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UNSOLVED PROBLEMS IN BCK-ALGEBRAS

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ABSTRACT We present old unsolved problems on BCK-sequences connected with convex congruences on BCK-algebras We posed also some new problems on subalgebras

1. Introduction

Note that a *BCK-algebra* is an algebra $\mathcal{G}=(G,\cdot,0)$ of type $(2,0)$ in which the following axioms are satisfied.

(1) $(xy \cdot xz) \cdot zy = 0$,
(2) $(x \cdot xy)y = 0$,
(3) $xx = 0$,
(4) $0x = 0$,
(5) $xy = yx = 0 \Longrightarrow x = y$,

where (1) means $((x \cdot y) \cdot (x \cdot z)) \cdot (z \cdot y) = 0$. In the sequel dots we use only to avoid repetitions of brackets.

The concept of a BCK-algebra was introduced by K. Iséki and S. Tanaka (cf. [15]) and it is easy to see that a BCK-multiplication xy generalizes, in particular, the ideal-quotient of commutative rings with unity.

The class of all BCK-algebras do not form a variety (cf. [27]), but many important subclasses of this class form a variety. A typical example is the class of *commutative BCK-algebras*, i.e. BCK-algebras

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with the condition $x \cdot xy = y \cdot yx$ (cf. [25]). Another variety was introduced by B. Bosbach later referred as commutative complementary semigroups (cf. [1], p.267). This variety was rediscovered by W. H. Cornish (cf. [5]) who called it the class of BCK-algebras with supremum. This variety contains the class of Brouwerian semilattices (cf. [1]). Other important varieties of BCK-algebras are the Heyting algebras and MV-algebras of C. C. Chang (cf. [2]).

On any BCK-algebra one can define the *natural order* $\leq$ by putting

(6) $x \leq y$ if and only if $xy = 0$.

It is not difficult to see that $\leq$ is a partial order with 0 as the smallest element. A BCK-algebra $\mathcal{G}$ in which for every $a, b \in G$ the set

$$\{x \in G \mid xa \leq b\}$$

has a greatest element, denoted by $a + b$, is called a *BCK-algebra with condition* (S). Every such BCK-algebra is a commutative semigroup with respect to the operation $+$ and 0 is its zero element. If it satisfies also the condition $xz \cdot yz = xy \cdot z$, then it is equivalent to implicative semilattice (cf. [3] or [21]). Moreover, as proved J. Meng (cf. [20]), BCK-algebras with condition (S), commutative residual pomonoids with the identity as the greatest element and implicative commutative semigroups are categorically equivalent to each other.

As a simple consequence of the above system of axioms we obtain

(7) $x0 = x$,
(8) $xy \cdot z = xz \cdot y$,
(9) $x(x \cdot xy) = xy$,
(10) $x \leq y$ implies $xz \leq yz$ and $zy \leq zx$.

An important role in the theory of BCK-algebras and related systems play ideals defined as a subset I of G such that $0 \in I$ and $y, xy \in I$ imply $x \in I$. Such ideals are ideals in the sense of ordered sets, i.e. $x \leq y$ and $y \in I$ imply $x \in I$. Note that such defined ideals (called also *BCK-ideals* (cf. [12])) can be characterized (cf. [9]) as subsets I with 0 such that $xy, y(yx \cdot z) \in I$ for all $y \in G$ and $x, z \in I$.

2. Convex congruences

Convex congruences have been successfully used in developing the theory of partially ordered algebraic systems (cf. [13]) such as partially ordered groups (cf. [19]) and BCK-algebras (cf. [23], [24]).

A congruence Θ on a partially ordered set G is called *convex* if for all $x, y, z \in G$ the following implication holds

(11) $x \leq y \leq z$ and $(x, z) \in \Theta$ imply $(x, y) \in \Theta$.

Of course, in lattices every congruence is convex. However congruences on BCK-algebras may not be necessarily convex, but in finite BCK-algebras all congruences are convex (cf. [26]).

We say that a relation Θ_I defined on a BCK-algebra $\mathcal{G}$ is induced by a subset $I \subseteq G$, if

$$(x, y) \in \Theta_I \iff xy,\ yx \in I.$$

A relation induced by an ideal is a congruence, but there are congruences which are not induced by an ideal. In finite BCK-algebras all congruences are induced by ideals (cf. [12]).

PROPOSITION 2 1 *A congruence induced by an ideal is convex.*

PROOF Indeed, if $x \leq y \leq z$ and $(x, z) \in \Theta_I$ then $xy = yz = 0 \in I$ and $yx = yx \cdot 0 = yx \cdot yz \leq zx \in I$ by (1) and (6). Hence $yx \in I$. Thus $xy, yx \in I$ and, in the consequence, $(x, y) \in \Theta_I$

In [9] the following result is proved.

PROPOSITION 2 2 *All congruences of a BCK-algebra, which belongs to a some variety, are induced by ideals.*

As a simple consequence of the above results we obtain

PROPOSITION 2 3 *If a BCK-algebra belongs to a some variety, then all its congruences are convex.*

T. Traczyk proved (cf. [23]) that all congruences of a given BCK-algebra are convex if every strongly decreasing sequence of elements of this BCK-algebra is finite.

3. n-commutative BCK-algebras

We say that $\{x_k\}$ is a *BCK-chain* if $x_0 \geq x_1$ and $x_{k+2} = x_k \cdot x_k x_{k+1}$ for all $k \geq 0$.

It is not difficult to see that $x_k \geq x_{k+1}$ for all $k \geq 0$. If $x_k = x_{k+1}$ for some k, then also $x_n = x_k$ for all $n \geq k$. Obviously $x_k = 0$ implies $x_{k+1} = 0$. In a finite BCK-algebra always there is $k < Card(G)$ such that $x_k = x_{k+1}$ for all its BCK-chains.

In connection with this the following problem seems to be interesting.

PROBLEM 1. Find the necessary and sufficient condition under which all BCK-chains of a given BCK-algebra terminate at finite step.

A BCK-algebra $\mathcal{G}$ is called an *n-commutative* if n is a minimal positive integer such that $x_n = x_{n+1}$ for all its BCK-chains.

A finite BCK-algebra is n-commutative for some $n < Card(G)$.

T. Traczyk proved (cf. [23]) that the class V_n of all n-commutative BCK-algebras is a variety and the sequence $V_1 \subset V_2 \subset V_3 \subset \ldots$ is strongly increasing.

To prove this fact, for arbitrary elements x, y of a given BCK-algebra $\mathcal{G}$, he defined the following two BCK-sequences

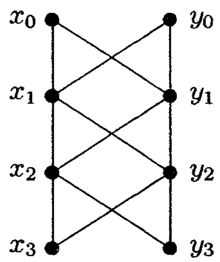
$x_0 = x,\ x_1 = y \cdot yx,\ x_2 = x_0 \cdot x_0 x_1, \ldots, x_k = x_{k-2} \cdot x_{k-2} x_{k-1}, \ldots$

$y_0 = y,\ y_1 = x \cdot xy,\ y_2 = y_0 \cdot y_0 y_1, \ldots, y_k = y_{k-2} \cdot y_{k-2} y_{k-1}, \ldots$ and proved that

(12) $x_0 \geq y_1 \geq x_2 \geq y_3$,

(13) $y_0 \geq x_1 \geq y_2 \geq x_3$

(see the diagram).



In connection with this he posed

PROBLEM 2. Prove or disprove that for any BCK-algebra the sequences of inequalities (12) and (13) can be prolonged.

For $x_3 = y_3$ we have $x_3 \geq y_4$ and $y_3 \geq x_4$, which shows that in this case (12) and (13) can be extend to the form $x_0 \geq y_1 \geq x_2 \geq y_3 \geq x_4$

and $y_0 \geq x_1 \geq y_2 \geq x_3 \geq y_4$. For $x_3 = y_3 = 0$ this extension is infinite. If $x_k = y_k$, $x_{k+1} = y_{k+1}$ for some k, then also $x_{k+t} = y_{k+t}$ for every $t \geq 0$. This means that in the case $x_2 = y_2$, $x_3 = y_3$ the sequences of inequalities (12) and (13) can be prolonged.

But there are infinite BCK-algebras in which the sequences of inequalities (12) and (13) can be prolonged to infinite strongly decreasing sequences.

EXAMPLE 3 1. Let $G = N \cup A \cup B$, where $N = \{0, 1, 2, ...\}$, $A = \{a_n | n \in N\}$ *and* $B = \{b_n | n \in N\}$.

On the set G we define the operation $*$ as follows:

for $m, n \in N$,

$$m * n = \begin{cases} 0 & \text{if } m \leq n, \\ m - n & \text{if } m > n, \end{cases}$$

$$m * a_n = m * b_n = 0,$$

$$b_m * n = b_{m+n},$$

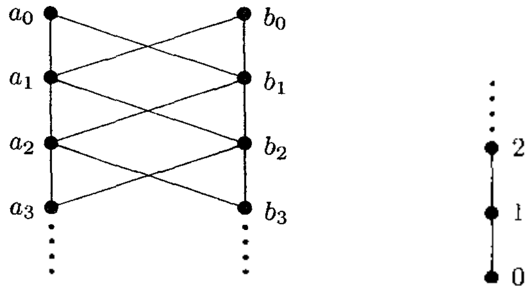
$$a_m * n = a_{m+n},$$

$$a_m * a_n = b_m * b_n = \begin{cases} 0 & \text{if } n \leq m, \\ n - m & \text{if } n > m, \end{cases}$$

$$a_m * b_n = a_m * a_{n+1},$$

$$b_m * a_n = b_m * b_{n+1}.$$

One can prove (for detail see [27]) that the set G with this operation is a BCK-algebra. Its natural order is represented by the following diagram.



It is not difficult to see that in this BCK-algebra the pairs of elements a_m, b_m, $m \in N$, are incomparable. For all other pairs we have $x \leq y$ or $y \leq x$.

Starting from $x_0 = a_m$, $y_0 = b_m$ we obtain two infinite BCK-sequences $\{x_k\}$ and $\{y_k\}$, where

$$x_k = \begin{cases} a_{m+k} & \text{for } k = 2n, \\ b_{m+k} & \text{for } k = 2n+1, \end{cases}$$

and

$$y_k = \begin{cases} b_{m+k} & \text{for } k = 2n, \\ a_{m+k} & \text{for } k = 2n+1. \end{cases}$$

In these sequences for all $k \in N$ we have $x_k > y_{k+1} > x_{k+2} > y_{k+3}$, which means that in this case the sequence of inequalities (12) and (13) can be prolonged. From Lemma 3.2c given below it follows that these inequalities can be prolonged also in the case when elements x_0 and y_0 are compared.

Note by the way, that in this BCK-algebra, the equivalence relation Θ corresponding to the partition $\{N, A, B\}$ is a congruence. Moreover, $N = [0]_\Theta$ is an ideal, but the congruence Θ_N induced by N has only two equivalence classes: N and $A \cup B$. Thus $\Theta \neq \Theta_N$.

Now we prove some general properties of BCK-sequences.

LEMMA 3 2 *Let $\mathcal{G}$ be a fixed BCK-algebra. Then for every $k \geq 0$*

(a) $x_k x_{k+2} = x_k x_{k+1}$,

(b) $x_{k+2} = x_k \cdot x_k x_{k+2}$,

(c) $x_k = y_{k+1}$ *if* $x_0 \leq y_0$.

PROOF Applying (9) to the definition of x_{k+2} we obtain

$$x_k x_{k+2} = x_k(x_k \cdot x_k x_{k+1}) = x_k x_{k+1},$$

which gives (a).

(b) follows from (a).

Now, if $x = x_0 \leq y_0 = y$, then

$$x_0 = x = x0 = x \cdot xy = y_1,$$

$$x_1 = y \cdot yx = y_0 \cdot y_0 y_1 = y_2,$$

and, by the induction

$$x_{k+2} = x_k \cdot x_k x_{k+1} = y_{k+1} \cdot y_{k+1} y_{k+2} = y_{k+3},$$

which proves (c).

COROLLARY 3 3 *If $x_0 = x_1$ then (12) and (13) can be prolonged.*

PROOF Indeed, if this case $y_0 \geq x_1 = x_0$, which by Lemma 3.2c gives $y_2 = x_1 = x_0$ and $y_3 = x_2 = x_0 \cdot x_0 x_1 = x_0$. Thus, by induction, $y_{k+1} = x_k = x_0$ for all $k \geq 0$. This means that (12) and (13) can be prolonged.

COROLLARY 3 4 *In linearly ordered BCK-algebras the sequences of inequalities (12) and (13) can be prolonged. Moreover, in the case $x_0 \leq y_0$ we have*

$$x_0 = y_1 \geq x_2 = y_3 \geq x_4 = y_5 \geq x_6 = y_7 \geq \cdots,$$

and

$$y_0 \geq x_1 = y_2 \geq x_3 = y_4 \geq x_5 = y_6 \geq \cdots.$$

PROPOSITION 3 5 *In commutative BCK-algebras the sequences of inequalities (12) and (13) can be prolonged.*

PROOF In commutative BCK-algebras $x_1 = y \cdot yx = x \cdot xy = y_1$ and $x_2 = x_0 \cdot x_0 x_1 = x_1 \cdot x_1 x_0 = (y \cdot yx) \cdot ((y \cdot yx)x) = (y \cdot yx) \cdot (yx \cdot yx) = x_1$, by (8), (3) and (7). Similarly we obtain $y_2 = y_1 = x_1$.

Hence $x_3 = x_1 \cdot x_1 x_2 = x_1$, $y_3 = y_1 \cdot y_1 y_2 = y_1 = x_1$, and, by the induction $x_k = x_1 = y_1 = y_k$ for all $k > 1$. This proves that the sequences of inequalities (12) and (13) can be prolonged.

Thus the Traczyk's problem can be reformulated as.

PROBLEM 3 Prove or disprove that in non-commutative BCK-algebras the sequences of inequalities (12) and (13) with incomparable starting elements can be prolonged.

PROPOSITION 3 6 *If (12) and (13) can be prolonged, then $x_k = y_k$ implies $x_{k+2n} = y_{k+2n}$ for every $n \geq 0$.*

PROOF. The proof is based on the following inequality

(14) $xy \cdot xz \leq x(x \cdot zy)$,

which holds in all BCK-algebras.

To prove it observe that, by (8) and (9), in all BCK-algebras we have

$$xy \cdot xz = x(x \cdot xy) \cdot xz = (x \cdot xz) \cdot (x \cdot xy).$$

Hence

$$\begin{aligned}(xy \cdot xz) \cdot x(x \cdot zy) &= ((x \cdot xz) \cdot (x \cdot xy)) \cdot x(x \cdot zy) \\ &= ((x(x(x \cdot zy))) \cdot (x \cdot xy)) \cdot xz && \text{by} \quad (8) \\ &= ((x \cdot zy) \cdot (x \cdot xy)) \cdot xz && \text{by} \quad (9) \\ &= ((x(x \cdot xy) \cdot zy) \cdot xz && \text{by} \quad (8) \\ &= (xy \cdot zy) \cdot xz = (xy \cdot xz) \cdot zy = 0 && \text{by} \quad (9), (8), (1)\end{aligned}$$

which completes the proof of (14).

Now, if $x_k = y_k$ for some fixed k, then

$$\begin{aligned}x_{k+2}y_{k+2} &= (x_k \cdot x_k x_{k+1})(y_k \cdot y_k y_{k+1}) \\ &= (x_k \cdot x_k x_{k+1})(x_k \cdot x_k y_{k+1}) \\ &= (x_k(x_k \cdot x_k y_{k+1}))(x_k x_{k+1}) && \text{by} \quad (8) \\ &= (x_k y_{k+1})(x_k x_{k+1}) && \text{by} \quad (9) \\ &= (x_k y_{k+1})(x_k x_{k+2}) && \text{by} \quad \text{Lemma 3.2c} \\ &\leq x_k(x_k \cdot x_{k+2} y_{k+1}) = 0 && \text{by} \quad (14),\end{aligned}$$

because of the assumption $x_{k+2} \leq y_{k+1}$. Thus $x_{k+2} \leq y_{k+2}$.

By the symmetry we obtain the inverse inequality. Hence $x_{k+2} = y_{k+2}$, and, by the induction, $x_{k+2n} = y_{k+2n}$ for every natural number n. This completes the proof.

PROBLEM 4 Find the necessary and sufficient conditions under which the sequences of inequalities (12) and (13) terminates at finite step.

As it is well known that in any BCK-algebra one can define a new operation $\wedge$ by putting $x \wedge y = x \cdot xy$. It is clear that $x \wedge y \leq x$ and $x \wedge y \leq y$, but in general $x \wedge y \neq y \wedge x$. Also $(x \wedge y) \wedge z \neq x \wedge (y \wedge z)$.

Using this operation, we can define a BCK-sequence as a sequence $\{x_{k+2}\}$, where $x_0 = x$, $y_0 = y$ are given, $x_1 = y \wedge x$, $y_1 = x \wedge y$ and $x_{k+2} = x_k \wedge x_{k+1}$ for all $k \geq 0$.

T. Traczyk proved (cf. [23]) that the variety V_1 coincides with the variety of all commutative BCK-algebras. This variety is uniquely determined by the identity $x_1 = y_1$, i.e by the identity $y \wedge x = x \wedge y$. The variety V_2 is determined by $x_2 = y_2$, i e. by the identity $x \wedge (y \wedge x) = y \wedge (x \wedge y)$, which is known in the literature as the Cornish's *condition (J)* (cf. [4]).

The problem of V_3 is open. But some known results suggest that this variety can be determined by the identity $x_3 = y_3$, i e. by

$$(y \wedge x) \wedge (x \wedge (y \wedge x)) = (x \wedge y) \wedge (y \wedge (x \wedge y)).$$

In connection with this the following problem (posed in [23]) seems to be interesting.

PROBLEM 5 Prove or disprove that the variety V_n of all n-commutative BCK-algebras is determined by the identity $x_n = y_n$.

4. Subalgebras

A subset S of a BCK-algebra G is a *subalgebra* if and only if it is closed under BCK-operation. Of course, every subalgebra contains 0. Moreover, as it is not difficult to see that every subset containing 0 and one nonzero element is a subalgebra. On the other hand, J Hao proved (cf. [14]) that every BCK-algebra of order $n \geq 2$ contains at least one subalgebra of the order $i = 1, 2, \ldots, n-1$. In particular, a BCK-algebra of order $n \geq 2$ contains at least one subalgebra of the order $n-1$. This means that every BCK-algebra of the order $n \geq 2$

may be considered as one-element extension of a some BCK-algebra of order $n-1$.

Let $N(i)$ denotes the number of subalgebras of the order i. Obviously $1 \leq N(i) \leq C_{n-1}^{i-1}$ for every BCK-algebra of order $n \geq 2$, where C_{n-1}^{i-1} denotes the number of ways for selecting $i-1$ elements from $n-1$ nonzero elements. Obviously, $N(2) = n-1$ for every BCK-algebra of the order $n \geq 2$. In general $N(i) < C_{n-1}^{i-1}$, but there are BCK-algebras in which $N(i) = C_{n-1}^{i-1}$ for all $i = 2, 3, \ldots, n$. A simple example of such BCK-algebra is the set $G_n = \{0, 1, \ldots, n-1\}$ with the operation $*$ defined by $x * y = x$ for $x > y$ and $x * y = 0$ otherwise.

PROBLEM 6. Describe the structure of finite BCK-algebras in which $N(i) = C_{n-1}^{i-1}$ for all $i = 2, 3, \ldots, n$.

PROBLEM 7. Describe the class of BCK-algebras in which every subset containing 0 is a subalgebra (an ideal).

The partial answer to the Problem 6 gives

PROPOSITION 4.1. *Let $\mathcal{G}$ be a BCK-algebra of the order $n > 3$. If $N(i) = C_{n-1}^{i-1}$ for some fixed $3 \leq i < n$, then every subset of G containing 0 is a subalgebra.*

PROOF. Let $M = \{0, a_1, a_2, ..., a_{i+1}\}$ be an arbitrary subset of a BCK-algebra $\mathcal{G}$, where i is as in the assumption.

Then $S_1 = \{0, a_2, a_3, ..., a_{i+1}\}$, $S_2 = \{0, a_1, a_3, ..., a_{i+1}\}$ and $S_3 = \{0, a_1, a_2, a_4, ..., a_{i+1}\}$ are subalgebras. Thus for all $x, y \in M = S_1 \cup S_2 \cup S_3$ we have $xy \in M$, which proves that M is a subalgebra. Hence, by induction, every subset containing 0 and $j \geq i$ nonzero elements is a subalgebra.

All subsets containing 0 and $j < i$ nonzero elements are subalgebras too. Indeed, if some $S_j = \{0, a_1, ..., a_j\}$ is not a subalgebra, then there exist $x, y \in S_j$ such that $xy = z \neq a_k$ for every $a_k \in S_j$. Thus $M = S_j \cup (\{a_{j+1}, ..., a_i\} \setminus \{z\})$ containing 0 and i nonzero elements is not a subalgebra, which is a contradiction.

COROLLARY 4 2 *In a BCK-algebra of order $n > 3$ all subsets containing 0 are subalgebras if and only if* $N(3) = \binom{n-1}{2}$.

COROLLARY 4.3 *In a BCK-algebra of the order $n > 3$ for every $i = 3, ..., n-1$ we have either $N(i) = C_{n-1}^{i-1}$ or $N(i) < C_{n-1}^{i-1}$.*

Let $\mathcal{G}$ be a *bounded* BCK-algebra, i.e. a BCK-algebra in which there exists an element 1 such that $x \leq 1$ for all $x \in G$. A subalgebra S of such BCK-algebra is called *extremal* if it contains 1. Such subalgebra has at least two elements: 0 and 1.

PROPOSITION 4 4 *If in a bounded BCK-algebra $\mathcal{G}$ all subsets of the form $\{0, a_1, a_2, ..., a_i, 1\}$, where $i \geq 2$ is fixed, are subalgebras, then all subset of G containing $0, 1$ and at least two elements are subalgebras.*

PROOF A modification of the proof of Proposition 4.1.

Let $N_e(i)$ denotes the number of extremal subalgebras of the order $i \geq 2$. Since every such subalgebra contains 0 and 1, then $N_e(i) \leq \binom{n-2}{i-2}$ for all bounded BCK-algebras of order $n \geq 2$.

COROLLARY 4 5 *In a bounded BCK-algebra of order $n > 3$ for every $i = 3, ..., n$ we have either $N_e(i) = C_{n-2}^{i-2}$ or $N_e(i) < C_{n-2}^{i-2}$.*

PROBLEM 8 Describe the class of bounded BCK-algebras in which every subset containing 0 and 1 is a subalgebra

PROBLEM 9 Describe the structure of finite BCK-algebras in which $N_e(i) = C_{n-2}^{i-2}$ for every $i = 3, 4, \ldots, n$.

Finally, we point out that these results are not true for finite BCI-algebras (i.e for BCK-algebras in which (4) is not satisfied). A BCI-algebra without subalgebras of order 3 is given in [14].

5. BCC-algebras

In connection with some problems on BCK-algebras (posed by K. Iséki) Y. Komori introduced in [16] a notion of BCC-algebras and proved (using some Gentzen-type system LC) that the smallest variety containing the class of all BCC-algebras is finitely based, but the class of all BCC-algebras is not a variety (cf. [17]).

Now in the literature BCC-algebras are defined as algebras $\mathcal{G} = (G, \cdot, 0)$ of type (2,0) satisfying

(15) $(xy \cdot zy) \cdot xz = 0$,

(3), (4) and (5).

This definition is a dual form of the ordinary definition given by Y. Komori. In this convention any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras [7]. Such BCC-algebras are called *proper*. The smallest proper BCC-algebra has four elements. For every $n \geq 4$ there exists at least one proper BCC-algebra with n elements. For $n = 4$ there is eight such non-isomorphic BCC-algebras (cf. [7]), but for $n = 5$ such BCC-algebras is 346.

Obviously, any subalgebra containing at most 3 elements is a BCK-algebra. Moreover, as was mentioned in [6], there are proper BCC-algebras in which all subalgebras are BCK-algebras. Also there are finite proper BCC-algebras in which all subsets containing 0 are subalgebras.

In connection with this the following two problems (posed in [6]) seems to be interesting.

PROBLEM 10. Find a characterization of proper BCC-algebras in which all subalgebras are BCK-algebras.

PROBLEM 11 Find a characterization of BCC-algebras of finite order n in which $N(i) = C_{n-1}^{i-1}$ for all $i = 1, 2, \ldots, n$.

The partial characterization is given by

PROPOSITION 5.1 *A BCC-algebra in which any subset containing 0 is a subalgebra is a BCK-algebra.*

PROOF. The proof is based (for details see [11]) on the simple fact (proved in [7]) that a BCC-algebra is a BCK-algebra if and only if it satisfies (2) or, equivalently, (8).

Note by the way, that any BCC-algebra satisfies (7) and (10), where $\leq$ is the natural order defined by (6). A commutative BCC-algebra is a

BCK-algebra. But there are positive implicative BCC-algebras which are not BCK-algebras. Similarly BCC-algebras with condition (J). Moreover, in some BCC-algebras holds also (9). Such BCC-algebras are called *special* (cf. [7]) and have many interesting properties.

PROBLEM 12 Describe the class of special BCC-algebras.

BCC-algebras are a generalization of BCK-algebras. So-called *weak BCC-algebras* (described in [8]), are a common generalization of BCC- and BCI-algebras. The class of all weak BCC-algebras is a quasivariety defined by the independent axioms system: (5), (7) and (15). Any weak BCC-algebra $\mathcal{G}$ satisfies also (3), but (2) or (8) are satisfied only in the case when $\mathcal{G}$ is a BCI-algebra. A weak BCC-algebra satisfying (4) is a BCC-algebra. A weak BCC-algebra which is not either a BCC-algebra or a BCI-algebra is called *proper*. The smallest proper weak BCC-algebra has four elements. There are two such non-isomorphic algebras.

The general theory of weak BCC-algebras is similar to the theory of BCI-algebras, but the problem of characterizations of congruences by well-defined ideals is open.

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