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INJECTIVE PROPERTY OF LAURENT POWER SERIES MODULE

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ABSTRACT Northcott and McKerrow proved that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-1}]$ is an injective left $R[x]$-module. Park generalized Northcott and McKerrow's result so that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-S}]$ is an injective left $R[x^S]$-module, where S is a submonoid of $\mathbb{N}$ ($\mathbb{N}$ is the set of all natural numbers) In this paper we extend the injective property to the Laurent power series module so that if R is a ring and E is an injective left R-module, then $E[[x^{-1}, x]]$ is an injective left $R[x^S]$-module

1. Introduction

Northcott [3] considered the module $k[x^{-1}]$ of inverse polynomial over the polynomial ring $k[x]$ (with k a field), and Northcott and McKerrow [1] proved that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-1}]$ is an injective left $R[x]$-module. In [6] Park generalized Northcott and McKerrow's result so that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-S}]$ is an injective left $R[x^S]$-module, where S is a submonoid of $\mathbb{N}$ ($\mathbb{N}$ is the set of all natural numbers) In this paper we extend the injective property to the Laurent power series module so that if R is a ring and E is an injective left R-module, then $E[[x^{-1}, x]]$ is an injective left

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$R[x^S]$-module. Inverse polynomial modules were developed in [4], [5] and recently in [2], [7].

DEFINITION 1.1. Let $S = \{0, k_1, k_2, k_3, \cdots\}$ be a submonoid of the natural numbers and M be a left R-module, then the Laurent power series $M[[x^{-1}, x]]$ is a left $R[x^S]$-module such that

$$r(\cdots + m_i x^i + \cdots + m_1 x + m_0 + n_1 x^{-1} + \cdots + n_i x^{-i} + \cdots)$$
$$= \cdots + r m_i x^i + \cdots + r m_1 x + r m_0 + r n_1 x^{-1} + \cdots + r n_i x^{-i} + \cdots$$

and such that

$$x^{k_i}(\cdots + m_j x^j + \cdots + m_1 x + m_0 + n_1 x^{-1} + \cdots + n_j x^{-j} + \cdots)$$
$$= \cdots + m_j x^{j+k_i} + \cdots + m_1 x^{1+k_i} + m_0 x^{k_i} + n_1 x^{-1+k_i} + \cdots + n_j x^{-j+k_i} + \cdots .$$

LEMMA 1.2. *Let $T = \{x_i | i \in S\} \subset R[x^S]$ (S is submonoid of the set of all natural numbers $\mathbb{N}$), then the localization $T^{-1}R[x^S]$ is a flat $R[x^S]$-module.*

PROOF. Assume $f : M'[x^{-S}] \to M[x^{-S}]$ is monic. We must show $1 \otimes f : T^{-1}M'[x^{-S}] \to T^{-1}M[x^{-S}]$ is monic,

$$\text{where} \quad \begin{cases} T^{-1}M'[x^{-S}] = T^{-1}R[x^S] \otimes_{R[x^S]} M'[x^{-S}] \\ T^{-1}M[x^{-S}] = T^{-1}R[x^S] \otimes_{R[x^S]} M[x^{-S}]. \end{cases}$$

Let

$$\begin{aligned}(1 \otimes f)&(t^{-1} \otimes m_0 + m_1 x^{-k_1} + \cdots + m_j x^{-k_j}) \\ &= t^{-1} \otimes f(m_0 + m_1 x^{-k_1} + \cdots + m_j x^{-k_j}) \\ &= 0 \qquad \text{in} \quad T^{-1}M[x^{-S}].\end{aligned}$$

Then multiplying by t gives

$$1 \otimes f(m_0 + m_1 x^{-k_1} + \cdots + m_j x^{-k_j}) = 0.$$

Since T is multiplicative closed , $T = \bar{T}$ and

$$\ker\theta_{M[x^{-S}]} = \{m_0 + m_1x^{-k_1} + \cdots + m_jx^{-k_j} \in M[x^{-S}] \mid \sigma(m_0 + m_1x^{-k_1} + \cdots + m_jx^{-k_j}) = 0 \text{ for some } \sigma \in T\},$$

there exists $\sigma \in T$ with $\sigma f(m_0 + m_1x^{-k_1} + \cdots + m_jx^{-k_j}) = 0$.
But

$$\sigma f(m_0 + m_1x^{-k_1} + \cdots + m_jx^{-k_j}) = f(\sigma(m_0 + m_1x^{-k_1} + \cdots + m_jx^{-k_j})).$$

Since f is monic, $\sigma(m_0 + m_1x^{-k_1} + \cdots + m_jx^{-k_j}) = 0$ (where σ is a unit in $T^{-1}R[x^S]$). However

$$\begin{aligned} 0 &= t^{-1} \otimes \sigma(m_0 + m_1x^{-k_1} + \cdots + m_jx^{-k_j}) \\ &= \sigma(t^{-1} \otimes (m_0 + m_1x^{-k_1} + \cdots + m_jx^{-k_j})), \end{aligned}$$

$$t^{-1} \otimes (m_0 + m_1x^{-k_1} + \cdots + m_jx^{-k_j}) = 0.$$

Therefore, $1 \otimes f$ is monic.

THEOREM 1 3. *Let R be a ring. The localization $T^{-1}R[x^S]$ and the Laurent power series module $R[x^{-1}, x]$ are isomorphic as $R[x^S]$-modules.*

PROOF Let $T = \{x^i \mid i \in S\} \subset R[x^S]$ and $S = \{0, k_1, k_2, \cdots\}$. Define $\phi : T^{-1}R[x^S] \longrightarrow R[x^{-1}, x]$ by

$$\begin{aligned} &\phi(r_0 + r_1x^{k_1} + \cdots + r_ix^{k_i}/x^{k_j}) \\ &= r_0x^{-k_j} + r_1x^{k_1-k_j} + \cdots + r_ix^{k_i-k_j} \in R[x^{-1}, x]. \end{aligned}$$

Let $r_0 + r_1x^{k_1} + \cdots + r_ix^{k_i}/x^{k_j}$,

$$s_0 + s_1x^{k_1} + \cdots + s_mx^{k_m}/x^{k_j} \in T^{-1}R[x^S] \quad (i \geq m).$$

Then

$$\{(r_0 + r_1x^{k_1} + \cdots + r_ix^{k_i}) - (s_0 + s_1x^{k_1} + \cdots + s_mx^{k_m})\}/x^{k_j} = 0.$$

$$(r_0 + r_1x^{k_1} + \cdots + r_ix^{k_i}) - (s_0 + s_1x^{k_1} + \cdots + s_mx^{k_m}) = 0.$$

$$(r_0 - s_0) + (r_1 - s_1)x^{k_1} + \cdots + (r_m - s_m)x^{k_m} + \cdots + r_ix^{k_i} = 0.$$

$$r_0 = s_0 , \ r_1 = s_1 , \ \cdots , \ r_m = s_m , \ \cdots , \ r_i = 0.$$

Thus

$$\begin{aligned}&\phi(r_0+r_1x^{k_1}+\cdots+r_ix^{k_i}/x^{k_j})\\&=r_0x^{-k_j}+r_1x^{k_1-k_j}+\cdots+r_ix^{k_i-k_j}\\&=s_0x^{-k_j}+s_1x^{k_1-k_j}+\cdots+s_mx^{k_m-k_j}\\&=\phi(s_0+s_1x^{k_1}+\cdots+s_mx^{k_m}/x^{k_j}).\end{aligned}$$

Therefore, ϕ is well-defined.

Now,

$$\begin{aligned}&\phi\{(r_0+r_1x^{k_1}+\cdots+r_ix^{k_i}/x^{k_j})+(s_0+s_1x^{k_1}+\cdots+s_mx^{k_m}/x^{k_j})\}\\&=\phi((r_0+s_0)+(r_1+s_1)x^{k_1}+\cdots+(r_m+s_m)x^{k_m}+\cdots\\&\quad+r_ix^{k_i}/x^{k_j})\\&=(r_0+s_0)x^{-k_j}+(r_1+s_1)x^{k_1-k_j}+\cdots+(r_m+s_m)x^{k_m-k_j}\\&\quad+\cdots+r_ix^{k_i-k_j}\\&=(r_0x^{-k_j}+r_1x^{k_1-k_j}+\cdots+r_ix^{k_i-k_j})+(s_0x^{-k_j}+s_1x^{k_1-k_j}\\&\quad+\cdots+s_mx^{k_m-k_j})\\&=\phi(r_0+r_1x^{k_1}+\cdots+r_ix^{k_i}/x^{k_j})+\phi(s_0+s_1x^{k_1}+\cdots\\&\quad+s_mx^{k_m}/x^{k_j}).\end{aligned}$$

And

$$\begin{aligned}&\phi\{x^{k_m}(r_0+r_1x^{k_1}+\cdots+r_ix^{k_i}/x^{k_j})\}\\&=\phi(r_0x^{k_m}+r_1x^{k_1+k_m}+\cdots+r_ix^{k_i+k_m}/x^{k_j})\\&=r_0x^{k_m-k_j}+r_1x^{k_1+k_m-k_j}+\cdots+r_ix^{k_i+k_m-k_j}\\&=x^{k_m}(r_0x^{-k_j}+r_1x^{k_1-k_j}+\cdots+r_ix^{k_i-k_j})\\&=x^{k_m}\{\phi(r_0+r_1x^{k_1}+\cdots+r_ix^{k_i}/x^{k_j})\}.\end{aligned}$$

Therefore, ϕ is an $R[x^S]$-linear map.

Let $r_0+r_1x^{k_1}+\cdots+r_ix^{k_i}/x^{k_j}$ be an element of $\ker\phi$, then

$$\begin{aligned}&\phi(r_0+r_1x^{k_1}+\cdots+r_ix^{k_i}/x^{k_j})\\&=r_0x^{-k_j}+r_1x^{k_1-k_j}+\cdots+r_ix^{k_i-k_j}\\&=0.\end{aligned}$$

Thus,

$$r_0 = r_1 = \cdots = r_i = 0.$$
$$r_0 + r_1 x^{k_1} + \cdots + r_i x^{k_i}/x^{k_j} = 0.$$

Therefore, ϕ is an injective $R[x^S]$-linear map Let $n_i x^i + \cdots + n_1 x + m_0 + m_1 x^{-1} + \cdots + m_j x^{-j} \in R[x^{-1}, x]$.
Choose, properly large α and β such that $\alpha - \beta = i$, and

$$n_i x^\alpha + \cdots + n_1 x^{\beta+1} + m_0 x^\beta + m_1 x^{\beta-1} + \cdots + m_j x^{\beta-j} \in R[x^S].$$

Then

$$\phi(n_i x^\alpha + \cdots + n_1 x^{\beta+1} + m_0 x^\beta + m_1 x^{\beta-1} + \cdots + m_j x^{\beta-j}/x^\beta)$$
$$= n_i x^i + \cdots + n_1 x + m_0 + m_1 x^{-1} + \cdots + m_j x^{-j}.$$

Therefore, ϕ is a surjective $R[x^S]$-linear map.
Hence, $T^{-1}R[x^S] \cong R[x^{-1}, x]$ as left $R[x^S]$-modules.

2. Injective Property of Laurent Power Series Module

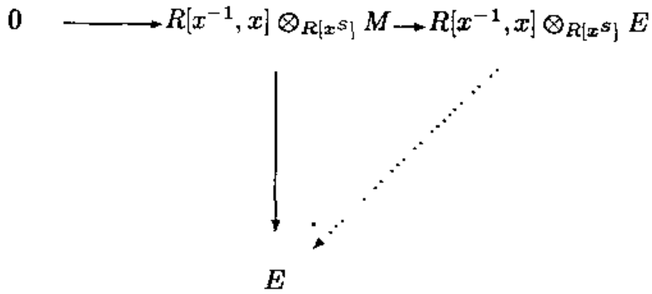
THEOREM 2 1 *Let R be a ring and E be an injective left R-module. Then $Hom_R(R[x^{-1}, x], E)$ is an injective left $R[x^S]$-module.*

PROOF Since $R[x^{-1}, x]$ is an $R - R[x^S]$ bimodule and $R[x^S]$ -flat module by Lemma 1,1 $\mathrm{Hom}_R(R[x^{-1}, x], E)$ is an injective $R[x^S]$ -module. Since $R[x^{-1}, x]$ is flat , we have

$$0 \to R[x^{-1}, x] \otimes_{R[x^S]} M \to R[x^{-1}, x] \otimes_{R[x^S]} E.$$

is exact for ${}_{R[x^S]}M \subset_{R[x^S]} E$. And $R[x^{-1}, x] \otimes_{R[x^S]} M$ is a left R-module, so is $R[x^{-1}, x] \otimes_{R[x^S]} E$.

Since E is an injective left R-module, we the have following commutative diagram.



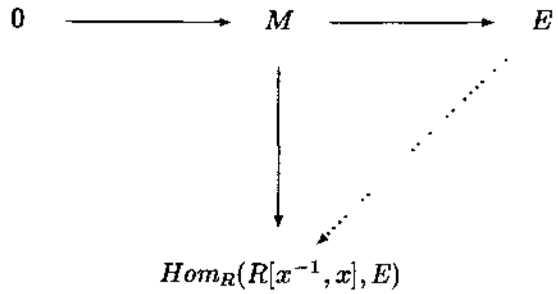
That is

$$\mathrm{Hom}_R(R[x^{-1},x] \otimes_{R[x^S]} E, E) \to \mathrm{Hom}_R(R[x^{-1},x] \otimes_{R[x^S]} M, E) \to 0$$

is exact. But by the adjoint isomorphism

$$\mathrm{Hom}_{R[x^S]}(E, \mathrm{Hom}_R(R[x], E)) \to \mathrm{Hom}_{R[x^S]}(M, \mathrm{Hom}_R(R[x], E)) \to 0$$

is exact. So the following diagram



can be completed. Therefore, $\mathrm{Hom}_R(R[x^{-1},x], E)$ is an injective left $R[x^S]$-module.

THEOREM 2.2 *Let R be a ring and E be an injective left R-module. Then $Hom_R(R[x^{-1},x], E)$ and $E[[x^{-1},x]]$ are isomorphic as left $R[x^S]$–modules.*

Proof. Define $\phi : \mathrm{Hom}_R(R[x^{-1},x],E) \longrightarrow E[[x^{-1},x]]$ by

$$\phi(f) = \cdots + f(x^{-2})x^{-2} + f(x^{-1})x^{-1} + f(1) + f(x)x + f(x^2)x^2 + \cdots .$$

Let $f,\ g \in\ \mathrm{Hom}_R(R[x^{-1},x],E)$. If $f = g$, then

$$\begin{aligned}\phi(f) &= \cdots + f(x^{-1})x^{-1} + f(1) + f(x)x + \cdots \\ &= \cdots + g(x^{-1})x^{-1} + g(1) + g(x)x + \cdots \\ &= \phi(g).\end{aligned}$$

Therefore, ϕ is well-defined. Now

$$\begin{aligned}\phi(f+g) &= \cdots + ((f+g)(x^{-1}))x^{-1} + (f+g)(1) + ((f+g)(x))x + \cdots \\ &= \cdots + (f(x^{-1}) + g(x^{-1}))x^{-1} + (f(1) + g(1)) \\ &\qquad + (f(x) + g(x))x + \cdots \\ &= \{\cdots + f(x^{-1})x^{-1} + f(1) + f(x)x + \cdots\} \\ &\qquad + \{\cdots + g(x^{-1})x^{-1} + g(1) + g(x)x + \cdots\} \\ &= \phi(f) + \phi(g).\end{aligned}$$

And

$$\begin{aligned}\phi(x^{k_i}f) &= \cdots + (x^{k_i}f)(x^{-1})x^{-1} + (x^{k_i}f)(1) + (x^{k_i}f)(x)x + \cdots \\ &= \cdots + f(x^{k_i-1})x^{-1} + f(x^{k_i}) + f(x^{k_i+1})x + \cdots , \\ x^{k_i}\phi(f) &= x^{k_i}\{\cdots + f(x^{-1})x^{-1} + f(1) + f(x)x + \cdots\} \\ &= \cdots + f(x^{k_i-1})x^{-1} + f(x^{k_i}) + f(x^{k_i+1})x + \cdots .\end{aligned}$$

Thus ϕ is an $R[x^S]$-linear map. Let f be an element of $\ker(\phi)$, then

$$\phi(f) = \cdots + f(x^{-1})x^{-1} + f(1) + f(x)x + \cdots = 0,$$

implies $f = 0$. Therefore, ϕ is one-to-one. Let $\cdots + m_2x^{-2} + m_1x^{-1} + e_0 + e_1x + e_2x^2 + \cdots$ be an element of $E[[x^{-1},x]]$. Then choose $f \in \mathrm{Hom}_R(R[x^{-1},x],E)$ such that

$$\begin{aligned}\cdots\ \ f(x^{-2}) &= m_2,\ \ f(x^{-1}) = m_1,\ \ f(1) = e_0, \\ f(x) &= e_1,\ \ f(x^2) = e_2,\ \ \cdots .\end{aligned}$$

Then

$$\begin{aligned}\phi(f) &= \cdots + f(x^{-1})x^{-1} + f(1) + f(x)x + \cdots \\ &= \cdots + m_1 x^{-1} + e_0 + e_1 x + \cdots .\end{aligned}$$

Therefore, ϕ is onto. Hence, $\mathrm{Hom}_R(R[x^{-1}, x], E) \cong E[[x^{-1}, x]]$ as left $R[x^S]$-modules.

THEOREM 2.3 *Let E be an injective R-module, then the Laurent power series $E[[x^{-1}, x]]$ is an injective left $R[x^S]$-module.*

PROOF. First, the localization $T^{-1}R[x^S]$ is a flat $R[x^S]$-module by lemma 1.2. And by theorem 1.3, $T^{-1}R[x^S] \cong R[x^{-1}, x]$ as flat $R[x^S]$-modules. Now by theorem 2.1, $\mathrm{Hom}_R(R[x^{-1}, x], x)$ is an injective left $R[x^S]$-module, and $\mathrm{Hom}_R(R[x^{-1}, x], E) \cong E[[x^{-1}, x]]$ by theorem 2.2, we conclude that $E[[x^{-1}, x]]$ is an injective left $R[x^S]$-module.

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