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AN EQUIVALENT FORMULATION TO AN ERDÖS' PROBLEM ON A SET HAVING DISTINCT SUBSET SUMS

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ABSTRACT We give an equivalent formulation to the Erdös' conjecture on the lower bound of the greatest element in a set having distinct subset sums On this basis we suggest a possible approach towards Erdos' conjecture Also we reproduce L. Moser's result by means of an analytic method

1. Introduction

A set having distinct subset sums is a set of numbers such that no two finite subsets have the same sum. To be precise, we give a formal definition.

DEFINITION 1 1

(i) Let A be a set of real numbers. We say that A *has the subset-sum-distinct property* (briefly *SSD-property*) if for any two finite subsets X, Y of A,

$$\sum_{x\in X} x = \sum_{y\in Y} y \qquad \Longrightarrow \qquad X = Y.$$

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Also, we say that A is *SSD* or A is an *SSD-set* if it has the SSD-property.

(ii) An increasing sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ is called a *subset-sum-distinct sequence* (or briefly, an *SSD-sequence*) if it has the SSD-property.

One of the most interesting and natural SSD-sequences is
$t = \{1, 2, 2^2, 2^3, \cdots\}$.

Now, for a given SSD-sequence $\{a_n\}_{n=1}^{\infty}$ how one can compare the size of this sequence with t ? The following basic lemma will give some insight.

LEMMA 1 2 *Let $\{a_n\}_{n=1}^{\infty}$ be an SSD-sequence. Then*

$$a_1 + a_2 + \cdots + a_n \geq 2^n - 1$$

for every $n \geq 1$.

PROOF Let

$$A = \{a_1, a_2, \cdots, a_n\} \text{ and } J = \{\sum_{b \in B} b \ : \ \phi \neq B \subset A\}.$$

Note that all the elements of J are positive integers. Since A has the SSD-property,

$$B, B' \subset A \text{ and } B \neq B' \quad \Longrightarrow \quad \sum_{b \in B} b \neq \sum_{b' \in B'} b'.$$

Hence $|J| = 2^n - 1$. Because $a_1 + a_2 + \cdots + a_n$ is the greatest element in J, we have

$$a_1 + a_2 + \cdots + a_n \geq 2^n - 1.$$

As $\{1, 2, 2^2, 2^3, \cdots\}$ suggests, SSD-sequences are quite sparce. It seems very natural to ask how dense they can be. We will consider a question of this flavor in this paper.

As a way of obtaining finite "dense" SSD-sets, one can use the Conway-Guy sequence (see [10], [11]). Here we explain the construction of the Conway-Guy sequence. First, define an auxiliary sequence u_n by

$$u_0 = 0, \quad u_1 = 1 \qquad \text{and} \qquad u_{n+1} = 2u_n - u_{n-r}, \quad n \geq 1,$$

where $r =< \sqrt{2n} >$, the nearest integer to $\sqrt{2n}$. Now, for a given positive integer n we define

$$a_i = u_n - u_{n-i}, \qquad 1 \leq i \leq n.$$

The well-known Conway-Guy conjecture is that $\{a_i : 1 \leq i \leq n\}$ was SSD for any positive integer n. This was resolved affirmatively in 1996 by T. Bohman (see[2]).

2. An equivalent formulation and lower bounds

Let $a = \{a_n\}_{n=1}^{\infty}$ be an SSD-sequence. It is an old problem to find a lower bound for a_n (see [1, pp.47-48], [3], [5], [6, p.467], [7, pp.59-60], [10] and [13]). This problem can be stated inversely: Under the condition $a_n \leq x$, find an upper bound on n. N. Elkies mentioned the inter-relation between a lower bound on a_n and an upper bound on n in terms of x (see [4]). A lower bound of the form

$$a_n \geq Cn^{-s}2^n(1 + o(1)) \tag{2.1}$$

corresponds to an upper bound of the form

$$n \leq \log_2 x + s \log_2 \log_2 x + \log_2 \frac{1}{C} + o(1). \tag{2.2}$$

The famous conjecture of Erdös is that (3.1) and (3.2) hold with $s = 0$ (see [9, p.64, problem C8]). In this context, we prove

THEOREM 2.1. *Erdös' conjecture is equivalent to*

$$\sum_{i=1}^{m} {a_i}^s \geq C^s \sum_{i=1}^{m} 2^{(i-1)s} = C^s \frac{1 - 2^{ms}}{1 - 2^s} \tag{2.3}$$

for all positive integers m, for all positive s, and for some positive constant C.

PROOF. It is clear that $a_m \geq C2^{m-1}$ for all m implies (2.3). In the other direction, (2.3) implies $ma_m{}^s \geq C^s 2^{(m-1)s}$ or $a_m \geq m^{-(1/s)}C2^{m-1}$. Since s can be as large as desired, we have $a_m \geq C2^{m-1}$ for all m.

Lemma 1.2 shows that (2.3) is true for $s = C = 1$. Moreover, L. Moser proved (2.3) holded for $s = 2$, $C = 1$ (see [10]). In his paper [4], Elkies also showed that

$$a_n \geq 2^{-n}\binom{2n}{n} \sim \frac{1}{\sqrt{\pi}} n^{-\frac{1}{2}} 2^n$$

by an analytic method. But we point out that from L. Moser's result we can derive the better bound, $a_n > \frac{1}{\sqrt{3}} n^{-\frac{1}{2}} 2^n$ for $n \geq 2$, very easily:

$$(*) \qquad \begin{aligned} &\sum_{i=1}^{n} a_i{}^2 \geq \sum_{i=1}^{n} 2^{(i-1)2} = \frac{1}{3}(4^n - 1) \\ &\Longrightarrow \quad na_n{}^2 > \frac{1}{3}4^n \quad \Longrightarrow \quad a_n > \frac{1}{\sqrt{3}} n^{-\frac{1}{2}} 2^n . \end{aligned}$$

In this way, it is very important to estimate an effective lower bound on $\sum_{i=1}^{m} a_i{}^s$ for large s. In Theorem 2.4 below we give a new proof of Moser's result $(*)$ using an analytic method. Though Moser's proof is simple enough, we give another because it might give some further insight.

First, we need the following two lemmas. The first one is quite simple; the second is known as Laplace's method for estimating integrals.

LEMMA 2.2 *Let $0 < a < b$ be fixed. Then for any $y > 0$, we have*

$$y^a + y^{-a} \leq y^b + y^{-b}.$$

PROOF Note that

$$\begin{aligned} 0 < y < 1 \quad &\Longrightarrow \quad y^b - y^a < 0 \quad \text{and} \quad y^a y^b - 1 < 0, \\ y \geq 1 \quad &\Longrightarrow \quad y^b - y^a \geq 0 \quad \text{and} \quad y^a y^b - 1 \geq 0. \end{aligned}$$

Hence, for any $y > 0$, $(y^b - y^a)(y^a y^b - 1) \geq 0$. This implies that $y^a y^{2b} + y^a \geq y^{2a} y^b + y^b$ and the result follows upon dividing by y^{a+b}.

LEMMA 2.3. *(Laplace's method) Assume that two real valued functions $\varphi(x)$ and $f(x)$, defined on $(-\infty, \infty)$, satisfy the following four conditions:*

(i) *$\varphi(x)(f(x))^N$ is absolutely integrable on $(-\infty, \infty)$, $N = 0, 1, 2, \cdots$.*

(ii) *$f(x) \geq 0$ for all x and $f(x)$ attains its maximum at $x = \xi$. Furthermore*

$$\sup\{f(x) \ : \ x \in C\} \quad < \quad f(\xi)$$

for any closed subset C of $(-\infty, \infty)$ not containing ξ.

(iii) *$f''(x)$ exists and is continuous on $(-\infty, \infty)$ and $f''(\xi) < 0$.*

(iv) *$\varphi(x)$ is continuous at $x = \xi$, and $\varphi(\xi) \neq 0$.*

Then

$$\int_{-\infty}^{\infty} \varphi(x)(f(x))^N dx \quad \sim \quad \varphi(\xi)(f(\xi))^{N+\frac{1}{2}} \sqrt{-\frac{2\pi}{N f''(\xi)}}$$

as $N \longrightarrow \infty$.

PROOF See [12, Vol. I, Part II, Problem 201].

THEOREM 2.4

$$\sum_{i=1}^{n} {a_i}^2 \ \geq \ \frac{1}{3}(4^n - 1).$$

PROOF Let

$$A = \left\{ \sum_{j=1}^{n} \epsilon_j a_j \ : \ \epsilon_j = +1 \ \text{ or } \ -1 \right\}.$$

Clearly every element in A has the same parity and $a \in A$ implies $-a \in A$. Since $\{a_j\}_{j=1}^{\infty}$ is an SSD-sequence, we also have $0 \notin A$. Note that no integers can be expressed in more than one way in the form $\sum_{j=1}^{n} \epsilon_j a_j$, where $\epsilon_j = \pm 1$. For

$$\sum_{j=1}^{n} \epsilon'_j a_j = \sum_{j=1}^{n} \epsilon''_j a_j \, , \quad \epsilon'_j = \pm 1 \, , \quad \epsilon''_j = \pm 1$$

is equivalent to

$$2 \sum_{\epsilon_j' = -\epsilon_j'' = 1} a_j = 2 \sum_{\epsilon_j' = -\epsilon_j'' = -1} a_j .$$

Hence $|A| = 2^n$. Now, by Lemma 2.2, we have

$$\sum_{a \in A} y^a \geq \sum_{j=1}^{2^{n-1}} (y^{2j-1} + y^{-(2j-1)})$$

for all $y > 0$. Let $y = e^x$. Then

$$\begin{aligned}\prod_{j=1}^{n} \left(e^{a_j x} + e^{-a_j x}\right) &= \sum_{a \in A} e^{a x} \\ &\geq \sum_{j=1}^{2^{n-1}} \left((e^x)^{2j-1} + (e^x)^{-(2j-1)}\right) = \frac{\sinh (2^n x)}{\sinh x}.\end{aligned}$$

Divide by 2^n, take reciprocals, and raise both sides to the power $2m$ to obtain

$$\left(\prod_{j=1}^{n} \frac{1}{\cosh (a_j x)}\right)^{2m} \leq 4^{mn} \left(\frac{\sinh x}{\sinh (2^n x)}\right)^{2m} .$$

Then integration from $-\infty$ to ∞ yields

$$\text{(2.4)} \quad \int_{-\infty}^{\infty} \left(\prod_{j=1}^{n} \frac{1}{\cosh (a_j x)}\right)^{2m} dx \leq 4^{mn} \int_{-\infty}^{\infty} \left(\frac{\sinh x}{\sinh (2^n x)}\right)^{2m} dx .$$

To estimate the integral on the right side of (2.4), apply Lemma 2.3 with $N = 2m$, $\xi = 0$,

$$\varphi(x) = 1 \quad \text{and} \quad f(x) = \frac{\sinh x}{\sinh (2^n x)} .$$

Then, since $f(0) = 2^{-n}$ and $f''(0) = \frac{1}{3}(2^{-n} - 2^n)$,

$$\int_{-\infty}^{\infty} \left(\frac{\sinh x}{\sinh (2^n x)} \right)^{2m} dx \sim (2^{-n})^{2m+\frac{1}{2}} \sqrt{\frac{3 \cdot 2\pi}{2m\,(2^n - 2^{-n})}}$$

as $m \longrightarrow \infty$. Thus

$$4^{mn} \int_{-\infty}^{\infty} \left(\frac{\sinh x}{\sinh (2^n x)} \right)^{2m} dx \sim 2^{-n} \sqrt{\frac{3\pi}{m\,(1 - 2^{-2n})}} \tag{2.5}$$

as $m \longrightarrow \infty$. Now, in order to estimate the integral on the left side of (2.4), apply Lemma 2.3 with $N = 2m, \xi = 0$,

$$\varphi(x) = 1 \quad \text{and} \quad f(x) = \prod_{j=1}^{n} \frac{1}{\cosh (a_j x)} \, .$$

Note that here $f(0) = 1$ and $f''(0) = -\sum_{j=1}^{n} a_j{}^2$. Hence

$$\int_{-\infty}^{\infty} \left(\prod_{j=1}^{n} \frac{1}{\cosh (a_j x)} \right)^{2m} dx \sim \sqrt{\frac{\pi}{m \sum_{j=1}^{n} a_j{}^2}} \tag{2.6}$$

as $m \longrightarrow \infty$.
From (2.4), (2.5) and (2.6) it follows that

$$\sum_{i=1}^{n} a_i{}^2 \geq \frac{1}{3} (4^n - 1) \, .$$

Finally, we sketch how this method might be used to obtain more detailed information on the lower bound of a_n. The (admittedly still rough) idea is first to obtain detailed information on all power sums $\sum_{j=1}^{n} a_j{}^{2k}$. To do this, introduce the generating function

$$\prod_{j=1}^{n} (e^{\omega_1 a_j x} + e^{\omega_2 a_j x} + \cdots + e^{\omega_{2k} a_j x})$$

where $\omega_1, \omega_2, \cdots, \omega_{2k}$ are all the $2k$-th roots of unity, and use the following generalized Laplace method:

THEOREM 2.5 *(Generalized Laplace Method) Assume that a real valued function $f(x)$ defined on $(-\infty,\infty)$ satisfies the following:*
(i) $(f(x))^N$ *is integrable on* $(-\infty,\infty), N = 0,1,2,\cdots$.
(ii) $f(x) \geq 0$ *for all* x *and* $f(x)$ *attains its maximum at* $x = \xi$.
Furthermore

$$\sup\{f(x) \ : \ x \in C\} \ < \ f(\xi)$$

for any closed subset C *of* $(-\infty,\infty)$ *not containing* ξ.
(iii) $f^{(2m)}(x)$ *exists and is continuous on* $(-\infty,\infty)$.
(iv) $f^{(l)}(\xi) = 0$ *for* $l < 2m$ *and* $f^{(2m)}(\xi) < 0$.
Then

$$\int_{-\infty}^{\infty} (f(x))^N \, dx \ \sim \ (f(\xi))^{N+\frac{1}{2m}} \ \frac{\Gamma(1/(2m))}{m} \left(\frac{(2m)\,!}{-N\, f^{(2m)}(\xi)} \right)^{1/(2m)}$$

as $N \longrightarrow \infty$.

PROOF Imitate the proof of the Lemma 2.3 and use the fact (see [8, p.355, #3.326])

$$\int_{-\infty}^{\infty} e^{-x^{2m}} dx \ = \ \frac{\Gamma(1/(2m))}{m}.$$

Then take

$$f(x) \ = \ \prod_{j=1}^{n} \frac{2k}{\left(e^{\omega_1 a_j x} + e^{\omega_2 a_j x} + \cdots + e^{\omega_{2k} a_j x}\right)}.$$

We have $f(0) = 1, f^{(l)}(0) = 0$ for $l < 2k$ and

$$f^{(2k)}(0) \ = \ -\sum_{j=1}^{n} {a_j}^{2k}.$$

Upon applying Theorem 2.5, it follows that

$$\int_{-\infty}^{\infty} f(x)^N \, dx \ \sim \ \frac{\Gamma(1/(2k))}{k} \left(\frac{(2k)!}{N \sum_{j=1}^{n} {a_j}^{2k}} \right)^{1/2k}$$

as $N \longrightarrow \infty$.

Of course, to make this approach successful, one needs to find an appropriate upper bound for the function $f(x)$.

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