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A NOTE ON THE ESSENTIAL SPECTRUM OF AN IRREDUCIBLE P-HYPONORMAL OPERATOR

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Abstract In this paper, we have the extended result of Bunce's theorem And we show that if T is an irreducible p-hyponormal operator such that $T^*T - TT^*$is compact, then $\sigma_{ap}(T) = \sigma_e(T)$ and $\sigma_p(\phi(T)) = \sigma_e(\phi(T))$

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space. The $*$-algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$ and $K(\mathcal{H})$ is the ideal of all compact operators on $\mathcal{H}$ For an operator T, we denote the spectrum, the essential spectrum and the approximate point spectrum by $\sigma(T)$, $\sigma_e(T)$ and $\sigma_{ap}(T)$, respectively.

Let $U|T|$ be the polar decomposition of T, where U is partial isometry, $|T|$ is a positive square root of T^*T and $\ker |T| = \ker U$ An operator T is said to be a *p-hyponormal operator* if $(T^*T)^p - (TT^*)^p \geq 0$. If $p = 1$, T is called *hyponormal* and if $p = \frac{1}{2}$, T is called *semi-hyponormal.* It is well known that a p-hyponormal operator is q-hyponormal operator for $q \leq p$.

If $\mathcal{A}$ is a C^*-algebra with identity, $\Phi_{\mathcal{A}}$ is the set of nonzero homomorphism of $\mathcal{A}$ into $\mathbb{C}$, and M is the commutator ideal of $\mathcal{A}$ (that is, M is the norm closed ideal generated by the set of all elements of

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$\mathcal{A}$ of the form $ab - ba$), then $M = \cap\{h^{-1}(0) : h \in \Phi_{\mathcal{A}}\}$ and $\Phi_{\mathcal{A}}$ is the maximal ideal space of $\mathcal{A}/M$. With the above statement, we have $\mathcal{A}/M \cong C(\Phi_{\mathcal{A}/M})$ under the Gel'fand transform, $a + M \to \hat{a}$ where $\hat{a}(h) = h(a)$ for a in $\mathcal{A}$ and h in $\Phi_{\mathcal{A}/M}$([5], [6]).

In [2], one of the authors constructed an extension $\mathcal{K}$ of $\mathcal{H}$ by means of weakly convergent sequences in $\mathcal{H}$ and the Banach Limit and obtained the faithful $*$-representation ϕ of $\mathcal{B}(\mathcal{H})$ on $\mathcal{K}$.

PROPOSITION 1 1 ([2]) *There exists a faithful $*$-representation ϕ of $\mathcal{B}(\mathcal{H})$ on $\mathcal{K}$ with the following properties:*

(1) $\|\phi(T)\| = \|T\|$
(2) $\sigma(T) = \sigma(\phi(T))$
(3) $\sigma_{ap}(T) = \sigma_p(\phi(T))$
(4) *If T is a compact operator on $\mathcal{H}$, then $\phi(T)$ is a compact operator on $\mathcal{K}$*
(5) *If T is a Fredholm operator on $\mathcal{H}$, $\phi(T)$ is a Fredholm operator on $\mathcal{K}$.*

For $T \in \mathcal{B}(\mathcal{H})$, $C^*(T)$ is the C^*-algebra generated by a single operator T and identity.

PROPOSITION 1 2 ([3])

(1) *The C^*-algebra $C^*(T)$ is isometrically $*$-isomorphic to the C^*-algebra $C^*(\phi(T))$.*
(2) *If M is the maximal ideal of $C^*(T)$, then $\phi(M)$ is the maximal ideal of $C^*(\phi(T))$.*
(3) *Let $\Phi_{C^*(T)}$ and $\Phi_{C^*(\phi(T))}$ be the maximal ideal space of $C^*(T)$ and $C^*(\phi(T))$, respectively. then $\Phi_{C^*(T)}$ and $\Phi_{C^*(\phi(T))}$ are isometrically isomorphic.*

PROPOSITION 1 3 ([3])

(1) $M = \cap\{f^{-1}(0) : f \in \Phi_{C^*(T)}\} \cong N = \cap\{h^{-1}(0) : h \in \Phi_{C^*(\phi(T))}\}$.
(2) $C^*(T)/M \cong C^*(\phi(T))/N$.

An operator is said to be *reducible* if it has a nontrivial reducing subspace. If an operator is not reducible, then it is called *irreducible*.

PROPOSITION 1.4 ([6]) *If T is an irreducible operator such that $T^*T - TT^*$ is compact, then the commutator ideal M of $C^*(T)$ is $K(\mathcal{H})$*

PROPOSITION 1.5 ([3]) *If T is an irreducible operator,then $\phi(T)$ is an irreducible operator.*

In this paper, we will improve the Bunce's theorem for hyponormal operator to p-hyponormal operator. Also for an irreducible p-hyponormal operator T, we investigate the relationship between the point spectrum and the essential spectrum of $\phi(T)$, and obtain that $\sigma_p(\phi(T)) = \sigma_e(\phi(T))$.

2. Main results

A point $z \in \mathbb{C}$ is the *joint approximate point spectrum* $\sigma_{ja}(T)$ if there exists a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that $(T-z)x_n \to 0$ and $(T-z)^*x_n \to 0$. M. Cho and T. Huruya showed that $\sigma_{ap}(T) = \sigma_{ja}(T)$ for p-hyponormal operator T([4]).

Bunce proved the following proposition and corollary for hyponormal operator([1]) In this paper, we have the same results for p-hyponormal operator.

PROPOSITION 2.1 *Let $T = U|T|$ be p-hyponormal. Then $\lambda \in \sigma_{ap}(T)$ if and only if there is a $*$-homomorphism $\psi : C^*(T) \to \mathbb{C}$ such that $\psi(T) = \lambda$.*

PROOF Suppose $\psi : C^*(T) \to \mathbb{C}$ is a $*$-homomorphism such that $\psi(T) = \lambda$. If $\lambda \notin \sigma_{ap}(T)$, then there is a constant $c > 0$ such that $\|(T-\lambda)x\| \geq c\|x\|$ for all x in $\mathcal{H}$. This implies that $T^*T - \lambda T^* - \bar{\lambda}T + \bar{\lambda}\lambda - c^2$ is a positive operator. Hence $0 \leq \psi(T^*T - \lambda T^* - \bar{\lambda}T + \bar{\lambda}\lambda) - c^2 = -c^2$, a contradiction. Hence $\lambda \in \sigma_{ap}(T)$.

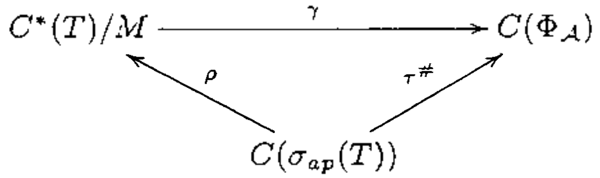
Conversely, suppose $\lambda \in \sigma_{ap}(T)$. Let $\{x_n\}$ be a sequence of unit vectors in $\mathcal{H}$ such that $\|(T-\lambda)x_n\| \to 0$. Let LIM denote a Banach limit and define $\psi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ by $\psi(B) = LIM < Bx_n, x_n >$. If $B \in \mathcal{B}(\mathcal{H})$ then $\|B(T-\lambda)x_n\| \to 0$. So $\psi(B(T-\lambda)) = LIM < B(T-\lambda)x_n, x_n >= 0$. Since T is p-hyponormal, $\sigma_{ap}(T) = \sigma_{ja}(T)$, thus $\|(T-\lambda)^*x_n\| \to 0$. Hence $\psi(B(T-\lambda)^*) = 0$ for every $B \in \mathcal{B}(\mathcal{H})$ and $\psi(I) = LIM\|x_n\|^2 = 1$. Therefore if $p(T, T^*)$ is any non-commuting polynomial in T and T^* that has no constant term, $\psi(p(T,T^*)+\alpha) = \alpha$, for all α in $\mathbb{C}$. This implies that ψ is multiplicative on a dense subalgebra of $C^*(T)$. Hence $\psi|_{C^*(T)}$ is multiplicative and $\psi(T-\lambda) = 0$ and

$$\begin{aligned} 0 = \psi(T-\lambda) &= LIM < (T-\lambda)x_n, x_n > \\ &= LIM < Tx_n, x_n > + LIM < -\lambda x_n, x_n > \\ &= \psi(T) - \lambda. \end{aligned}$$

So $\psi(T) = \lambda$ and $\psi(T^*) = LIM < T^*x_n, x_n > = \{LIM < Tx_n, x_n > \}^* = \psi(T)^*$. Therefore ψ is a $*$-homomorphism such that $\psi(T) = \lambda$.

COROLLARY 2.2. *If T is p-hyponormal, there is an isometric $*$-isomorphism of $C^*(T)/M$ onto $C(\sigma_{ap}(T))$, where $A+M$ is mapped to the function z.*

PROOF. Let $\tau : \Phi_{\mathcal{A}} \to \sigma_{ap}(T)$ be defined by $\tau(\psi) = \psi(T)$. By Proposition 2.1, this map is surjective. On the other hand, if $\psi, \psi' \in \Phi_{\mathcal{A}}$ and $\psi(T) = \psi'(T)$, then $\psi = \psi'$. Since $\Phi_{\mathcal{A}}$ is compact and map is continuous, τ is a homeomorphism and $\tau^{\#} : C(\sigma_{ap}(T)) \to C(\Phi_{\mathcal{A}})$ is defined by $\tau^{\#}(f) = f \circ \tau$. Note that $\tau^{\#}$ is an isometric $*$-isomorphism. We define a map $\rho : C(\sigma_{ap}(T)) \to C^*(T)/M$ so that the following diagram commutes :



where the Gel'fand transform $\gamma : C^*(T)/M \to C(\Phi_{\mathcal{A}})$ is an isometric $*$-isomorphism of $C^*(T)/M$ onto $C(\Phi_{\mathcal{A}})$.

We show that the $*$-representation ϕ preserves the p-hyponormality.

PROPOSITION 2.3 *Let $T = U|T|$ be p-hyponormal, $\phi(T)$ is a p-hyponormal operator.*

PROOF We need only to prove that $p = \frac{1}{2^n}$ for some n. Since $|\phi(T)|^2 = \phi(T)^*\phi(T) = \phi(T^*T) = \phi(|T|^2) = \phi(|T|)^2$. By the uniqueness of the square root of a positive operator, we have $\phi(|T|) = |\phi(T)|$. Similarly, $\phi(|T^*|) = |\phi(T^*)|$

By the assumption, we have

$$(T^*T)^{\frac{1}{2^n}} - (TT^*)^{\frac{1}{2^n}} \geq 0.$$

Thus,

$$\begin{aligned}
&(\phi(T^*)\phi(T))^{\frac{1}{2^n}} - (\phi(T)\phi(T^*))^{\frac{1}{2^n}} \\
&= \phi((T^*T))^{\frac{1}{2^n}} - \phi((TT^*))^{\frac{1}{2^n}} \\
&= \phi((T^*T)^{\frac{1}{2^n}}) - \phi((TT^*)^{\frac{1}{2^n}}) \\
&= \phi((T^*T)^{\frac{1}{2^n}} - (TT^*)^{\frac{1}{2^n}}) \geq 0.
\end{aligned}$$

With the notation of Proposition 1.1 , Proposition 1.3 and Corollary 2.2, we have following

PROPOSITION 2 4 *If T is a p-hyponormal operator, then* $C^*(T)/M \cong C^*(\phi(T))/N \cong C(\sigma_p(\phi(T)))$.

In [3], One of the authors proved the following two theorems for hyponormal operator. Now we have same results for p-hyponormal operator.

THEOREM 2 5 *If T is an irreducible p-hyponormal operator such that $T^*T - TT^*$ is compact, then $\sigma_{ap}(T) = \sigma_e(T)$ and $\sigma_p(\phi(T)) = \sigma_e(\phi(T))$.*

PROOF The fact that $\sigma_{ap}(T) = \sigma_e(T)$ follows immediately from Proposition 1.4 and Corollary 2.2. The second assertion is clear from Proposition 1.4 and Proposition 1.5.

COROLLARY 2 6 *If T is an irreducible p-hyponormal operator such that $T^*T - TT^*$ is compact, then $\sigma_e(T) = \sigma_e(\phi(T))$.*

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