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STRONG CONVERGENCE OF SOLUTIONS OF NONLINEAR VOLTERRA EQUATIONS IN BANACH SPACES

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ABSTRACT The strong convergences of solutions to a nonlinear Volterra equation are studied in Banach space As an application, a nonlinear heat flow in a homogeneous bar of unit length of a material with memory is investigated

1. Introduction

Let X be a real Banach space and let A be a m-accretive operator in X. We shall consider the strong convergences as $t \to \infty$ of solutions to abstract nonlinear Volterra equation

$$(\mathrm{V}_{b,g,f}) \quad u(t)+\int_0^t b(t-s)(Au(s)+g(s)u(s))ds \ni f(t), \quad t \in \mathbb{R}^+ = [0,\infty)$$

where $b : \mathbb{R}^+ \to \mathbb{R}$, $g : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : \mathbb{R}^+ \to X$ are given, and the integral is taken in the sense of Bochner.

The asymptotic behavior of solutions of $(\mathrm{V}_{b,g,f})$ has primarily been studied in the case that $g \equiv 0$. See [2, 4, 5, 10, 11, 13, 14, 15, 19]. In particular, Kato [15] investigated the "unbounded behavior" of solution

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$u(t)$ of $(\mathrm{V}_{b,0,f})$, that is, the behavior when $u(t)$ is allowed to be unbounded, and improved the results of due to Clément [4] and Clément and Nohel [5] for the convergence of the solution $u(t)$ of $(\mathrm{V}_{b,0,f})$ itself as $t \to \infty$. The unbounded behavior of the solution $u(t)$ of $(\mathrm{V}_{b,g,f})$ was given in [1, 12].

In this paper, we study the strong convergence of the solution $u(t)$ of $(\mathrm{V}_{b,g,f})$ itself as $t \to \infty$ in connection with the results of Clément [4], Clément and Nohel [5] and Kato [15]. An application of the physical problem is also investigated. Our study can be viewed as an attempt to extend earlier corresponding results obtained in [4, 5, 15] in the case that $g \equiv 0$.

2. Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$, and dual $(X^*, \|\cdot\|_*)$. The duality pairing between X and X^* will be denoted by $(\cdot,\cdot)$. Let A be a set-valued operator in X with domain $D(A)$ and range $R(A)$. A is said to be *accretive* if $[y_2 - y_1, x_2 - x_1]_+ \geq 0$ for $y_i \in Ax_i$, $i = 1, 2$, where $[y,x]_+ = \lim_{\lambda \downarrow 0}(\|x+\lambda y\| - \|x\|)\lambda^{-1}$ for $x, y \in X$. We say that A is *m-accretive* if it is accretive and $R(I+\lambda A) = X$ for all $\lambda > 0$, where I stands for the identity on X. If A is m-accretive, one can define its Yosida approximation A_λ by $A_\lambda = \lambda^{-1}(I - J_\lambda)$, with $J_\lambda = (I + \lambda A)^{-1}$, $\lambda > 0$. Also recall that the Yosida approximation A_λ $(\lambda > 0)$ of a m-accretive operator A is strictly accretive; i.e., $[A_\lambda x - A_\lambda y, x - y]_- \geq 0$ for $x, y \in X$, where $[y,x]_- = \lim_{\lambda \uparrow 0}(\|x+\lambda y\| - \|x\|)\lambda^{-1}$ for $x, y \in X$. (cf. [6, 19]).

As in [1], we assume throughout this paper that A is an m-accretive operator on X and consider equation $(\mathrm{V}_{\mathrm{b,g,f}})$ under the following minimal assumptions:

$$(\mathrm{H}_b) \qquad b \in AC_{loc}(\mathbb{R}^+;\mathbb{R}), \quad b(0) = 1, \quad b' \in BV_{loc}(\mathbb{R}^+;\mathbb{R}),$$

$$(\mathrm{H}_g) \qquad g \in C(\mathbb{R}^+;\mathbb{R}^+)$$

$$(\mathrm{H}_f) \qquad f \in W^{1,1}_{loc}(\mathbb{R}^+;X), \quad f(0) \in \overline{D(A)}.$$

Here $\overline{D(A)}$ is the closure of $D(A)$. According to [1, 5] under these assumptions, the equation ($\mathrm{V}_{b,g,f}$) is (a.e on $\mathbb{R}^+$) equivalent to

$$\text{(E)} \qquad \frac{du(t)}{dt} + \frac{d}{dt}(k * u)(t) + Au(t) + g(t)u(t) \ni k(t)f(0) + F(t),$$

with $u(0) = f(0)$. Here $*$ denotes the convolution $k * z(t) = \int_0^t k(t-s)z(s)ds$, k satisfies

$$\text{(2.1)} \qquad b(t) + (k * b)(t) = 1, \quad t \geq 0$$

and F is (a.e on $\mathbb{R}^+$) given by

$$\text{(2.2)} \qquad F(t) = f'(t) + (k * f')(t).$$

Note that (2.1) can be rewritten as $k + b' * k = -b'$, so that, by (H_b), k is uniquely determined in $BV_{loc}(\mathbb{R}^+; \mathbb{R})$. It also follows (see ($\mathrm{H}_f$)) that $F \in L^1_{loc}(\mathbb{R}^+; X)$. The strong solutions are considered as functions in $W^{1,1}_{loc}(\mathbb{R}^+; X) \bigcap C(\mathbb{R}^+; \overline{D(A)})$ for which the equations ($\mathrm{V}_{b,g,f}$) and (E) hold a.e $t \in \mathbb{R}^+$. A function $u \in C(\mathbb{R}^+; \overline{D(A)})$ is said to be a *generalized solution* of the equation ($\mathrm{V}_{b,g,f}$) if $\lim_{\lambda \downarrow 0} u_\lambda = u$ in $C([0,T]; X)$ for any $0 < T < \infty$, where u_λ is the strong solution of the approximating equation ($\mathrm{V}_{\lambda b,g,f}$) (equivalently, ($\mathrm{E}_\lambda$)) in which A is replaced by the Yosida approximation A_λ for each $\lambda > 0$ (cf. [1, 6, 8]).

To study the convergence of generalized solutions of ($\mathrm{V}_{b,g,f}$), we need the concept of a complete positivity of the kernel b. This concept is defined as follows (cf. [5, 15]). For $b \in L^1_{loc}(\mathbb{R}^+; \mathbb{R})$, define $s(b)$ and $r(b) \in L^1_{loc}(\mathbb{R}^+; \mathbb{R})$ by the equations

$$\text{(2.3)} \qquad s(b)(t) + b * s(b)(t) = 1$$

$$\text{(2.4)} \qquad r(b)(t) + b * r(b)(t) = b(t)$$

respectively. We say that b is *completely positive* on $\mathbb{R}^+$ if $s(\lambda b)$ and $r(\lambda b)$ are nonnegative on $\mathbb{R}^+$ for every $\lambda > 0$. It is well-known [2, 5] that if (H_b) is satisfied and k, defined by (2.1), is nonnegative and nonincreasing on $\mathbb{R}^+$, then the kernel b is completely positive on $\mathbb{R}^+$. Furthermore, in this case b satisfies $0 \le b(t) \le 1$ for $t \ge 0$, and $\lim_{t\to\infty} b(t) = b(\infty)$ exists with $b(\infty) = (1 + \int_0^\infty k(s)ds)^{-1}$ if $k \in L^1(\mathbb{R}^+;\mathbb{R})$, and $b(\infty) = 0$ if $k \notin L^1(\mathbb{R}^+;\mathbb{R})$. It follows from this fact that $b(\infty) > 0$ if and only if $k \in L^1(\mathbb{R}^+;\mathbb{R})$. Also, in this case, $b \notin L^{(}\mathbb{R}^+;\mathbb{R})$.

Finally, we recall the following important property of completely positive kernel.

PROPOSITION 2 1 [11, 13] *Let b be completely positive and $w \in W^{1,1}_{loc}(\mathbb{R}^+;X)$. Then $k*w$ and $k*\|w\|$ are locally absolutely continuous and differentiable a.e. on $\mathbb{R}^+$. Furthermore,*

$$\left[\frac{d}{dt}(k*w)(t), w(t)\right]_+ \ge \frac{d}{dt}(k*\|w\|)(t) \tag{2.5}$$

for almost all $t > 0$.

3. Main results

Now, we study the convergence of solutions $u(t)$ of $(\mathrm{V}_{b,g,f})$ itself as $t \to \infty$.

First, we have the following main result along with a slightly different method from that of result of Kato [15] in the case that $g \equiv 0$.

THEOREM 3 1 *Let (H_b), (H_g) and (H_f) be satisfied and let F be associated to f by (2.2). Suppose that b is completely positive. Let $b \in L^1(\mathbb{R}^+;\mathbb{R})$ and let $g \in L^1(\mathbb{R}^+)$. Then for the generalized solution u of $(\mathrm{V}_{b,g,f})$,*

$$\begin{aligned}\|u(t) - u^\infty\| \le \frac{1}{\bar{b}}\left(\int_t^\infty b(t-s)ds\right)\|u_0 - u^\infty\| \\ - \int_0^t b(t-s)g(s)\|u(s) - u^\infty\|ds \\ + \int_0^t b(t-s)[F(s) - F^\infty, u(s) - u^\infty]_+ds,\end{aligned} \tag{3.1}$$

where $u^\infty = J_{\bar b}(u_0 + \bar b F^\infty)$, $\bar b = \int_0^\infty b(t)dt > 0$ *and* $F^\infty \in X$.

In particular, let $F = F_1 + F_2$. *Suppose either that* $F_1 \in L^\infty(\mathbb{R}^+; X)$ *and* $\lim_{t\to\infty} F_1(t) = F^\infty$ *or that* $F_1 - F^\infty \in L^p(\mathbb{R}^+; X)$, $1 \le p < \infty$, *and suppose that* $F_2 \in L^1(\mathbb{R}^+; X) + L^p(\mathbb{R}^+; X)$, $1 < p < \infty$. *Then* $\lim_{t\to\infty} u(t) = u^\infty$.

PROOF Since A_λ is m-accretive, there exists a $u_\lambda^\infty \in X$ such that $u_\lambda^\infty + \bar b A_\lambda u_\lambda^\infty = u_0 + \bar b F^\infty$. The m-accretiveness of A guarantees that $\lim_{\lambda\downarrow 0} u_\lambda^\infty = u^\infty$. Let u_λ be a strong solution of $(\mathrm{V}_{\lambda b,g,f})$ Then, by strictly accretiveness of A_λ and the inequality (2.5), we have

$$\begin{aligned}
0 &\le [A_\lambda u_\lambda(t) - A_\lambda u_\lambda^\infty, u_\lambda(t) - u_\lambda^\infty]_- \\
&= \Big[-\frac{d}{dt}u_\lambda(t) - \frac{d}{dt}(k * u_\lambda)(t) - g(t)(u_\lambda(t) - u_\lambda^\infty) \\
&\quad + k(t)(u_0 - u_\lambda^\infty) + F(t) - \frac{1}{\bar b}(u_0 - u_\lambda^\infty) - F^\infty, u_\lambda(t) - u_\lambda^\infty\Big]_- \\
&\le -\left(\frac{d}{dt}[\|u_\lambda(t) - u_\lambda^\infty\| + k * \|u_\lambda - u_\lambda^\infty\|(t)] + g(t)\|u_\lambda(t) - u_\lambda^\infty\|\right) \\
&\quad + \left(k(t) - \frac{1}{\bar b}\right)\|u_0 - u_\lambda^\infty\| + [F(t) - F^\infty, u_\lambda(t) - u_\lambda^\infty]_+
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{d}{dt}\Big[\|u_\lambda(t) - u_\lambda^\infty\| + k * \|u_\lambda - u_\lambda^\infty\|(t)\Big] + g(t)\|u_\lambda(t) - u_\lambda^\infty\| \\
&\le \left(k(t) - \frac{1}{\bar b}\right)\|u_0 - u_\lambda^\infty\| + [F(t) - F^\infty, u_\lambda(t) - u_\lambda^\infty]_+.
\end{aligned} \tag{3.2}$$

Define, for fixed $\lambda > 0$,

$$x(t) = \|u_\lambda(t) - u_\lambda^\infty\| - \|u_0 - u_\lambda^\infty\|,$$

$$\varphi(t) = -\frac{1}{\bar b}\|u_0 - u_\lambda^\infty\| + [F(t) - F^\infty, u_\lambda(t) - u_\lambda^\infty]_+ - g(t)\|u_\lambda(t) - u_\lambda^\infty\|$$

and

$$\psi(t) = x(t) + (k * x)(t). \tag{3.3}$$

Then $\psi(0) = 0$ and

(3.4)

$$\frac{d}{dt}\psi(t) = \frac{d}{dt}[\|u_\lambda(t) - u_\lambda^\infty\| + k * \|u_\lambda - u_\lambda^\infty\|(t)] - k(t)\|u_0 - u_\lambda^\infty\| \leq \varphi(t).$$

Using (2.1) (or $k + k * b' = -b'$) and means of the variation of constants formula together with (3.3), we obtain

$$(3.5)\quad x(t) = \psi(t) + (b' * \psi)(t) = \psi(t) + \int_0^t b'(t-s)\psi(s)ds, \qquad t \geq 0.$$

An integration by parts shows that (3.5) is equivalent to

$$(3.6)\qquad x(t) = \int_0^t b(t-s)\psi'(s)ds.$$

Since $b \geq 0$, we deduce from (3.4) and (3.6) that

$$x(t) \leq \int_0^t b(t-s)\varphi(s)ds,$$

and hence we obtain

(3.7)

$$\begin{aligned}\|u_\lambda(t) - u_\lambda^\infty\| - \|u_0 - u_\lambda^\infty\| \leq -\frac{1}{\bar{\bar{b}}}\left(\int_0^t b(s)ds\right)\|u_0 - u_\lambda^\infty\| \\ + \int_0^t b(t-s)[F(s) - F^\infty, u_\lambda(s) - u_\lambda^\infty]_+ ds \\ - \int_0^t b(t-s)g(s)\|u_\lambda(s) - u_\lambda^\infty\|ds.\end{aligned}$$

Letting $\lambda \downarrow 0$ in (3.7), we obtain(3.1).

For the proof of the last assertion, since $\int_0^t b(t-s)g(s)ds \geq 0$ for $t \geq 0$, it is enough to show that either if $\lim_{t\to\infty} y(t) = y^\infty$ or if $y - y^\infty \in L^p(\mathbb{R}^+; X)$, $1 \leq p < \infty$, then $\lim_{t\to\infty}\int_0^t b(t-s)\|y - y^\infty\|(s)ds = 0$. But one can easily check these facts since $b \in L^1(\mathbb{R}^+;\mathbb{R})$and $b(\infty) = 0$.

THEOREM 3 2 *Let* (H_b), (H_g) *and* (H_f) *be satisfied and let* F *be associated to* f *by* (2.2). *Suppose that* b *is completely positive and that there exists an* $\omega > 0$ *such that* $A - \omega I$ *is accretive in* X. *Let* $r(b)$, $s(b)$ *be defined by* (2.3), (2.4), *respectively, and let* $g \in L^1(\mathbb{R}^+)$. *Then, for the generalized solution* u *of* ($V_{b,g,f}$),

$$\begin{aligned}\|u(t) - u^\infty\| \leq &\left(1 - \int_0^t r(\omega b)(\tau)d\tau\right)\|u_0 - u^\infty\| \\ &- \frac{1}{\omega}\int_0^t r(\omega b)(t-\tau)g(\tau)\|u(\tau) - u^\infty\|d\tau \\ &+ \int_0^t r(\omega b)(t-\tau)[F(\tau) - F^\infty, u(\tau) - u^\infty]_+ d\tau,\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&\|u(t) - u^\infty\| + k * \|u - u^\infty\|(t) \\ &\leq \left(1 + \int_0^t k(\tau)d\tau - \omega\int_0^t s(\omega b)(\tau)d\tau\right)\|u_0 - u^\infty\| \\ &- \int_0^t s(\omega b)(t-\tau)g(\tau)\|u(\tau) - u^\infty\|d\tau \\ &+ \int_0^t s(\omega b)(t-\tau)[F(\tau) - F^\infty, u(\tau) - u^\infty]_+ d\tau\end{aligned} \tag{3.9}$$

where $F^\infty \in X$ *and* u^∞ *is the unique element in* X *satisfying* $Au^\infty \ni F^\infty$.

In particular, let $F = F_1 + F_2$. *Suppose either that* $b \notin L^1(\mathbb{R}^+;\mathbb{R})$ *and* $F_1 \in L^\infty(\mathbb{R}^+;X)$ *and* $\lim_{t\to\infty} F_1(t) = F^\infty$ *or that* $k \in L^1(\mathbb{R}^+;\mathbb{R})$ (*this implies* $b \notin L^1(\mathbb{R}^+)$) $F_1 - F^\infty \in L^p(\mathbb{R}^+;X)$, $1 \leq p < \infty$, *and suppose that* $F_2 \in L^1(\mathbb{R}^+;X) + L^p(\mathbb{R}^+;X)$, $1 < p < \infty$. *Then* $\lim_{t\to\infty} u(t) = u^\infty$.

PROOF Let u_λ be a strong solution of ($V_{\lambda b,g,f}$) and $\hat{u}_\lambda$ be the one corresponding to $\hat{u}_0$ and $\hat{f}$. Since $A - \omega I$ is accretive, $A_\lambda - \delta$, $\delta = \frac{\omega}{1+\lambda\omega}$

is (strictly) accretive. Thus, by the inequality (2.5), we have

$$\begin{aligned}\delta\|u_\lambda(\tau)-\hat{u}_\lambda(\tau)\| &\le [A_\lambda u_\lambda(\tau)-A_\lambda\hat{u}_\lambda(\tau),u_\lambda(\tau)-\hat{u}_\lambda(\tau)]_- \\ &\le -\frac{d}{d\tau}[\|u_\lambda(\tau)-\hat{u}_\lambda(\tau)\|+k*\|u_\lambda-\hat{u}_\lambda\|(\tau) \\ &+g(\tau)\|u_\lambda(\tau)-\hat{u}_\lambda(\tau)\|] \\ &+k(\tau)\|u_0-\hat{u}_0\|+[F(\tau)-\hat{F}(\tau),u_\lambda(\tau)-\hat{u}_\lambda]_+.\end{aligned}$$

Set $\alpha(t)=\frac{d}{dt}\|u_\lambda(t)-\hat{u}_\lambda(t)\|$. Then it follows that
(3.10)

$$\begin{aligned}\alpha(\tau)+k*\alpha(\tau) &\le -(g(\tau)+\delta)\|u_\lambda(\tau)-\hat{u}_\lambda(\tau)\| \\ &+[F(\tau)-\hat{F}(\tau),u_\lambda(\tau)-\hat{u}_\lambda(\tau)]_+ \quad \text{a.e. } \tau\in\mathbb{R}^+.\end{aligned}$$

Now to show (3.8), observe that $b+k*b=1$ and $r(\omega b)+\omega b*r(\omega b)=\omega b$ yield the relation $r(\omega b)+k*r(\omega b)=\omega(1-1*r(\omega b))$. Since $r(\omega b)\ge 0$ (by completely positivity of b), we multiply (3.10) by $r(\omega b)*$ to obtain
(3.11)

$$\begin{aligned}\|u_\lambda(t)-\hat{u}_\lambda(t)\| &\le \left(1-\int_0^t r(\omega b)(\tau)\right)\|u_0-\hat{u}_0\| \\ &+\left(1-\frac{\delta}{\omega}\right)r(\omega b)*\|u_\lambda-\hat{u}_\lambda\|(t) \\ &-\frac{1}{\omega}\int_0^t r(\omega b)(t-\tau)g(\tau)\|u_\lambda(\tau)-\hat{u}_\lambda(\tau)\|d\tau \\ &+\frac{1}{\omega}\int_0^t r(\omega b)(t-\tau)[F(\tau)-\hat{F}(\tau),\hat{u}_\lambda(\tau-\hat{u}_\lambda(\tau)]_+d\tau.\end{aligned}$$

Since $1-\frac{\delta}{\omega}=\frac{\lambda\omega}{1+\lambda\omega}$, letting $\lambda\downarrow 0$ in (3.11) yields

$$\begin{aligned}\|u(t)-\hat{u}(t)\| &\le \left(1-\int_0^t r(\omega b)(\tau)d\tau\right)\|u_0-\hat{u}_0\| \\ &-\frac{1}{\omega}\int_0^t r(\omega b)(t-\tau)g(\tau)\|u(\tau)-\hat{u}(\tau)\|d\tau \\ &+\frac{1}{\omega}\int_0^t r(\omega b)(t-\tau)[F(\tau)-\hat{F}(\tau),u(\tau)-\hat{u}(\tau)]_+d\tau\end{aligned}$$

where $\hat{u}$ is a generalized solution of $(\mathrm{V}_{b,g,\hat{f}})$. Thus, in particular, if $Au^\infty \ni F^\infty$, we obtain (3.8).

Next, to prove (3.9), observe that $b+k*b=1$ and $s(\omega b)+\omega*s(\omega b)=1$ implies $s(\omega b)+k*(\omega b)=1+1*k-\omega*s(\omega b)$. Since $s(\omega b)\geq 0$ (by completely positivity of b), we multiply (3.10) by $s(\omega b)*$ to get

(3.12)

$$\begin{aligned}&\|u_\lambda(t)-\hat{u}_\lambda(t)\|+k*\|u_\lambda-\hat{u}_\lambda\|(t)-\omega s(\omega b)*\|u_\lambda-\hat{u}_\lambda\|(t)\\ &\leq\left(1+\int_0^t k(\tau)-\omega\int_0^t s(\omega b)(\tau)d\tau\right)\|u_0-\hat{u}_0\|\\ &-\frac{\omega}{1+\lambda\omega}s(\omega b)*\|u_\lambda-\hat{u}_\lambda\|(t)-\int_0^t s(\omega b)(t-\tau)g(\tau)\|u_\lambda(\tau)\\ &-\hat{u}_\lambda(\tau)\|d\tau+\int_0^t s(\omega b)(t-\tau)[F(\tau)-\hat{F}(\tau),u_\lambda(\tau)-\hat{u}_\lambda(\tau)]_+d\tau.\end{aligned}$$

Then, letting $\lambda\downarrow 0$ in (3.12), we have

$$\begin{aligned}&\|u(t)-\hat{u}(t)\|+k*\|u-\hat{u}\|(t)\\ &\leq\left(1+\int_0^t k(\tau)d\tau-\omega\int_0^t s(\omega b)(\tau)d\tau\right)\|u_0-\hat{u}_0\|\\ &-\int_0^t s(\omega b)(t-\tau)g(\tau)\|u(\tau)-\hat{u}(\tau)\|d\tau\\ &+\int_0^t s(\omega b)(t-\tau)[F(\tau)-\hat{F}(\tau),u(\tau)-\hat{u}(\tau)]_+d\tau.\end{aligned}$$

In particular, if $Au^\infty\ni F^\infty$, we obtain (3.9). Since

$$\int_0^t r(\omega b)(t-\tau)g(\tau)\|u(\tau)-u^\infty\|d\tau\ \geq\ 0$$

and

$$\int_0^t s(\omega b)(t-\tau)g(\tau)\|u(\tau)-u^\infty\|d\tau\geq 0\quad\text{for } t\geq 0,$$

the last statement follows from the fact that if $b\notin L^1(\mathbb{R}^+;\mathbb{R})$, then for every $\omega>0$, $\lim_{t\to\infty}s(\omega b)(t)=0$ and $\int_0^\infty r(\omega b)(\tau)d\tau=1$ ([5, Proposition 2.1]). Note also that $k \in L^1(\mathbb{R}^+;\mathbb{R})$ implies $b \notin L^1(\mathbb{R}^+;\mathbb{R})$ and then

$$\omega \int_0^\infty s(\omega b)(\tau)d\tau = \frac{1}{b(\infty)} = 1 + \int_0^\infty k(\tau)d\tau.$$

REMARK 1

(1) Theorem 3.1 is an improvement of Theorem 2 of Clément [4], Theorem 3.2 of Clément and Nohel [5] and Theorem 3.1 of Kato [15] in the case that $g \equiv 0$.

(2) Theorem 3.3 in [5] and Theorem 3.2 in [15] is also a special case of Theorem 3.2 if $g \equiv 0$.

(3) It follows from Theorem 3.2 that if A is a strongly m-accretive operator, then the estimates of convergence can be obtained (here $A^{-1}0$ is not assumed bounded compact and X is arbitrary in comparison with [21]). Some other strong convergence results were obtained in [13, 14] in the case that $g \equiv 0$.

4. An example

In this section, as in [1, 5, 19], we consider a nonlinear heat flow in a homogeneous bar of unit length of a material with memory.

Let $u(t,x)$, $e(t,x)$, $q(t,x)$ and $\mu(t,x)$ denote, respectively, the temperature, internal energy, heat flux, and external heat supply at time t and position x ($-\infty < t < \infty$, $0 \le x \le 1$). Let the ends of the bar at $x = 0$ and $x = 1$ be maintained at zero temperature. For simplicity and without loss of generality let the history of u be prescribed as zero when $t < 0$ and $0 \le x \le 1$. The equation satisfied by u is derived from the assumptions that in such materials the internal energy e and the heat flux q are functionals of u and of the gradient of u respectively (rather than functions of u and u_x). Specifically, according to the theory developed by, e.g., Gurtin and Pipkin [9], Nunziato [20] for heat flow in materials of fading memory type, we assume that e and q are taken respectively as the functionals

(4.1)

$$e(t,x) = b_0 u(t,x) + \int_0^t \beta(t-s)u(s,x)ds + \int_0^t \alpha(t-s)g(s)u(s,x)ds,$$

$$q(t,x) = -c_0\sigma(u_x(t,x)) + \int_0^t \gamma(t-s)\sigma(u_x(s,x))ds, \tag{4.2}$$

for $t \geq 0$ and $0 < x < 1$.

Here $b_0 > 0$, $c_0 > 0$ are positive constants and the functions β, $\gamma : \mathbb{R}^+ \to \mathbb{R}$ are given sufficiently smooth functions called the *internal energy* and *heat flux relaxation functions*, respectively. The given function σ is a real function satisfying

$$\sigma \in C^1(\mathbb{R}), \quad \sigma(0) = 0, \quad \sigma'(s) \geq p_0 > 0, \quad s \in \mathbb{R}, \tag{4.3}$$

for some $p_0 > 0$, $\alpha(t) = c_0 - \int_0^t \gamma(s)ds$, and $g \in C(\mathbb{R}^+;\mathbb{R}^+)$. In the physical literature the relaxation functions β, γ are usually taken as finite linear combinations of decaying exponentials with positive coefficients. For physical reasons (see [5], [19]), we should require at least that β, $\gamma \in L^1(\mathbb{R}^+)$ and

$$b_0 + \int_0^\infty \beta(t)dt > 0 \quad \text{and} \quad c_0 - \int_0^\infty \gamma(t)dt > 0.$$

The law of balance of heat requires that the equation $e_t = -q_x + \mu$ ($= -\mathrm{div}q + \mu$) should hold. If $u(0,x) = u_0(x)(0 < x < 1)$ is the initial temperature distribution in the rod, we obtain, in view of (4.1), (4.2) and the assumption that the temperature at the ends of the rod is zero, the following initial-boundary value problem to be satisfied by the temperature u:

$$\begin{aligned}
&\frac{\partial}{\partial t}[b_0u(t,x) + (\beta * u)(t,x) + (\alpha * gu)(t,x)] \\
&= c_0\sigma(u_x(t,x))_x - (\gamma * \sigma(u_x)_x)(t,x) + \mu(t,x), \\
&\quad 0 < t < \infty, \ \ 0 < x < 1, \\
&u(t,0) = u(t,1) = 0, \quad t > 0, \\
&u(0,x) = u_0(x), \quad 0 < x < 1.
\end{aligned} \tag{4.4}$$

We remark that if the history of u for $t < 0$ is not zero, the integrals in (4.1) and (4.2) range over the interval $(-\infty, t)$ (rather than $(0,t)$), and the resulting equation corresponding to (4.4) would have additional known forcing terms stemming from the integrals over $(-\infty, 0)$ in (4.1) and (4.2).

As in [1, 5, 20], we transform this initial-boundary value problem to a Volterra integral equation in the space $X = L^2(0,1)$. Let

$$G(t,x) = u_0(x) + \int_0^t \mu(s,x)ds, \qquad t \geq 0, \;\; 0 < x < 1, \tag{4.5}$$

and note that

$$c_0\sigma(u_x)_x - \gamma * \sigma(u_x)_x = \frac{\partial}{\partial t}(\alpha * \sigma(u_x)_x).$$

Then (4.4) with $b_0 = 1$ and $c_0 = 1$ (without loss of generality) leads to the equation

$$u + \beta * u + \alpha * (Au + gu) = G, \quad t \geq 0, \;\; 0 < x < 1. \tag{4.6}$$

The nonlinear operator $A : D(A) \subset X \to X$ is defined by $Au = -\sigma(u_x)_x$, together with the boundary conditions $u(t,0) = u(t,1) \equiv 0$ and

$$D(A) = \{u \in H_0^1(0,1) : \sigma(u_x)_x \in X\}.$$

It is well known that if assumptions (4.3) are satisfied, then $A = \partial\phi$, where $\phi : L^2(0,1) \to (-\infty, \infty]$,

$$\phi(y) = \begin{cases} \displaystyle\int_0^1 W(\frac{dy}{dx})(x)dx & \text{if } y \in H_0^1(0,1) \\ +\infty & \text{otherwise,} \end{cases} \tag{4.7}$$

where $W(z) = \int_0^z \sigma(\xi)d\xi$. Thus ϕ is convex, lower-semicontinuous and proper on $L^2(0,1)$ (in fact, $\phi(y) \geq 0$), and A is maximal monotone and hence m-accretive on X (see Lemma 2.3 in [19] (cf. [3])).

Let $r(\beta)$ denote the resolvent kernel of β (i.e., $r(\beta) + \beta * r(\beta) = \beta$). Clearly, if $\beta \in L^1(\mathbb{R}^+)$, then $r(\beta) \in L^1_{\text{loc}}(\mathbb{R}^+)$ (at least). Next, define $b : \mathbb{R}^+ \to \mathbb{R}$ and $f : \mathbb{R}^+ \to X$ by

$$b = \alpha - r(\beta) * \alpha, \tag{4.8}$$

and

$$f = G - r(\beta) * G, \tag{4.9}$$

respectively. Then the variation of constants formula shows that (4 6) is equivalent to

$$u + b * (Au + gu) = f \ (= u_0 + 1 * (\mu - r(\beta) * \mu - u_0 r(\beta))), \tag{4.10}$$

which is an equation of the form $(V_{b,g,f})$.

Now, we can apply the theory developed in Section 3 to discuss the strong convergence of the generalized solution of (4.10) (equivalent to the heat flow problem (4.4) with $b_0 = 1$ and $c_0 = 1$). We assume that $u_0 \in L^2(0,1)$ and that the forcing function $\mu \in L^1_{\text{loc}}(\mathbb{R}^+; L^2(0,1))$ Then, by (4.5), (4.9), and since the resolvent of β satisfies $r(\beta) \in L^1_{\text{loc}}(\mathbb{R}^+)$ under the condition $\beta \in L^1(\mathbb{R}^+)$, it is easily seen that $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^+; L^2(0,1))$ under same condition. Also note that $D(A)$ is dense in X, so that all of (H_f) is satisfied. If moreover, we take (H_b), (H_g) to hold, then (4.10) has a unique generalized solution u on $\mathbb{R}^+$.

Before giving the convergence result for (4.10), we now state a lemma which is essentially [5, Lemma 4.2].

LEMMA 4 1 *Let β be bounded, nonnegative, nonincreasing and convex on $\mathbb{R}^+$. Let γ be positive, nonincreasing, log-convex, and bounded on $\mathbb{R}^+$. Suppose that*

$$\alpha(\infty) = 1 - \int_0^\infty \gamma(t)dt > 0, \quad \textit{and} \quad \beta'(t) + \gamma(0)\beta(t) \le 0 \ \textit{a.e.}\ t > 0.$$

Then b (given by (4.8)) is completely positive and satisfies (H_b), and k associated with b in (2.1) satisfies $k \in L^1(\mathbb{R}^+ : \mathbb{R})$ with

$$\int_0^\infty k(\tau)d\tau = \left[\frac{1}{\alpha(\infty)}\overline{\gamma}(1+\beta) + \overline{\beta}\right], \tag{4.11}$$

where $\overline{\beta} = \int_0^\infty \beta(t)dt$, $\overline{\gamma} = \int_0^\infty \gamma(t)dt$. Moreover, $b \notin L^1(\mathbb{R}^+;\mathbb{R})$ and $b' \in L^1(\mathbb{R}^+, \mathbb{R})$.

Now we give the result of applying Theorem 3.2, combined with Lemma 4.1, to the heat flow problem (4.1) with $b_0 = 1$ and $c_0 = 1$.

THEOREM 4 2 *Let the assumption of Lemma 4.1 be satisfied. Let* $u_0 \in L^2(0,1)$, $\mu \in L^1_{\text{loc}}(\mathbb{R}^+; L^2(0,1))$ *Let the assumption (4.3) be satisfied and let* k, F, b, f *be defined by (2.1), (2.2) and (4.8). Suppose that* g *satisfies* (H_g) *and* $g \in L^1(\mathbb{R}^+)$. *Let* $A = \partial\phi$, *where* ϕ *is defined in (4.7).*

If, in addition, $\beta \in L^1(\mathbb{R}^+)$, *and* $\mu = \mu_1 + \mu_2$ *(where* $\mu_1 \in L^\infty(\mathbb{R}^+; L^2(0,1))$ *and there exists* $\mu^\infty \in L^2(0,1)$ *such that* $\lim_{t\to\infty} \|\mu_1(t) - \mu^\infty(t)\|_{L^2(0,1)} = 0$, *and where* $\mu_2 \in L^P(\mathbb{R}^+; L^2(0,1))$ *for some* $p \geq 1$*), then equation (4.10) has a unique generalized solution* u *such that converges strongly in* $L^2(0,1)$ *as* $t \to \infty$ *to the element* $u^\infty \in L^2(0,1)$; u^∞ *is the unique solution of the limit equation* $Au^\infty = F^\infty$, *where*

$$F^\infty = \mu_1^\infty\left(1 + \frac{\overline{\gamma}}{\sigma(\infty)}\right), \qquad \overline{\gamma} = \int_0^\infty \gamma(t)dt.$$

In particular, if $\mu_1^\infty = 0$, *then* $u^\infty = 0$.

PROOF The result follows from the last statement of Theorem 3.2 and the proof of [5, Theorem 4.1].

REMARK 2 Theorem 4.2 is also an improvement of [5, Theorem 4.1, 3] in the case that $g \equiv 0$.

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