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ON THE SPECTRAL GEOMETRY FOR THE JACOBI OPERATORS OF HARMONIC MAPS INTO KENMOTSU MANIFOLDS

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ABSTRACT When the target manifold is certain Kenmotsu manifolds, we characterize invariant immersions, tangential anti-invariant immersions and normal anti-invariant immersions by the spectra of the Jacobi operator

1. Introduction

The spectral geometry for the second order operators arising in Riemannian geometry has been studied by many authors. Among them, the spectral geometry for the Jacobi operator of the energy of a harmonic map was studied in [4,7] (for manifolds) and [6] (for Riemannian foliations), and for the Jacobi operator of the area functional was studied in [1,3]. The Jacobi operator of a harmonic map arises in the second variation formula of the energy of a harmonic map This formula can be expressed in terms of an elliptic differential operator (called the *Jacobi operator*) defined on the space of cross sections of the induced bundle of the target manifold.

In this paper, we shall study the spectral goemetry for the Jacobi operator of a harmonic map when the target manifold is Kenmotsu manifolds.

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2. Preliminaries

Let (M, g) be an m-dimensional closed (i.e., compact without boundary) Riemannian manifold with the metric g and (N, h) be an n-dimensional Riemannian manifold with the metric h.

A smooth map $f : (M, g) \longrightarrow (N, h)$ is said to be *harmonic* if it is a critical point of the energy functional E, which is defined by $E(f) := \int_M e(f) dv_g$, where the *energy density* $e(f)$ of f is defined to be $e(f) := \frac{1}{2}\sum_{i=1}^m h(f_* e_i, f_* e_i)$ (f_* is the differential of f, $\{e_1 \cdots e_m\}$ a local orthonomal frame field on M, and dv_g the volume element with respect to g).

Let us consider the Jacobi operator J_f for a harmonic map f defined by $J_f V = \tilde{\Delta}_f V - R_f V$ for $V \in \Gamma(E)$ (the space of smooth sections of the induced bundle $f^*TN =: E$ of the tangent bundle TN), where $\tilde{\Delta}$ is the rough Laplacian associated to the induced connection $\tilde{\nabla}$ of E defined by $\tilde{\nabla}_X V := \nabla^h_{f_* X} V$ (for any vector field X on M, ∇^h the Levi-Civita connection of the metric h), and $R_f V := \sum_{i=1}^m R_h(V, f_* e_i) f_* e_i$ (R_h is the Riemannian curvature tensor of (N, h)). In this paper, we take the convention $R_h(\tilde{X}, \tilde{Y}) := [\nabla^h_{\tilde{X}}, \nabla^h_{\tilde{Y}}] - \nabla^h_{[\tilde{X}, \tilde{Y}]}$, where $\tilde{X}, \tilde{Y}$ are vector fields on N. Then J_f is self-adjoint, elliptic of second order and has a discrete spectrum as a conseqence of the compactness of M.

Consider the semigroup e^{-tJ_f} given by

$$e^{-tJ_f} V(x) = \int_M K(t, x, y, J_f) V(y) dv_g(y),$$

where $K(t, x, y, J_f) \in Hom(E_y, E_x)$ is the kernel function ($x, y \in M$, E_x is the fibre of E over x). Then we have asymptotic expansions for the L^2-trace

$$(2.1) \qquad Tr(e^{-tJ_f}) = \sum_{i=1}^{\infty} e^{-t\lambda_i} \sim (4\pi t)^{-\frac{m}{2}} \sum_{n=0}^{\infty} t^n a_n(J_f) \quad (t \downarrow 0^+),$$

where each $a_n(J_f)$ is the spectral invariant of J_f, which depends only on the discrete spectrum ;

$$Spec(J_f) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \cdots \uparrow +\infty\}.$$

Applying the Jacobi operator J_f of a harmonic map f to Gilkey's results([2]), H.Urakawa([7]) obtained

$$a_0(J_f) = n \cdot Vol(M,g), \tag{2.2}$$

$$a_1(J_f) = \frac{n}{6}\int_M \tau_g dv_g + \int_M Tr(R_f)dv_g, \tag{2.3}$$

$$\begin{aligned} a_2(J_f) = {} & \frac{n}{360}\int_M (5\tau_g^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2)dv_g \\ & + \frac{1}{360}\int_M [-30\|R^{\tilde{\nabla}}\|^2 + 60\tau_g Tr(R_f) \\ & + 180Tr({R_f}^2)]dv_g, \end{aligned} \tag{2.4}$$

where $R^{\tilde{\nabla}}$ is the curvature tensor of the connection $\tilde{\nabla}$ on E, which is defined by $R^{\tilde{\nabla}} := f^*R_h$, and R_g, ρ_g, τ_g are the curvature tensor, Ricci tensor, scalar curvature on M, respectively

3. The caculation of spectral invariants

Let (ϕ, ξ, η, h) be the almost contact metric structure([5,9]) of the almost contact Riemannian manifold N. This means that

$$\begin{aligned} & \phi^2 = -I + \xi \otimes \eta, && \phi(\xi) = 0 = \eta \circ \phi, \\ & \eta(\xi) = 1, && h(\phi\tilde{X}, \tilde{Y}) = -h(\tilde{X}, \phi\tilde{Y}), \\ & \eta(\tilde{X}) = h(\tilde{X}, \xi), && \end{aligned} \tag{3.1}$$

where ϕ is a tensor field of type (1,1), ξ a vector field, η a 1-form, I the identity transformation, h a Riemannian metric and $\tilde{X}$, $\tilde{Y}$ vector fields on N. Define a 2-form Φ on N by $\Phi(\tilde{X}, \tilde{Y}) := h(\tilde{X}, \phi\tilde{Y})$ for any vector fields $\tilde{X}$, $\tilde{Y}$ on N.

In an almost contact Riemannian manifold N, if the Ricci tensor ρ_h satisfies $\rho_h = ah + b\eta \otimes \eta$, where a and b are smooth fuctions on N, then it is called an *η-Einstein manifold.*

PROPOSITION 1 *Let f, f' be harmonic maps of compact Riemannian manifold (M,g) into an η-Einstein manifold (N, ϕ, ξ, η, h) whose*

Ricci tensor ρ_h is of the form ; $\rho_h = ah + b\eta \otimes \eta$ with $a(\neq 0)$ and b are some smooth functions on N. If $Spec(J_f) = Spec(J_{f'})$ and the structure vector field ξ is normal to both $f(M)$ and $f'(M)$, then $E(f) = E(f')$.

PROOF. From (2.3), we get

$$2aE(f) + b\int_M \|f^*\eta\|^2 dv_g = 2aE(f') + b\int_M \|f'^*\eta\|^2 dv_g.$$

But $\|f^*\eta\|^2 = 0 = \|f'^*\eta\|^2$ because of the normality of the structure vector field ξ, which completes the proof.

K.Kenmotsu([5]) studied a class of almost contact Riemannian manifolds which satisfies the following conditions ;

$$\begin{aligned}
&(\nabla^h_{\tilde{X}}\phi)\tilde{Y} = -\eta(\tilde{Y})\phi\tilde{X} - h(\tilde{X}, \phi Y)\xi,\\
&\nabla^h_{\tilde{X}}\xi = \tilde{X} - \eta(\tilde{X})\xi
\end{aligned}$$

for any vector fields $\tilde{X}, \tilde{Y}$ on N. One call such manifolds *Kenmotsu manifolds*. It has been shown([5]) that a Kenmotsu manifold (N, ϕ, ξ, η, h) has constant ϕ-sectional curvature k if and only if

(3.2)

$$\begin{aligned}
h(R(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W}) = \alpha\{&h(\tilde{Y},\tilde{Z})h(\tilde{X},\tilde{W}) - h(\tilde{X},\tilde{Z})h(\tilde{Y},\tilde{W})\}\\
&+ \beta\{\eta(\tilde{X})\eta(\tilde{Z})h(\tilde{Y},\tilde{W}) + \eta(\tilde{Y})\eta(\tilde{W})h(\tilde{X},\tilde{Z})\\
&- \eta(\tilde{X})\eta(\tilde{W})h(\tilde{Z},\tilde{Y}) - \eta(\tilde{Z})\eta(\tilde{Y})h(\tilde{X},\tilde{W})\\
&+ \Phi(\tilde{X},\tilde{Z})\Phi(\tilde{W},\tilde{Y}) - \Phi(\tilde{X},\tilde{W})\Phi(\tilde{Z},\tilde{Y})\\
&- 2\Phi(\tilde{X},\tilde{Y})\Phi(\tilde{Z},\tilde{W})\}
\end{aligned}$$

for any vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}$ on N, where $\alpha = \frac{k-3}{4}, \beta = \alpha + 1 = \frac{k+1}{4}$.

It is known([5]) that a Kenmotsu manifold with constant ϕ-sectional curvature k is nothing but the hyperbolic space with constant curvature $-1 = k$.

Now consider another example i.e., a warped product space $N = L \times_f F$, where F denotes a Kaehlerian manifold and $f(t) = ce^t$ a warping function on a real line L(c is a nonzero constant). Then N admits an almost constact metric structure. In particular, when F is the complex projective space with constant holomorphic sectional curvature 4, the curvature tensor of $N = L \times_f F$ is obtained by putting

$$\alpha = \frac{1}{f^4} - 1, \quad \beta = \alpha + 1 = \frac{1}{f^4} \quad \text{in} \quad (3.2).$$

This is an example of Kenmotsu manifolds, not hyperbolic spaces, except for $\frac{1}{f^4} \neq 0$([5]).

In this context, throughout this paper, $N(\alpha, \beta)$ will denote a $(2n+1)$-dimensional Kenmotsu manifold whose curvature tensor is the form (3.1) with smooth functions α and $\beta = \alpha + 1$ on N. Obviously, $N(\alpha, \beta)$ is an η-Einstein manifold.

For a harmonic map $f : (M, g) \longrightarrow N(\alpha, \beta)$, we obtain from (3.1) and (3.2)

$$\begin{aligned} Tr(R_f) &= \sum_{i=1}^{m} \sum_{a=1}^{2n+1} h(R_h(v_a, f_*e_i)f_*e_i, v_a) \\ &= 4(\alpha n + \beta)e(f) - 2\beta(n+1)\|f^*\eta\|^2, \end{aligned} \tag{3.3}$$

(3.4)

$$\begin{aligned} Tr(R_f{}^2) = &\sum_{i,j=1}^{m} \sum_{a=1}^{2n+1} h(R_h(v_a, f_*e_i)f_*e_i, R_h(v_a, f_*e_j)f_*e_j) \\ = &\{(2n-1)\alpha^2 + 4\alpha\beta + \beta^2\}(tr f^*h)^2 \\ &+ (\alpha^2 + 9\beta^2)\|f^*h\|^2 - 6\alpha\beta\|f^*\Phi\|^2 \\ &+ (-4\alpha\beta - 16\beta^2) \sum_{i,j=1}^{m} \eta(f_*e_i)\eta(f_*e_j)h((f_*e_i, f_*e_j) \\ &+ 2(n+7)\beta^2\|f^*\eta\|^4, \end{aligned}$$

(3.5)

$$\|R^{\tilde{\nabla}}\|^2 = \sum_{i,j=1}^{m} \sum_{a,b=1}^{2n+1} h(R_h(f_*e_i, f_*e_j)v_a, v_b)h(R_h(f_*e_i, f_*e_j)v_a, v_b)$$
$$= -2(\alpha^2+\beta^2)\|f^*h\|^2 + 8\alpha\beta \sum_{i,j=1}^{m} \eta(f_*e_i)\eta(f_*e_j)h(f_*e_i, f_*e_j)$$
$$+ 2(\alpha^2+\beta^2)(trf^*h)^2 - 8\alpha\beta(trf^*h)\|f^*\eta\|^2$$
$$+ \{12\alpha\beta + 8\beta^2(n+1)\}\|f^*\Phi\|^2,$$

where $m = \dim M$, $\|f^*\eta\|^2 := \sum_{i=1}^m \eta(f_*e_i)\eta(f_*e_i)$, $\|f^*\Phi\|^2 := \sum_{i,j=1}^m h(f_*e_i, \phi f_*e_j)^2$, $\|f^*h\|^2 := \sum_{i,j=1}^m h(f_*e_i, f_*e_j)^2$, $\{e_i : i = 1, \cdots, m\}$ is a local orthonormal frame field on M, and $\{v_a : a = 1, \cdots, 2n+1\}$ is a local orthonormal frame field on $N(\alpha, \beta)$.

Thus substituting (3.3) $\sim$ (3.5) into (2.2) $\sim$ (2.4), we get

PROPOSITION 2 *For a harmonic map $f : (M, g) \longrightarrow N(\alpha, \beta)$ of an m-dimensional compact Riemannian manifold (M, g) into a $(2n+1)$-dimensional Kenmotsu manifold $N(\alpha, \beta)$. Then the coefficients $a_0(J_f)$, $a_1(J_f)$ and $a_2(J_f)$ of the asymptotic expansion for the Jacobi operator J_f are respectively given by*

$$a_0(J_f) = (2n+1)Vol(M, g), \tag{3.6}$$

(3.7)

$$a_1(J_f) = \frac{(2n+1)}{6} \int_M \tau_g dv_g - 2\beta(n+1) \int_M \|f^*\eta\|^2 dv_g$$
$$+ 4(\alpha n + \beta)E(f),$$

(3.8)

$$
\begin{aligned}
a_2(J_f) = \frac{2n+1}{360} \int_M & [5{\tau_g}^2 - 2\|\rho_g\|^2 + 2\|R_g\|^2] dv_g \\
& + \frac{1}{12} \int_M [8(\alpha^2 + 7\beta^2)\|f^*h\|^2 - 32(\alpha\beta + 3\beta^2) \sum_{i,j=1}^{m} \eta(f_*e_i)\eta(f_*e_j) \\
& \times h(f_*e_i, f_*e_j) + 16\{(3n-2)\alpha^2 + 6\alpha\beta + \beta^2\}e(f)^2 \\
& + 16\alpha\beta\|f^*\eta\|^2 e(f) - 8\{6\alpha\beta + \beta^2(n+1)\}\|f^*\Phi\|^2 \\
& + 12(n+7)\beta^2\|f^*\eta\|^4] dv_g + \frac{2}{3} \int_M (\alpha n + \beta)\tau_g e(f) dv_g \\
& - \frac{1}{3} \int_M \beta(n+1)\|f^*\eta\|^2 \tau_g dv_g .
\end{aligned}
$$

4. Isometric minimal immersions

Let (N, h) be a $(2n+1)$-dimensional Kenmotsu manifold and $f : (M, g) \longrightarrow (N, h)$ be an isometric immersion of a Riemannian manifold (M, g) into (N, h). f is called an *invariant immersion* if $\phi(f_*TM) \subset f_*TM$ and ξ is tangent to $f(M)$ everywhere on M If f is an invariant immersion, then the immersion f is minimal. f is called an *tangential(normal* resp.)*anti-invariant immersion* if $\phi(f_*TM) \perp f_*TM$ and ξ is tangent(normal resp) to $f(M)$ everywhere on M.

PROPOSITION 3 *Let f and f' be isometric minimal immersions of compact Riemannian manifolds (M, g) and (M', g') into an η-Einstein manifold, respectively. Assume that $Spec(J_f) = Spec(J_{f'})$ and the structure vector field ξ is normal (or tangent) to both $f(M)$ and $f'(M')$ Then we have*

(i) $$dim(M) = dim(M'),$$

(ii) $$Vol(M, g) = Vol(M', g'),$$

(iii) $$\int_M \tau_g dv_g = \int_{M'} \tau_{g'} dv_{g'} .$$

PROOF (i) follows from the asymptotic expansion (2.1), (ii) $\sim$ (iii) from (2.2) and (2.3).

PROPOSITION 4 *Let f, f' be invariant immersions of compact Riemannian manifolds (M, g) and (M', g') into a Kenmotsu manifold $N(\alpha, \beta)$. Assume that $Spec(J_f) = Spec(J_{f'})$. If f is a totally geodesic immersion, then so is f'.*

PROOF Since the invariant immersion f is minimal, using the structure equation of Gauss and (3.2), we see that the scalar curvature is given by

$$\tau_g = (m-1)(m\alpha + \beta) + \|B\|^2,$$

where $\|B\|^2$ denotes the square of the norm of the second fundamental form B of the immersion f. Hence from (iii) of Proposition 3, we get

$$\int_M \|B\|^2 \, dv_g = \int_{M'} \|B'\|^2 \, dv_{g'},$$

where B' denotes the second fundamental form of the immersion f', which gives the proof.

PROPOSITION 5 *Let f, f' be tangential or normal anti-invariant, minimal immersions of compact Riemannian manifolds (M, g), (M', g') into a Kenmotsu manifold $N(\alpha, \beta)$. Assume that $Spec(J_f) = Spec(J_{f'})$. If f is a totally geodesic immersion, then so is f'.*

PROOF. If f is minimal, tangential(resp. normal) anti-invariant, then the scalar curvature is given by $\tau_g = (m-1)(m\alpha - 2\beta) + \|B\|^2$ (resp. $\tau_g = m(m-1)\alpha + \|B\|^2$). Then one can argue as in the preceding proof.

LEMMA 6 *Let f, f' be isometric minimal immersions of compact Riemannian manifolds (M, g) into a Kenmotsu manifold $N(\alpha, \beta)(\beta \neq 0)$. Assume that $Spec(J_f) = Spec(J_{f'})$. Then ξ is tangent(resp. normal) to $f(M)$ if and only if ξ is tangent(resp. normal) to $f'(M)$.*

PROOF. Since f and f' are isometric immersions, $e(f) = \frac{1}{2}\dim(M) = e(f')$. It is clear from (3.6) and (3.7) that

$$\int_M \|f^*\eta\|^2 dv_g = \int_M \|f'^*\eta\|^2 dv_g. \tag{4.1}$$

First, if ξ is tangent to $f(M)$, then the equation (4.1) implies that

$$\int_M dv_g = \int_M \|f'^*\eta\|^2 dv_g$$

since $\|f^*\eta\|^2 = 1$. From this equation, we get $1 = \|f'^*\eta\|^2$.

Now we put $\xi = \xi^t + \xi^n$, where ξ^t(resp. ξ^n) denotes the tangential(resp. normal) component of ξ with respect to $f'(M)$ and the metric h. Then the above equation gives

$$h(\xi,\xi) = 1 = \sum_{i=1}^{m} \eta(f'_*e_i)^2 = \sum_{i=1}^{m} h(f'_*e_i,\xi^t)^2 = h(\xi^t,\xi^t),$$

which implies that $\xi^n = 0$, i.e., ξ is tangent to $f'(M)$, where $\{e_i\ ;\ i = 1,\cdots,m\}$ is a local frame field on M. The converse is similar.

Next, if ξ is normal to $f(M)$, then the left hand side of the equation (4.1) vanishes. Hence $\|f'^*\eta\|^2 = 0$ if and only if $\sum_{i=1}^{m} f'^*\eta(e_i)^2 = \sum_{i=1}^{m} h(f'_*e_i,\xi)^2 = 0$. This implies that ξ is normal to $f'(M)$.

LEMMA 7 *Let (N,ϕ,ξ,η,h) be a $(2n+1)$-dimensional almost contact metric manifold. Let f be an isometric immersion of an m-dimensional compact Riemannian manifold (M,g) into (N,h). Then we have the inequality*

$$0 \le \int_M \|f^*\Phi\|^2 dv_g \le \lambda\, Vol(M,g)$$

where λ denotes $m-1$ or m according as ξ is tangent or normal to $f(M)$, respectively.

Moreover,

(i) the equality

$$\int_M \|f^*\Phi\|^2 \, dv_g = (m-1)\, Vol(M,g)$$

holds if and only if f is an invariant immersion, and
(ii) the equality

$$0 = \int_M \|f^*\Phi\|^2$$

holds if and only if f is an anti-invariant immersion.

PROOF. First, to prove the inequality, we only have to show

$$0 \leq \|f^*\Phi\|^2 \leq \lambda,$$

at each point of M. Take an orthonormal basis $\{e_i \,;\, i = 1, \cdots, m\}$ of the tangent space T_xM at $x \in M$. Then we get

$$\begin{aligned} \|f^*\Phi\|^2 &= \sum_{i,j=1}^{m} h(\phi f_* e_i, f_* e_j) h(\phi f_* e_i, f_* e_j) \\ &= \sum_{i,j=1}^{m} h(P\phi f_* e_i, f_* e_j) h(P\phi f_* e_i, f_* e_j) \\ &= \sum_{i=1}^{m} h(P\phi f_* e_i, P\phi f_* e_i), \end{aligned}$$

where P is the orthogonal projection of $T_{f(x)}N$ onto f_*T_xM with respect to the metric h. Hence we obtain

$$0 \leq \|f^*\Phi\|^2 \leq \sum_{i=1}^{m} h(\phi f_* e_i, \phi f_* e_i).$$

Since $\{f_* e_j \,;\, j = 1, \cdots, m\}$ is an orthonormal basis of f_*T_xM, we get

$$\begin{aligned} \sum_{i=1}^{m} h(\phi f_* e_i, \phi f_* e_i) &= \sum_{i=1}^{m} \{h(f_* e_i, f_* e_i) - \eta(f_* e_i)\eta(f_* e_i)\} \\ &= m - \sum_{i=1}^{m} h(f_* e_i, \xi) h(f_* e_i, \xi) = \lambda. \end{aligned}$$

Next, if $\int_M \|f^*\Phi\|^2 dv_g = (m-1)\mathrm{Vol}(M,g)$, we have

$$\begin{aligned}\|f^*\Phi\|^2 = m-1 &\iff P\phi f_* e_i = \phi f_* e_i,\ i = 1, \cdots, m\\ &\iff \phi f_* T_x M \subset f_* T_x M, \text{ at each point } x \in M,\\ &\iff \text{the immersion } f \text{ is invariant.}\end{aligned}$$

If $0 = \int_M \|f^*\Phi\|^2$, we have

$$\begin{aligned}\|f^*\Phi\|^2 = 0 &\iff P\phi f_* e_i = 0, i = 1, \cdots, m\\ &\iff h(f_*X, \phi f_*Y) = 0 \text{ for any vector fields } X, Y \text{ on } M\\ &\iff f \text{ is anti-invariant.}\end{aligned}$$

THEOREM 8 *Let f, f' be isometric minimal immersions of a compact Riemannian manifold (M,g) into a Kenmotsu manifold $N(\alpha,\beta)$ with $\beta \neq 0$. Assume that $Spec(J_f) = Spec(J_{f'})$. Then*

(a) *if f is an invariant immersion, then so is f',*

(b) *if f is a tangential anti-invariant immersion, then so is f',*

(c) *if f is a normal anti-invariant immersion, then so is f'.*

PROOF. Since f, f' are isometric immersions, we have

$$e(f) = e(f') = \frac{1}{2}m, \quad \|f^*h\|^2 = \|f'^*h\|^2 = m.$$

Moreover, if ξ is tangent or normal to $f(M)$, then

$$\begin{aligned}\sum_{i,j=1}^{m} \eta(f_*e_i)\eta(f_*e_j)h(f_*e_i, f_*e_j) &= \|f^*\eta\|^2 = \mu\\ = \|f'^*\eta\|^2 = \sum_{i,j=1}^{m} &\eta(f'_*e_i)\eta(f'_*e_j)h(f'_*e_i, f'_*e_j)\end{aligned}$$

because of Lemma 6, where $\mu = 1$ or 0 according as ξ is tangent or normal to $\tilde{f}(M)$, respectively, and $\{e_i : i = 1, \cdots, m\}$ is a local orthonormal frame field on M. Thus (3.8) implies that

$$\int_M \|f^*\Phi\|^2 dv_g = \int_M \|f'^*\Phi\|^2 dv_g. \tag{4.2}$$

Then (a) follows from (i) in Lemma 7, (4.2) and Lemma 6. The remained parts (b) and (c) also follow from (ii) in Lemma 7, (4.2) and Lemma 6. Hence we complete the proof.

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