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A PERTURBED ALGORITHM OF GENERALIZED QUASIVARIATIONAL INCLUSIONS FOR FUZZY MAPPINGS

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ABSTRACT In this paper, we introduce a class of generalized quasivariational inclusions for fuzzy mappings and show its equivalence with a class of fixed point problems Using this equivalence, we develop the Mann and Ishikawa type perturbed iterative algorithms for this class of generalized quasivariational inclusions Further, we prove the existence of solutions for the class of generalized quasivariational inclusions and discuss the convergence criteria for the perturbed algorithms

1. Introduction

In recent years, fuzzy set theory introduced by Zadeh[15] in 1965 has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of regional, physical, mathematical and engineering, management science, economics, transportation problems, and operations research, see [2,15,16] Motivated and inspired by the recent research work going on in these two different fields, Chang and Zhu[2], and Noor[8] introduced the concept of variational inequalities and complementarity problems for fuzzy mappings. Noor[8] has shown that the variational inequalities for fuzzy mappings are equivalent to fuzzy fixed

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point problems. This equivalence was used to suggest an iterative algorithm for solving variational inequalities.

We remark that one of the most important and difficult problems in variational inequality theory for fuzzy mappings is the development of an efficient and implementable iterative algorithm for solving various classes of variational inequalities for fuzzy mappings. The proximal method [5] is one of the most efficient methods. We also remark that the projection method and its various method can not be applied to study the existence of a solution and to develop the iterative algorithm for our considered class of generalized quasivariational inclusions for fuzzy mappings. Therefore, the aim of this paper is to study the existence theory and to develop the Mann and Ishikawa type perturbed iterative algorithm for the class of generalized quasivariational inclusions for fuzzy mappings. The convergence criteria for these algorithms is also discussed.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $<,>$. A fuzzy set in H is a function with domain H and values in $[0,1]$. If A is a fuzzy set in H and $x \in H$, the function value $A(x)$ is called the grade of membership of x in A. We denote the collection of all fuzzy sets on H by $\mathcal{F}(H)$. Let $A \in \mathcal{F}(H)$ and $\alpha \in (0,1]$. The α-level set of A, denoted $(A)_\alpha$, is defined by $(A)_\alpha = \{x : A(x) \geq \alpha\}$.

A mapping T from H into $\mathcal{F}(H)$ is called a *fuzzy mapping*. If $T : H \to \mathcal{F}(H)$ is a fuzzy mapping, then $T(u)$, for $u \in H$, is a fuzzy set in $\mathcal{F}(H)$ and $T(u,v)$, for $v \in H$, is the degree of membership of v in $T(u)$.

Now, let T and M be two fuzzy mappings from H into $\mathcal{F}(H)$, and $g, m : H \to H$ be two single valued mappings. Assume $\phi : H \to R \cup \{+\infty\}$ is a proper convex lower semi-continuous function and $\partial\phi$ is the subdifferential of ϕ. Then the generalized quasivariational inclusion problem for fuzzy mappings (FGQVIP) is to find $u \in H$, $x \in (T(u))_r$, $y \in (M(u))_r$ ($r \in (0,1]$) such that $(g-m)(u) \cap \mathrm{dom}\partial\phi \neq \phi$ and

$$(2.1) \quad < x-y, v-(g-m)(u) > \geq \phi((g-m)(u)) - \phi(v) \quad \text{for all} \quad v \in H,$$

where $g - m$ is defined as $(g - m)(u) = g(u) - m(u)$ for each $u \in H$ and some $r \in (0, 1]$.

The problem (2.1) has many important and significant applications in economics, transportation, control and optimization and network problems. For the recent state of the art, see Giannessi and Maugeri[4].

If $\phi = \delta_K$, the indicator function of the nonempty closed convex set K in H, then FGQVIP (2.1) is equivalent to finding $u \in H$, $x \in (T(u))_r$, $y \in (M(u))_r$ $(r \in (0,1])$ such that $g(u) \in K + m(u)$ such that

$$(2.2) \qquad < x - y, v - g(u) > \geq 0 \quad \text{for all} \quad v \in K(u),$$

where the set $K(u)$ is equal to $K + m(u)$, which is called the *completely generalized strongly quasivariational inequality problem* for fuzzy mappings[10].

Particularly, if g is the identity mapping, the problem (2.2) is said to be the *generalized strongly quasivariational inequality problem* for fuzzy mappings (see [9]). We remark that FGQVIP (2.1) also includes as special cases, the variational inequality problems for fuzzy mappings considered by [8].

3. Mann and Ishikawa type perturbed iterative algorithms

DEFINITION 3 1 Let X be a Banach space with the dual space X^* and let $\phi : X \to R \cup \{+\infty\}$ be a proper functional. ϕ is said to be *subdifferential* at a point $x \in X$ if there exists an $f^* \in X^*$ such that

$$\phi(y) - \phi(x) \geq < f^*, y - x >, \quad {}^{\forall} y \in X,$$

where f^* is called a *subgradient* of ϕ at x. The set of all subgradients of ϕ at x is denoted by $\partial\phi(x)$. The mapping $\partial\phi : X \to 2^{X^*}$ denoted by

$$\partial\phi(x) = \{f^* \in X^* : \phi(y) - \phi(x) \geq < f^*, y - x >, \quad {}^{\forall} y \in X\}$$

is said to be the *subdifferential* of ϕ.

First of all, we prove that FGQVIP (2.1) is equivalent to a fixed point problems.

LEMMA 3.1 *FGQVIP (2.1) has a solution if and only if for some given $\eta > 0$, the mapping $F : H \to 2^H$ defined by*
(3.1)

$$F(u) = \cup_{x\in(T(u))_r}\cup_{y\in(M(u))_r}[u-(g-m)(u)+J_\eta^\phi((g-m)(u)-\eta(x-y))],$$

where $J_\eta^\phi = (I + \eta\partial\phi)^{-1}$ is the so called proximal mapping, I stands for the identity on H, has a fixed point.

PROOF Let (u^*, x^*, y^*) be a solution of FGQVIP (2.1). Then we have $u^* \in H$, $x^* \in (T(u^*))_r$, $y^* \in (M(u^*))_r$ such that $(g-m)(u^*) \cap \mathrm{dom}\partial\phi \neq \phi$ and

$$(3.2) \quad < x^* - y^*, v - (g-m)(u^*) > \geq \phi((g-m)(u^*)) - \phi(v), \quad ^\forall v \in H.$$

Using the definition of $\partial\phi$, (3.2) can be written as

$$y^* - x^* \in \partial\phi((g-m)(u^*)),$$

and hence for any given $\eta > 0$,

$$\begin{aligned}(g-m)(u^*) - \eta(x^* - y^*) &\in (g-m)(u^*) + \eta\partial\phi((g-m)(u^*)) \\ &= (I + \eta\partial\phi)((g-m)(u^*)).\end{aligned}$$

From the definition of J_η^ϕ, we have

$$(g-m)(u^*) = J_\eta^\phi((g-m)(u^*) - \eta(x^* - y^*)),$$

and hence

$$\begin{aligned}u^* &= u^* - (g-m)(u^*) + J_\eta^\phi((g-m)(u^*) - \eta(x^* - y^*)) \\ &\in \cup_{x^*\in(T(u^*))_r}\cup_{y^*\in(M(u^*))_r}[u^* - (g-m)(u^*) \\ &\quad + J_\eta^\phi((g-m)(u^*) - \eta(x^* - y^*))] = F(u^*),\end{aligned}$$

i.e., $u^* \in H$ is a fixed point of F.

Conversely, if $u^* \in H$ is a fixed point of F, by definition of F, there exist $x^* \in (T(u^*))_r$ and $y^* \in (M(u^*))_r$ such that

$$u^* = u^* - (g-m)(u^*) + J_\eta^\phi((g-m)(u^*) - \eta(x^* - y^*)).$$

Hence, from the definition of J_η^ϕ, we have

$$(g-m)(u^*) - \eta(x^* - y^*) \in (g-m)(u^*) + \eta\partial\phi((g-m)(u^*)).$$

Note $\eta > 0$, and we have

$$y^* - x^* \in \partial\phi((g-m)(u^*)).$$

The definition of $\partial\phi$ yields

$$< x^* - y^*, v - (g-m)(u^*) >\geq \phi((g-m)(u^*)) - \phi(v), \quad {}^\forall v \in H,$$

and $\mathrm{Im}(g-m) \cap \mathrm{dom}\partial\phi \neq \phi$. Thus (u^*, x^*, y^*) is a solution of FGQVIP (2.1).

The transformation of FGQVIP (2.1) to the fixed point problem (3.1) is very useful in the approximation analysis of FGQVIP (2.1). One of the consequences of this transformation is that we can obtain an approximate solution by an iterative algorithm.

DEFINITION 3 2 [8] A fuzzy mapping $T : H \to \mathcal{F}(H)$ is said to be

(i) *F-strongly monotone* if for each $x, y, u, v \in H$ with $T(x,u) > 0$ and $T(y,v) > 0$, there exists a constant $\delta \in (0,1)$ such that

$$< u - v, x - y > \geq \ \delta\|x - y\|^2.$$

(ii) *F-Lipschitz continuous* if for each $x, y, u, v \in H$ with $T(x,u) > 0$ and $T(y,v) > 0$, there exists a constant $\rho \in (0,1)$ such that

$$\|u - v\| \leq \rho\|x - y\|.$$

DEFINITION 3 3 An operator $g : H \to H$ is said to be

(i) *α-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$< g(x) - g(y), x - y >\geq \alpha\|x - y\|^2 \quad \text{for all} \quad x, y \in H;$$

(ii) *β-Lipschitz continuous* if there exists a constant $\beta > 0$ such that

$$\|g(x) - g(y)\| \leq \beta\|x - y\| \quad \text{for all} \quad x, y \in H.$$

Mann type perturbed iterative algorithm (MTPIA)

Let $T, M : H \to \mathcal{F}(H)$ and $g, m : H \to H$. Given $u_0 \in H$, the iterative sequences $\{u_n\}$, $\{x_n\}$, and $\{y_n\}$ are defined by

$$u_{n+1} = (1-\alpha_n)u_n+\alpha_n[u_n-(g-m)(u_n)+J_\eta^{\phi_n}((g-m)(u_n)-\eta(x_n-y_n))]+e_n,$$

$$x_n \in (T(u_n))_r, \quad \text{and} \quad y_n \in (M(u_n))_r, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a real sequence satisfying $\alpha_0 = 1$, $0 \leq \alpha_n \leq 1$ for $n > 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$; $e_n \in H$ for all n is an error which is taken into account for a possible inexact computation of the proximal point; $\{\phi_n\}$ is the sequence approximating ϕ and $\eta > 0$ is a constant.

Ishikawa type perturbed iterative algorithm (ITPIA)

Let $T, M : H \to \mathcal{F}(H)$ and $g, m : H \to H$. Given $u_0 \in H$, the iterative sequences $\{u_n\}$, $\{x_n\}$, and $\{y_n\}$ are defined by

$$u_{n+1} = (1-\alpha_n)u_n+\alpha_n[v_n-(g-m)(v_n)+J_\eta^{\phi_n}((g-m)(v_n)-\eta(\bar{x}_n-\bar{y}_n))]+e_n,$$

$$v_n = (1-\beta_n)u_n+\beta_n[u_n-(g-m)(u_n)+J_\eta^{\phi_n}((g-m)(u_n)-\eta(x_n-y_n))]+\beta_n\gamma_n,$$

for $n \geq 0$, where $\bar{x}_n \in (T(v_n))_r$, $\bar{y}_n \in (M(v_n))_r$, $x_n \in (T(u_n))_r$, $y_n \in (M(u_n))_r$; e_n and γ_n in H for all $n \geq 0$ are errors; $\{\phi_n\}$ is the sequence approximating ϕ ; $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying $\alpha_0 = 1$, $0 \leq \alpha_n, \beta_n \leq 1$ for $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\eta > 0$ is a constant.

Next, we review some definitions and results which are needed in the sequel.

LEMMA 3 2 [7] *Let ϕ be a proper convex lower semicontinuous function. Then $J_\eta^\phi = (I + \eta\partial\phi)^{-1}$ is nonexpansive, i.e.,*

$$\|J_\eta^\phi(u) - J_\eta^\phi(v)\| \leq \|u - v\| \quad \textit{for all} \quad u, v \in H.$$

We remark that if $\phi = \delta_K$, the indicator function of a nonempty closed convex K in H, then $J_\eta^\phi(u) = P_K(u)$ for each $u \in H$ and $\eta > 0$ where P_K is the projection mapping of H onto K.

Several special cases of ITPIA are listed below.

(i) If $\beta_n = 0$ for all $n \geq 0$, ITPIA reduces to MTPIA.

(ii) If $\phi_n = \delta_K$ for all $n \geq 0$, the indicator function of a nonempty closed convex set K in H, $e_n = r_n = 0$ for all $n \geq 0$, $\beta_n = 0$ for all $n \geq 0$, $g = I$, the identity mapping, and m is a zero mapping, then ITPIA reduces to the Algorithm in Theorem 3.4 of Park and Jeong [9].

(iii) If $\phi_n = \delta_K$ for all $n \geq 0$, $g = I$, the identity mapping, $e_n = r_n = 0$ for all $n \geq 0$ and $\beta_n = 0$ for all $n \geq 0$, then ITPIA reduces to the Algorithm in Theorem 3.5 of Park and Jeong [9].

4. Existence and convergence result

In this section, we prove the existence of a solution of FGQVIP (2.1) and discuss the convergence criteria of ITPIA.

THEOREM 4.1 *Let $T : H \to \mathcal{F}(H)$ be F-Lipschitz continuous and F-strongly monotone, $M : H \to \mathcal{F}(H)$ be F-Lipschitz continuous, $(g - m) : H \to H$ be α-strongly monotone, and $g, m : H \to H$ be β-Lipschitz continuous and ξ-Lipschitz continuous, respectively. Assume that*

$$(4.1) \qquad < m(v) - m(u), g(u) - g(v) > \leq \lambda \|u - v\|^2, \quad \forall u, v \in H$$

for some constant λ such that $\lambda_0 \leq \lambda \leq \beta\xi$, where

$$\lambda_0 = \inf\{A :< m(v) - m(u), g(u) - g(v) > \leq A\|u - v\|^2, \ \forall u, v \in H\}.$$

If there exists a constant $\eta > 0$ such that $\rho\eta < 1 - k$,

$$(4.2) \qquad \frac{1}{\eta}\{1 - [1 - (k + \eta\rho)]^2 + \eta^2\rho^2\} < 2\delta < \frac{1}{\eta} + \rho^2\eta,$$

and $k = 2\sqrt{1 - 2\alpha + \xi^2 + \beta^2 + 2\lambda} < 1$, then (u^*, x^*, y^*) is a solution of FGQVIP (2.1). Moreover, if

$$\lim_{n\to\infty} \|J_\eta^{\phi_n}(v) - J_\eta^{\phi}(v)\| = 0, \ \forall v \in H,$$

and $\{u_n\}$, $\{\bar{x}_n\}$, *and* $\{\bar{y}_n\}$ *are defined by ITPIA with conditions*

(i) $\lim_{n\to\infty}\|e_n\| = 0 = \lim_{n\to\infty}\|r_n\|$,

(ii) $\sum_{i=0}^{n}\prod_{j=i+1}^{n}(1-\alpha_j(1-c))$ *converges*, $0 \le c < 1$, *then* $\{u_n\}$, $\{\bar{x}_n\}$, *and* $\{\bar{y}_n\}$ *strongly converge to* u^*, x^*, *and* y^*, *respectively.*

PROOF First we prove that the FGQVIP (2.1) has a solution (u^*, x^*, y^*) By Lemma 3.1, it is enough to show that the mapping $F : H \to 2^H$ defined by (3.1) has a fixed point u^*. For any $u, v \in H$, $p \in F(u)$, and $q \in F(v)$, there exist $x_1 \in (T(u))_r$, $x_2 \in (T(v))_r$, $y_1 \in (M(u))_r$, and $y_2 \in (M(v))_r$ such that

$$p = u - (g-m)(u) + J_\eta^\phi((g-m)(u) - \eta(x_1 - y_1))$$

and

$$q = v - (g-m)(v) + J_\eta^\phi((g-m)(v) - \eta(x_2 - y_2)).$$

By Lemma 3.2, we have

$$\begin{aligned} \|p-q\| &\le \|u - v - ((g-m)(u) - (g-m)(v))\| \\ &\quad + \|(g-m)(u) - (g-m)(v) - \eta(x_1 - x_2) + \eta(y_1 - y_2)\| \\ &\le 2\|u - v - ((g-m)(u) - (g-m)(v))\| \\ &\quad + \|u - v - \eta(x_1 - x_2)\| + \eta\rho\|u - v\|. \end{aligned} \tag{4.3}$$

By using the Lipschitz continuity of g and m, the strong monotonicity of $(g-m)$, and (4.1), we obtain

$$\begin{aligned} &\|u - v - ((g-m)(u) - (g-m)(v))\|^2 \\ &= \|u-v\|^2 - 2 < u - v, (g-m)(u) - (g-m)(v) > \\ &\quad + \|(g-m)(u) - (g-m)(v)\|^2 \\ &= \|u-v\|^2 - 2 < u - v, (g-m)(u) - (g-m)(v) > \\ &\quad + \|m(u) - m(v)\|^2 + \|g(u) - g(v)\|^2 \\ &\quad + 2 < m(v) - m(u), g(u) - g(v) > \\ &\le (1 - 2\alpha + \xi^2 + \beta^2 + 2\lambda)\|u-v\|^2. \end{aligned} \tag{4.4}$$

By using F-Lipschitz continuity and F-strong monotonicity of T, we have

$$\begin{aligned}
&\|u - v - \eta(x_1 - x_2)\|^2 \\
&\quad = \|u - v\|^2 - 2\eta < u - v, x_1 - x_2 > + \eta^2 \|x_1 - x_2\|^2 \\
&\quad \leq \|u - v\|^2 - 2\eta\delta\|u - v\|^2 + \eta^2\rho^2\|u - v\|^2 \\
&\quad = (1 - 2\eta\delta + \eta^2\rho^2)\|u - v\|^2. \qquad (4.5)
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
D(F(u), F(v)) &\leq \{2\sqrt{1 - 2\alpha + \xi^2 + \beta^2 + 2\lambda} \\
&\quad + \sqrt{1 - 2\eta\delta + \eta^2\rho^2} + \eta\rho\}\|u - v\| \\
&= \{k + t(\eta) + \eta\rho\}\|u - v\| \\
&= \theta\|u - v\|, \qquad (4.6)
\end{aligned}$$

where $k = 2\sqrt{1 - 2\alpha + \xi^2 + \beta^2 + 2\lambda}$, $t(\eta) = \sqrt{1 - 2\eta\delta + \eta^2\rho^2}$, and $\theta = k + t(\eta) + \eta\rho$. By condition (4.2), we see that $0 < \theta < 1$. It follows from (4.6) and Theorem 3.1 of Siddiqi and Ansari [13] that F has a fixed point $u^* \in H$. Hence by Lemma 3.1, there exist $x^* \in (T(u^*))_r$, and $y^* \in (M(u^*))_r$ such that (u^*, x^*, y^*) is a solution of FGQVIP (2.1).

Next we prove that the iterative sequences $\{u_n\}$, $\{\bar{x}_n\}$, and $\{\bar{y}_n\}$ defined by ITPIA strongly converge to u^*, x^*, and y^*, respectively.

Since FGQVIP (2.1) has a solution (u^*, x^*, y^*) then, by Lemma 3.1, we have

$$u^* = u^* - (g - m)(u^*) + J_\eta^\phi((g - m)(u^*) - \eta(x^* - y^*)).$$

By making use of the same arguments used for obtaining (4.4) and (4.5), we get

$$\|u_n - u^* - ((g-m)(u_n) - (g-m)(u^*))\| \leq \sqrt{1 - 2\alpha + \xi^2 + \beta^2 + 2\lambda}\|u_n - u^*\|$$

$$\|u_n - u^* - \eta(x_n - x^*)\| \leq \sqrt{1 - 2\eta\delta + \eta^2\delta^2}\|u_n - u^*\|,$$

$$\|v_n - u^* - ((g-m)(v_n) - (g-m)(u^*))\| \leq \sqrt{1 - 2\alpha + \xi^2 + \beta^2 + 2\lambda}\|v_n - u^*\|,$$

and

$$\|v_n - u^* - \eta(\bar{x}_n - x^*)\| \leq \sqrt{1 - 2\eta\delta + \eta^2\delta^2}\|v_n - u^*\|.$$

By setting

$$h(u^*) = (g - m)(u^*) - \eta(x^* - y^*)$$

and

$$h(v_n) = (g - m)(v_n) - \eta(\bar{x}_n - \bar{y}_n),$$

we have

$$\begin{aligned} \|u_{n+1} - u^*\| &\leq (1 - \alpha_n)\|u_n - u^*\| \\ &\quad + \alpha_n\|v_n - u^* - ((g - m)(v_n) - (g - m)(u^*))\| \\ &\quad + \alpha_n\|J_\eta^{\phi_n}(h(v_n)) - J_\eta^{\phi}(h(u^*))\| + \|e_n\|. \end{aligned} \tag{4.7}$$

Now, since $J_\eta^{\phi_n}$ is nonexpansive, we have

$$\begin{aligned} &\|J_\eta^{\phi_n}(h(v_n)) - J_\eta^{\phi}(h(u^*))\| \\ &\quad \leq \|h(v_n) - h(u^*)\| + \|J_\eta^{\phi_n}(h(u^*)) - J_\eta^{\phi}(h(u^*))\| \\ &\quad \leq \|v_n - u^* - ((g - m)(v_n) - (g - m)(u^*))\| \\ &\quad + \|v_n - u^* - \eta(\bar{x}_n - x^*)\| \\ &\quad + \eta\|\bar{y}_n - y^*\| + \|J_\eta^{\phi_n}(h(u^*)) - J_\eta^{\phi}(h(u^*))\| \\ &\quad \leq \sqrt{1 - 2\alpha + \xi^2 + \beta^2 + 2\lambda}\|v_n - u^*\| \\ &\quad + \sqrt{1 - 2\eta\delta + \eta^2\delta^2}\|v_n - u^*\| + \eta\rho\|v_n - u^*\| \\ &\quad + \|J_\eta^{\phi_n}(h(u^*)) - J_\eta^{\phi}(h(u^*))\|. \end{aligned} \tag{4.8}$$

On combining (4.7) and (4.8) and using the F-Lipschitz continuity of M, we get

$$\begin{aligned} &\|u_{n+1} - u^*\| \\ &\quad \leq (1 - \alpha_n)\|u_n - u^*\| + \alpha_n[2\sqrt{1 - 2\alpha + \xi^2 + \beta^2 + 2\lambda} \\ &\quad + \sqrt{1 - 2\eta\delta + \eta^2\delta^2} + \eta\rho]\|v_n - u^*\| \\ &\quad + \alpha_n\|J_\eta^{\phi_n}(h(u^*)) - J_\eta^{\phi}(h(u^*))\| + \|e_n\| \\ &\quad = (1 - \alpha_n)\|u_n - u^*\| + \alpha_n\theta\|v_n - u^*\| \\ &\quad + \alpha_n\varepsilon_n + \|e_n\|, \end{aligned} \tag{4.9}$$

where $\theta = 2\sqrt{1-2\alpha+\xi^2+\beta^2+2\lambda}+\sqrt{1-2\eta\alpha+\eta^2\beta^2}+\eta\rho$ and $\varepsilon_n = \|J_\eta^{\phi_n}(h(u^*)) - J_\eta^{\phi}(h(u^*))\|$.

Next

$$\begin{aligned}\|v_n-u^*\| &\le (1-\beta_n)\|u_n-u^*\| \\ &\quad + \beta_n\|u_n-u^*-((g-m)(u_n)-(g-m)(u^*))\| \\ &\quad + \beta_n\|J_\eta^{\phi_n}(h(u_n)) - J_\eta^{\phi}(h(u^*))\| + \beta_n\|r_n\|. \end{aligned} \tag{4.10}$$

By making use of the same arguments used for obtaining (4.8), we get

$$\begin{aligned}&\|J_\eta^{\phi_n}(h(u_n)) - J_\eta^{\phi}(h(u^*))\| \\ &\quad \le \|h(u_n)-h(u^*)\| + \|J_\eta^{\phi_n}(h(u^*)) - J_\eta^{\phi}(h(u^*))\| \\ &\quad \le \|u_n-u^*-((g-m)(u_n)-(g-m)(u^*))\| \\ &\quad + \|u_n-u^*-\eta(x_n-x^*)\| \\ &\quad + \eta\|y_n-y^*\| + \|J_\eta^{\phi_n}(h(u^*)) - J_\eta^{\phi}(h(u^*))\| \\ &\quad \le [\sqrt{1-2\alpha+\xi^2+\beta^2+2\lambda}+\sqrt{1-2\eta\delta+\eta^2\delta^2} \\ &\quad + \eta\rho]\|u_n-u^*\| + \varepsilon_n. \end{aligned} \tag{4.11}$$

On combining (4.10) and (4.11), we get

$$\begin{aligned}\|v_n-u^*\| &\le (1-\beta_n)\|u_n-u^*\| + \beta_n\theta\|u_n-u^*\| \\ &\quad + \beta_n\varepsilon_n + \beta_n\|r_n\| \\ &= (1-\beta_n(1-\theta))\|u_n-u^*\| + \beta_n(\varepsilon_n+\|r_n\|) \\ &\le \|u_n-u^*\| + \beta_n(\varepsilon_n+\|r_n\|) \end{aligned} \tag{4.12}$$

since $(1-\beta_n(1-\theta)) \le 1$. On combining (4.9) and (4.12), we get

$$
\begin{aligned}
\|u_{n+1} - u^*\| &\leq (1 - \alpha_n(1-\theta))\|u_n - u^*\| + \alpha_n \varepsilon_n \\
&\quad + \theta \alpha_n \beta_n (\varepsilon_n + \|r_n\|) + \|e_n\| \\
&\leq \prod_{i=0}^{n} (1 - \alpha_i(1-\theta))\|u_0 - u^*\| \\
&\quad + \sum_{j=0}^{n} \alpha_j \prod_{i=j+1}^{n} (1 - \alpha_i(1-\theta))\varepsilon_j \\
&\quad + \theta \sum_{j=0}^{n} \alpha_j \beta_j \prod_{i=j+1}^{n} (1 - \alpha_i(1-\theta))(\varepsilon_j + \|r_j\|) \\
&\quad + \sum_{j=0}^{n} \prod_{i=j+1}^{n} (1 - \alpha_i(1-\theta))\|e_j\|, \qquad (4.13)
\end{aligned}
$$

where $\prod_{i=j+1}^{n}(1 - \alpha_i(1-\theta)) = 1$ when $j = n$.

Now, let B denote the lower triangular matrix with entries $b_{nj} = \alpha_j \prod_{i=j+1}^{n}(1 - \alpha_i(1-\theta))$. Then B is multiplicative, see Rhoades [11], so that

$$\lim_{n\to\infty} \sum_{j=0}^{n} \alpha_j \prod_{i=j+1}^{n} (1 - \alpha_i(1-\theta))\varepsilon_j = 0,$$

$$\lim_{n\to\infty} \theta \sum_{j=0}^{n} \alpha_j \beta_j \prod_{i=j+1}^{n} (1 - \alpha_i(1-\theta))(\varepsilon_j + \|r_j\|) = 0,$$

since $\lim_{n\to\infty} \|r_n\| = 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$.

Let D be the lower triangular matrix with entries $d_{nj} = \prod_{i=j+1}^{n}(1 - \alpha_i(1-\theta))$. Condition (ii) implies that D is multiplicative, and hence

$$\lim_{n\to\infty} \sum_{j=0}^{n} \prod_{i=j+1}^{n} (1 - \alpha_i(1-\theta))\|e_j\| = 0,$$

since $\lim_{n\to\infty} \|e_n\| = 0$. Also,

$$\lim_{n\to\infty} \prod_{i=0}^{n} (1 - \alpha_i(1-\theta)) = 0,$$

since $\sum_{i=0}^{n} \alpha_i = \infty$. Hence, it follows from inequality(4.13) that $\lim_{n\to\infty}\|u_{n+1} - u^*\| = 0$, i.e., the sequence $\{u_n\}$ strongly converges to u^* in H. The inequality (4.12) implies that the sequence $\{v_n\}$ also converges to u^*. Since $\bar{x}_n \in T(v_n)$, $x^* \in T(u^*)$, and T is F-Lipschitz continuous, we have

$$\|\bar{x}_n - x^*\| \leq \rho\|v_n - u^*\| \to 0 \quad \text{as} \quad n \to \infty,$$

i.e., $\{\bar{x}_n\}$ strongly converges to x^*. Similarly, we can prove that $\{\bar{y}_n\}$ strongly converges to y^*.

We remark that if $\beta_n = 0$ for all $n \geq 0$, Theorem 4.1 gives the conditions under which the sequences $\{u_n\}$, $\{x_n\}$, and $\{y_n\}$ defined by MTPIA strongly converge to u^*, x^*, and y^*, respectively.

References

[1] C Baiocchi and A. Capelo, *Variational and quasivariational inequalities*, J Wiley and Sons, New York (1984)

[2] S Chang and Y Zhu, *On variational inequalities for fuzzy mappings*, Fuzzy Sets and Systems **32** (1989), 359 - 367

[3] S C Fang and E L. Peterson, *Generalized variational inequalities*, J Optim Theory Appl. **38** (1982), 363 - 383

[4] F Giannessi and A Maugeri, *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York (1995)

[5] A Hassouni and A Moudafi, *A perturbed algorithm for variational inclusions*, J Math Anal Appl. **185** (1994), 706 - 712

[6] D Kinderlehrar and G. Stampacchia, *An introduction to variational inequalities and applications*, Academic Press, New york (1980)

[7] G Minty, *Monotone nonlinear operators in Hilbert spaces*, Duke Math J **29** (1962), 341- 346.

[8] M A Noor, *Variational inequalities for fuzzy mappings (I)*, Fuzzy Sets and Systems **55** (1993), 309 - 312

[9] J Y Park and J U Jeong, *Generalized strongly quasivariational inequalities for fuzzy mappings*, Fuzzy Sets and Systems **99(1)** (1998), 115 - 120

[10] J.Y Park and J U. Jeong, *Completely generalized strongly quswvariational inequalities for fuzzy mappings*, Fuzzy Sets and Systems **110(1)** (2000), 91 - 99

[11] B E Rhoades, *Fixed point theorems and stability results for fixed point iteration procedures*, Indian J Pure Appl Math **21** (1990), 1 - 19

[12] M.H Shin and K K Tan, *Generalized quasivariational inequalities in locally convex spaces*, J Math Anal.Appl **108** (1985), 333 - 343
[13] A H Siddiqi and Q H. Ansari, *An iterative method for generalized variational inequalities*, Math Japonica **34** (1989), 475 - 481
[14] W. Takahashi, *Nonlinear variational inequalities and fixed point theorems*, J Math Soc Japan **28** (1976), 168 - 181
[15] L A Zadeh, *Fuzzy Sets*, Inform and Control **8** (1965), 338 - 353
[16] H J Zimmermann, *Fuzzy set theory and its applications*, Kluwer Academic Dordrecht-Boston (1988).

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