $x=0$, immediately yields (1.7).

References

[1] W.N Bailey, *Products of generalized hypergeometric series*, Proc London Math.Soc **28** (1928), 242 - 250

[2] E E Kummer, *Über die hypergeometrische Reihe ...*, J Reine Angew Math **15** (1836), 39 - 83

[3] C T Preece, *The product of two generalized hypergeometric functions*, Proc. London Math Soc **22** (1924), 370 - 380

[4] E D Rainville, *Special Functions*, The Macmillan Company, New York, 1960
[5] A K Rathie and J Choi, *Another proof of Kummer's second theorem*, Comm. Korean Math Soc **13** (1998), 933 - 936

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