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ON EXISTENCE THEOREMS FOR NONLINEAR INTEGRAL EQUATIONS IN BANACH ALGEBRAS VIA FIXED POINT TECHNIQUES

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Abstract In this paper an improved version of a fixed point theorem of the present author [3] in Banach algebras is obtained under the weaker conditions with a different method and using measure of non-compactness. The newly developed fixed point theorem is further applied to certain nonlinear integral equations of mixed type for proving the existence theorems and stability of the solution in Banach algebras

1. Introduction

Recently the present author [2, 3, 4] initiated the study of the nonlinear integral equations of mixed type in Banach algebras via fixed point techniques. The present author in [3] proved the existence theorem for the nonlinear integral equation (in short IE)of the form,

(1.1)

$$\begin{aligned} x(t) =& q(t) + \int_0^t k_1(t,s) f_1(s,x(s))ds + h(t)\int_0^a k_2(t,s) f_2(s,x(s))ds \\ &+ \left(\int_0^t k_1(t,s) f_1(s,x(s))ds\right)\left(\int_0^a k_2(t,s) f_2(s,x(s))ds\right) \end{aligned}$$

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under the mixed type of conditions, namely, Lipschitz type and compactness on the functions involved in the IE (1.1). The special cases of the IE (1.1) include Hammerstein, Fredholm and Hammerstein-Fredholm integral equations of mixed type [5,6]. Also the special cases of the IE (1.1) occur in several physical problems such as Chandrasekhar's integral equation in heat transfer [6], in some biological processes and queuing theory etc(cf.[2.6]). Therefore, it is of interest to study the integral equations of the type (1.1) for different aspects of the solution. In the present paper we study the existence and stability of certain nonlinear IE of the type (1.1) via the fixed point techniques. The following fixed point theorem is proved in Dhage [3].

THEOREM 1.1. *Let A, B and C be three operators on a nonempty, closed, convex and bounded subset S of a Banach algebra into itself such that*

(i) *A and C are contractions with contractions constants α and β respectively,*
(ii) *B is complete continuous, and*
(iii) *$AxBy + Cx \in S$, whenever $x, y \in S$.*

Then the operator equation

$$AxBx + Cx = x \tag{1.2}$$

has a solution in S, whenever $\alpha M + \beta < 1$, where

$$M = |B(S)| = \sup\{\|Bx\| : x \in S\}.$$

In the following section we shall prove an improved version of Theorem 1.1 under weaker conditions with a different method which will be further used for proving the existence and stability of solution of certain nonlinear integral equations which is more general than (1.1) in the subsequent part of this paper.

2. Fixed point theorem

Before proving the main fixed point theorem, we give some preliminaries which will be used in the sequel. Let X denote a Banach space and a the Kuratowskii [8, page 492] measure of non compactness in X defined by

$$\alpha(S) = \inf\{\gamma > 0 : S \subset \cup_{i=1}^{n} F_i, \ \ \mathrm{diam}(F_i) \le \gamma \ \text{ for all } \ i\ \} \tag{2.1}$$

for any bounded set S in X. Now we give a few properties of α:

(a) $\alpha(S) = 0 \Leftrightarrow S$ is precompact,
(b) $S_1 \subset S_2 \Rightarrow \alpha(S_1) \le \alpha(S_2)$,
(c) $\alpha(\lambda S) = |\lambda|\alpha(S)$, $\lambda \in \mathbb{R}$,
(d) $\alpha(S_1 + S_2) \le \alpha(S_1) + \alpha(S_2)$, and
(e) $\alpha(\mathrm{conv}(S)) = \alpha(\overline{S}) = \alpha(S)$,

where $\mathrm{conv}(S)$ and $\overline{S}$ denote the convex hull and closure of S respectively. The details of the Kuratowskii measure of non compactness may be found in Deimling [2] and Sadovskii [7].

DEFINITION 2.1. An operator $T : X \to X$ is said to be *ϕ–Lipschitzian* if there exists a continuous nondecreasing function $\phi_T : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\|Tx - Ty\| \le \phi_T(\|x - y\|) \tag{2.2}$$

for all $x, y \in X$. In particular if $\phi_T(\gamma) < \gamma$, $\gamma > 0$, T is called a *nonlinear contraction* on X. Further if $\phi_T(\gamma) = k\gamma$, $k > 0$, T is called a *Lipschitzian*, and moreover if $k < 1$, T is called a *contraction* with contraction constant k.

DEFINITION 2.2. A mapping T on a subset S of a Banach space X is said to be *α-condensing* if for any bounded subset F of S with $T(F)$ is bounded implies $\alpha(T(F)) < \alpha(F)$ for $\alpha(F) > 0$.

Now we state a key theorem of Sadovskii [7] which will be used in the proof of the fixed point theorem.

THEOREM 2.1 (SADOVSKII [17]). *Let S be a nonempty closed, convex and bounded subset of X and let $T : S \to S$ be a continuous and α-condensing mapping. Then T has a fixed point.*

THEOREM 2.2 *Let S be a nonempty closed, convex and bounded subset of a Banach algebra X and let $A, B, C : S \to X$ be three operators such that*

(A_1) *A and C are ϕ-Lipschitzians,*
(A_2) *B is complete continuous,*
(A_3) *$AxBx + Cx \in S$ for every $x \in S$.*

Then the operator equation (1.1) *has a solution in S whenever*

$$M\phi_A(\gamma) + M\phi_C(\gamma) < \gamma,$$

for $\gamma > 0$, where $M = \sup\{\|Bx\| : x \in S\}$.

PROOF Define a mapping T on S by

$$Tx = AxBx + Cx, \quad x \in S. \tag{2.3}$$

By (A_3), T maps S into itself and hence T is well defined. Since B is completely continuous $B(S)$ is precompact and hence bounded. Therefore, the number $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\}$ exists and so $\|Bx\| \leq M$ for all $x \in S$. We shall show that T is continuous and $\alpha-$condensing on S.

Let $x, y \in S$, by (A_1),

(2.4)
$$\begin{aligned}\|Tx - Ty\| =& \|Ax - Ay\|\|Bx\| + \|Ay\|\|Bx - By\| + \|Cx - Cy\| \\ &\leq M\phi_A(\|x - y\|) + \phi_C(\|x - y\|) + \|A(S)\|\|Bx - By\|.\end{aligned}$$

Now S is bounded, so by (A_1),

$$\|Ax\| \leq \|Ax_0\| + \|Ax - Ax_0\| \leq \|Ax_0\| + \phi_A(\text{diam}(S)) \tag{2.5}$$

for $x, x_0 \in S$, and so

$$\|A(S)\| = \sup\{\|Ax\| : x \in S\} < \infty.$$

First we show that T is a continuous mapping on S. For, let $\{x_n\}$ be a sequence in S such that $x_n \to x$ as $n \to \infty$. We prove that

$Tx_n \to Tx$ as $n \to \infty$. Since the operator B is completely continuous, it is continuous and so $Bx_n \to Bx$ as $n \to \infty$.

Now from inequality (2.4), we obtain

$$
\begin{aligned}
\|Tx_n - Tx\| &\le M\phi_A(\|x_n - x\|) + \phi_C(\|x_n - x\|) \\
&\quad + \|A(S)\|\|Bx_n - Bx\| \to 0 \\
&\quad as \;\; n \to \infty \; (\;\; since \; \|A(S)\| < \infty).
\end{aligned} \tag{2.6}
$$

This shows that the mapping T is continuous on S Again let $\epsilon > 0$ be given and $F \subset S$. Then there exists subsets $F_1, F_2, \cdots, F_n$ of F such that $F = \cup_{i=1}^n F_i$, and $\mathrm{diam}(F) < \alpha(F) + \epsilon$ for each $i = 1, 2, \cdots, n$. Since B is completely continuous, $B(F)$ is precompact and hence $\alpha(B(F)) < \frac{\delta}{\|A(S)\|}$, where $\delta > 0$ is arbitrary number. There exist subsets $G_1, G_2, \cdots, G_n$, of $B(F)$ such that $B(F) = \cup_{j=1}^m G_j$, and $\mathrm{diam}(G_j) < \frac{\delta}{\|A(S)\|}$ for all $j = 1, 2, \cdots, m$. Then

$$F = \cup_{j=1}^m B^{-1}(G_j)$$

and so

$$T(F) = \cup_{i=1}^n \cup_{j=1}^m T(F_i \cap B^{-1}(G_j)). \tag{2.7}$$

Now by definition of α, we get

$$\mathrm{diam}(F_i \cap B^{-1}(G_j)) < \alpha(F) + \epsilon \quad \text{for all } \; i = 1, 2, \cdots, n \, . \, j = 1, 2, \cdots, m. \tag{2.8}$$

From the inequalities (2.4)-(2.5), (2 7)-(2.8), we obtain,

$$\mathrm{diam}(T(F_i \cap B^{-1}(G_j))) < \delta + \max\{M\phi_A(\gamma) + \phi_C(\gamma) : \gamma \in [0, \alpha(F) + \epsilon]\}. \tag{2.9}$$

Since ϕ_A and ϕ_C are nondecreasing in $\mathbb{R}_+$, the maximum on the right hand side of inequality (2.9) is attained at $\gamma = \alpha(F) + \epsilon$

But we know

$$\alpha(T(F)) = \max_{i,j} \mathrm{diam} T(F_i \cap B^{-1}(G_j)) \tag{2.10}$$

and so from (2.9), we get

$$\alpha(T(F)) < \delta + \max\{M\phi_A(\gamma) + \phi_C(\gamma) : \gamma \in [0, \alpha(F) + \epsilon]\}.$$

Since ϵ and δ are arbitrary positive numbers, we have

$$\alpha(T(F)) = M\phi_A(\alpha(F)) + \phi_C(\alpha(F)) < \alpha(F) \tag{2.11}$$

for $\alpha(F) > 0$. This shows that T is a α-condensing on S. Now an application of Theorem 2.1 yields that the operator equation $Tx = x$ has a solution in S and consequently the operator equation (1.2) has a solution in S. This completes the proof.

REMARK 2.1. From hypothesis (A_1) of Theorem 2.2 we note that the operators A, B and C need not map the domain set S into itself and every contraction is ϕ-Lipschitzian. Therefore Theorem 2.2 is a generalization of Theorem 1.1 under the weaker hypothesis (i) and (iii) thereof.

When $C = 0$ in Theorem 2.2, we obtain the following result as a corollary which is again new and includes the main fixed point Theorems of Dhage [2,4] as special cases under weaker conditions

COROLLARY 2.1. *Let S be a nonempty, closed, convex and bounded subset of the Banach algebra X and let $A, B : S \to X$ be two operators such that*

(i) *A is ϕ-Lipschitzian,*
(ii) *B is completely continuous, and*
(iii) *$AxBx \in S$ for every $x \in S$.*

Then the operator equation

$$AxBx = x \tag{2.12}$$

has a solution in S whenever

$$M\phi_A(\gamma) < \gamma \quad \textit{for} \quad \gamma > 0,$$

where $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\}$.

In the following section we prove the existence and stability of the solution of certain nonlinear integral equations in Banach algebra which are more than (1.1) and by the application of Theorem 2.2.

3. Nonlinear integral equations

Let $\mathbb{R}$ denote the real line and let $I = [t_0, t_0 + a] \subset \mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $a > 0$, be a closed and bounded interval. Let E denote a Banach algebra with a norm $\|\cdot\|_E$. Now we consider the nonlinear IE

$$x(t) = q(t) + \int_{t_0}^{t} f(t, s, x(s))ds + \left(\int_{t_0}^{t} g(t, s, x(s))ds\right)\left(\int_{t_0}^{t_0+a} k(t, s, x(s))ds\right) \tag{3.1}$$

for $t \in I$, where $q : I \to E$ and $f, g, k : I \times I \times E \to E$ are continuous.

We seek the solution of the IE (3.1) in the space $X = BM(I, E)$ of all bounded and measurable E-valued functions on I. Define a norm $\|\cdot\|$ in $X = BM(I, E)$ by

$$\|x\| = \sup_{t \in I} \|x(t)\|_E. \tag{3.2}$$

Clearly X is a Banach algebra with this supremum norm. To prove the existence theorem for the IE (3.1), we need the following hypotheses:

(H_0) $q \in BM(I, E)$ and the functions $f(t, s, x)$, $g(t, s, x)$ and $k(t, s, x)$ are measurable in $t, s \in I$ for all $x \in E$.

(H_1) The functions f, g and k are bounded on $I \times I \times E$ with bounds K_1, K_2 and K_3 respectively.

(H_2) The functions $f(t, s, x)$, $g(t, s, x)$ and $k(t, s, x)$ satisfy the Lipschitz condition in x uniformly for $t, s \in I$, i.e. there exists constants $L_1 > 0$ and $L_2 > 0$ such that

$$\|f(t, s, x) - f(t, s, y)\|_E \le L_1 \|x - y\|_E$$
$$\|g(t, s, x) - g(t, s, y)\|_E \le L_2 \|x - y\|_E \quad \text{and}$$
$$\|k(t, s, x) - f(t, s, y)\|_E \le L_1 \|x - y\|_E$$

for all $(t,s,x),(t,s,y) \in I \times I \times E$.

(H_3) Given $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$\|k(t_1,s,x) - k(t_2,s,x)\|_E < \epsilon,$$

whenever $(t_1,s,x),(t_2,s,x) \in I \times I \times E$ and $|t_1 - t_2| < \delta$.

THEOREM 3 1 *Assume* (H_0)-(H_3). *Then IE* (3.1) *admits a solution, whenever* $(L_1 + L_2K_3a)a < 1$.

PROOF. Define a subset S of the Banach algebra X by

$$S = \{x \in X : \|x\| \leq K\}, \tag{3.3}$$

where $K = \|q\| + (K_1 + K_2K_3a)a$. Clearly S is a nonempty, closed, convex and bounded subset of the Banach algebra X. We define three operators A, B and C on S by

$$Ax(t) = \int_{t_0}^{t} g(t,s,x(s))ds, \quad t \in I, \tag{3.4}$$

$$Bx(t) = \int_{t_0}^{t_0+a} k(t,s,x(s))ds, \quad t \in I, \tag{3.5}$$

and

$$Cx(t) = q(t) + \int_{t_0}^{t} f(t,s,x(s))ds, \quad t \in I. \tag{3.6}$$

In view of the hypotheses (H_0) and (H_1), it follows that the operators A, B and C are well defined and Ax, Bx, $Cx \in BM(I,E)$ for all $x \in S$.

Now for any $x \in S$, by (H_1) we have

$$\begin{aligned}\|Ax(t)Bx(t) + Cx(t)\|_E &\leq \|Ax(t)\|_E\|Bx(t)\|_E + \|Cx(t)\|_E \\ &\leq \left(\int_{t_0}^{t} \|g(t,s,x(s))\|_E ds\right)\left(\int_{t_0}^{t_0+a} \|k(t,s,x(s))\|_E\right. \\ &\quad + \|q(t)\|_E + \int_{t_0}^{t} \|f(t,s,x(s))\|ds \\ &\leq \|q\| + (K_1 + K_2K_3a)a = K.\end{aligned}$$

This shows that $AxBx + Cx \in S$ for all $x \in S$. Again let $x, y \in S$, then by (H_2), one gets

$$\|Ax(t) - Ay(t)\|_E \leq \int_{t_0}^{t} \|g(t,s,x(s)) - g(t,s,y(s))\|_E ds = \phi_A(\|x-y\|),$$

where $\phi_A(\gamma) = L_2 a\gamma$.

Similarly, we have

$$\|Cx - Cy\| \leq \phi_C(\|x-y\|),$$

where $\phi_C(\gamma) = L_1 a\gamma$.

Further by (H_1), we obtain

$$\begin{aligned} M = \|B(S)\| &= \sup\{\|Bx\| : x \in S\} \\ &\leq \sup\left\{\int_{t_0}^{t_0+a} \|k(t,s,x(s))\|_E ds : x \in S\right\} \\ &\leq K_3 a. \end{aligned} \tag{3.7}$$

This shows that the set $B(S)$ is uniformly bounded. Next we show that the set $B(S)$ is equi-continuous on I. Let $x \in S$ be any element.

Then by (H_3), we have

$$\|Bx(t_1) - Bx(t_2)\|_E \leq \int_{t_0}^{t_0+a} \|k(t_1,s,x(s)) - k(t_2,s,y(s))\|_E ds \to 0$$
$$as \;\; t_1 \;\to\; t_2.$$

Therefore $B(S)$ is an equi-continuous set. Further the hypothesis (H_2) implies that B is continuous operator on S. Consequently B is completely continuous on S.

Now $M\phi_A(\gamma) + \phi_C(\gamma) = (L_1 + L_2 k_3 a)a\gamma < \gamma$ for $\gamma > 0$

Thus all the conditions of Theorem 2.2 are satisfied and hence an application of it yields that the operator equation $AxBx + Cx = x$ has a solution in S Consequently that IE (3.1) has a solution on I. This completes the proof.

Next we prove the stability of the solution of the IE (3.1) in the sense of the following definition

DEFINITION 3 1 The IE (3.1) is said to *have a stable solution* p on I if for $\epsilon > 0$ there exists $\delta > 0$ such that $\|p(t)\|_E < \epsilon$ whenever $\|q(t)\|_E < \delta$ for all $t \in I$.

We need the following hypotheses in the sequel.

(H_4) The functions $f(t,s,x)$ and $g(t,s,x)$ satisfy Lipschitz type conditions as follows: For $\epsilon > 0$, there exists a $\delta > 0$, $2a\delta < 1$, such that

$$\|f(t,s,x) - f(t,s,y)\|_E \leq \delta\|x-y\|_E$$

and

$$\|g(t,s,x) - g(t,s,y)\|_E \leq \delta\|x-y\|_E$$

for all $(t,s,x),(t,s,y) \in I \times I \times E$ with $\|x\|_E \leq \epsilon$, $\|y\|_E \leq \epsilon$.

(H_5) $f(t,s,0) = 0$ and $g(t,s,0) = 0$ for all $t,s \in I$, where 0 is the zero element of E.

(H_6) The assumptions (H_3) and the function k is bounded on $I \times I \times E$ with bound K_a and $K_3 a \leq 1$.

(H_7) The function $k(t,s,x)$ satisfies the Lipschitz condition in x uniformly for $t,s \in I$, i.e. there exists a constant $L_3 > 0$ such that

$$\|k(t,s,x) - k(t,s,y)\|_E \leq \mathrm{L}_3\|x-y\|_E$$

for all $(t,s,x),(t,s,y) \in I \times I \times E$.

THEOREM 3 2. *Assume* (H_0), (H_4)-(H_7). *Then IE* (3.1) *admits a stable solution on* I.

PROOF. Let $\epsilon > 0$ be given and δ be chosen as in (H_4). Let $b = 1 - 2a\delta$. Then define a subset $\overline{S}(\epsilon)$ of the Banach algebra X by

$$\overline{S}(\epsilon) = \{x \in X : \|x\| \leq \epsilon\}. \tag{3.8}$$

Clearly $\overline{S}(\epsilon)$ is a nonempty, closed, convex and bounded subset of the Banach algebra X. We define three operators A, B and C on $\overline{S}(\epsilon)$

by (3.4), (3.5) and (3.6) respectively. Now for any $x \in \overline{S}(\epsilon)$, we have

$$\begin{aligned}\|Ax(t)Bx(t) + Cx(t)\|_E &\leq \|Ax(t)\|_E\|Bx(t)\|_E + \|Cx(t)\|_E \\ &\leq \left(\int_{t_0}^{t} \|g(t,s,x(s))\|_E ds\right)\left(\int_{t_0}^{t_0+a} \|k(t,s,x(s))\|_E d\right. \\ &\quad + \|q(t)\|_E + \int_{t_0}^{t} \|f(t,s,x(s))\|_E ds \\ &\leq \delta a\|x\|K_3 a + b\epsilon + \delta a\|x\| = \epsilon.\end{aligned}$$

This shows that $AxBx + Cx \in \overline{S}(\epsilon)$ for all $x \in \overline{S}(\epsilon)$. Again as in the proof of Theorem 3.1, it can be shown that B is completely continuous and $\|B\overline{S}(t)\| = K_3 a \leq 1$. After simple computation, we get

$$\phi_A(\gamma) = \delta a \gamma = \frac{1-b}{2}\gamma$$

and

$$\phi_C(\gamma) = \delta a \gamma = \frac{1-b}{2}\gamma.$$

Since $K_3 a \leq 1$, $M\phi_A(\gamma) + \phi_C(\gamma) = 1 - b < 1$. Now an application of Theorem 2.1 yields that the opertor equation $AxBx + Cx = x$ has a solution in $\overline{S}(\epsilon)$. Consequently the IE (3.1) has a stable solution on I, i.e. there is a solution p of the IE (3.1) such that $\|p(t)\|_E \leq \epsilon$ whenever $\|q(t)\|_E \leq b\epsilon$ for some $b \in (0,1)$. This completes the proof.

THEOREM 3.3. *Assume* (H_0), (H_4)-(H_7). *Assume also that g is bounded on $I \times I \times E$ with bound K_2. Then IE* (3.1) *admits a unique stable solution on I, whenever $\|q(t)\|_E \leq b\epsilon$ and $L_3 \epsilon K_2 a^2 < b$, where $b = 1 - 2a\delta$.*

PROOF By Theorem 3.2, the IE (3.1) has a stable solution on I, whenever $\|q(t)\|_E \leq b\epsilon$ for some $b \in (0,1)$. To prove the uniqueness, let y be another stable solution of the IE (3.1) on I. Then by (H_4)-(H_7), we obtain

$$\begin{aligned}\|x(t)-y(t)\|_E \leq & \left(\int_{t_0}^{t} \|g(t,s,x(s))-g(t,s,y(s))\|_E ds\right) \\ & \left(\int_{t_0}^{t_0+a} \|k(t,s,x(s))\|_E ds\right) \\ & + \left(\int_{t_0}^{t} \|g(t,s,x(s))\|_E ds\right) \\ & \left(\int_{t_0}^{t_0+a} \|k(t,s,x(s))-k(t,s,y(s))\|_E ds\right) \\ & + \int_{t_0}^{t} \|f(t,s,x(s))-f(t,s,y(s))\|_E ds \\ \leq & \delta a\|x-y\|K_3 a + K_2 L_3 \epsilon a^2 \|x-y\| + \delta a\|x-y\| \\ = & [2a\delta + K_2 L_3 \epsilon a^2]\|x-y\| \\ = & [1-(b-L_3\epsilon a^2)]\|x-y\|,\end{aligned}$$

which is possible only when $x(t) = y(t)$ for all $t \in I$. Since $1-(b-L_3\epsilon a^2) < 1$, this completes the proof.

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