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PROPERTIES OF THE FIRST ALGEBRAIC LOCAL COHOMOLOGY GROUP AND GRÖBNER BASIS

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1. Introduction

In this paper, we study some properties of the first algebraic local cohomology groups and Gröbner basis for residue calculus based on the papers S. Tajima [6,7]. For a given meromorphic fuction $u(z)$, we can treat the principal part m of $u(z)$ as an element of the algebraic local cohomology group $\mathcal{H}^1_{[A]}(\mathcal{O}_X)$ by setting $m = u(z)\ mod\ \mathcal{O}_X$ where $\mathcal{O}_X$ is the sheaf of holomorphic functions. Since $\phi(z)$ is holomorphic in X, it suffices to consider $m = u(z)\ mod\ \mathcal{O}_X$ instead of $u(z)$ when we compute the residue of $\phi(z)u(z)dz$. S. Tajima and Y. Nakamura [5] investigated the residue calculus and Horowitz's algorithm for rational function with differential equation.

We investigate some properties of modules, extension groups and the algebraic local cohomology and introduce a Gröbner basis that is very useful for computing the residues.

2. Preliminaries

We introduce some tools from the theory of extension group induced by homomorphism theories: modules, extension groups, the first algebraic local cohomology group and Gröbner basis.

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THEOREM 2.1[3] *Every free module F over a ring Λ with identity is projective.*

Let Λ be a ring and A, B be Λ-modules. We let

$$Hom_\Lambda(A,B) := \{\varphi | \varphi : A \to B \text{ is a } \Lambda\text{-module homomorphism}\}.$$

THEOREM 2 2[2]. *(1) Let $0 \longrightarrow B' \xrightarrow{\mu} B \xrightarrow{\varepsilon} B''$ be an exact sequence of Λ-modules. For every Λ-module A, the induced sequence $0 \longrightarrow Hom_\Lambda(A,B') \xrightarrow{\mu_*} Hom_\Lambda(A,B) \xrightarrow{\varepsilon_*} Hom_\Lambda(A,B'')$ is exact.*

(2) Let $A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \longrightarrow 0$ be an exact sequence of Λ-modules. For every Λ-module B, the induced sequence $0 \longrightarrow Hom_\Lambda(A'',B) \xrightarrow{\varepsilon^*} Hom_\Lambda(A,B) \xrightarrow{\mu^*} Hom_\Lambda(A',B)$ is exact.

DEFINITION 2.3 A short exact sequence $0 \longrightarrow R \xrightarrow{\mu} P \xrightarrow{\varepsilon} A \longrightarrow 0$ of Λ-modules with P projective is called a *projective presentation* of A.

DEFINITION 2.4 To the Λ-modules A, B and to the projective presentation of A given in the Definition 2.4, we can associate the abelian group

$$\begin{aligned} Ext_\Lambda(A,B) &= coker(\mu^* : Hom_\Lambda(P,B) \longrightarrow Hom_\Lambda(R,B)), \\ &= Hom_\Lambda(R,B)/Im\mu^*. \end{aligned}$$

Now, we deduce two exact sequences connecting Hom and Ext.

THEOREM 2 5[2]. *Let A be a Λ-module and let $0 \longrightarrow B' \xrightarrow{\varphi} B \xrightarrow{\psi} B'' \longrightarrow 0$ be a short exact sequence of Λ-modules. Then there exists a "connecting homomorphism" $\omega : Hom_\Lambda(A,B'') \longrightarrow Ext_\Lambda(A,B)$ such that the following sequence is exact :*

$$\begin{aligned} 0 \longrightarrow Hom_\Lambda(A,B') \xrightarrow{\varphi_*} Hom_\Lambda(A,B) \xrightarrow{\psi_*} Hom_\Lambda(A,B'') \\ \xrightarrow{\omega} Ext_\Lambda(A,B') \xrightarrow{\varphi_*} Ext_\Lambda(A,B) \xrightarrow{\psi_*} Ext_\Lambda(A,B''). \end{aligned}$$

DEFINITION 2 6 (1) A *graded Λ-module* A (graded by the integers) is a family of Λ-modules $A = \{A_n\}$, $n \in \mathbb{Z}$.

(2) If A, B are graded Λ-modules (i.e. $A = \{A_n\}$, $B = \{B_n\}$, $n \in \mathbb{Z}$), a morphism $\varphi : A \to B$ of degree k is a family of Λ-module homomorphisms $\{\varphi_n : A_n \to B_{n+k}\}, n \in \mathbb{Z}$.

The category so defined is denoted by $\mathfrak{M}_{\Lambda}^{\mathbb{Z}}$, a graded (left) module.

DEFINITION 2 7 A cochain complex $\mathbf{C} = \{C_n, \delta_n\}$ is an object in $\mathfrak{M}_{\Lambda}^{\mathbb{Z}}$ together with an endomorphism $\delta : \mathbf{C} \to \mathbf{C}$ of degree $+1$ with $\delta\delta = 0$.

DEFINITION 2 8 A *cochain map* $\varphi : C \to D$ is a morphism of degree 0 in $\mathfrak{M}_{\Lambda}^{\mathbb{Z}}$ such that $\varphi\delta = \tilde{\delta}\varphi$ where $\tilde{\delta}$ is coboundary operator in D. Thus φ is a family $\{\varphi^n : C^n \to D^n\}, n \in \mathbb{Z}$ of homomorphisms such that the diagram

$$\begin{array}{ccc} C^n & \xrightarrow{\delta^n} & C^{n+1} \\ {\scriptstyle\varphi^n}\downarrow & & \downarrow{\scriptstyle\varphi^{n+1}} \\ D^n & \xrightarrow[\tilde{\delta}^n]{} & D^{n+1} \end{array}$$

is commutative.

DEFINITION 2 10 Given a cochain complex $\mathbf{C} = \{C^n, \delta^n\}$, we define its *cohomology module* $H(C) = \{H^n(C)\}$ by

$$H^n(C) = ker\delta^n / Im\delta^{n-1}, \quad n \in \mathbb{Z}.$$

If $\Lambda = \mathbb{Z}$, $H(C)$ is called *the cohomology group of* C. A cochain map $\varphi : C \to D$ induces a morphism of graded modules $H(\varphi) = \varphi^* : H(C) \to H(D)$.

3. Algebraic local cohomology group and Gröbner basis

In this section, we use the ideas and the results of M. Kashiwara [4] and N. Takayama [8].

THEOREM 3 1. *Using the above notations, we have*

$$\begin{cases} H^0(\mathbf{C}) = Hom_\Lambda(A,B), \\ H^1(\mathbf{C}) = Ext_\Lambda(A,B), \\ H^n(\mathbf{C}) = 0, \qquad n \neq 0,1. \end{cases}$$

PROOF. We have $H^0(\mathbf{C}) = ker\ \delta^0 \ /\ Im\ \delta^{-1} = ker\ \mu^*$. Since $0 \longrightarrow Hom_\Lambda(A,B) \xrightarrow{\varepsilon^*} Hom_\Lambda(P,B) \xrightarrow{\mu^*} Hom_\Lambda(R,B)$ is exact, from D. A. Cox [1], $ker\ \mu^* = Im\ \varepsilon^* \cong Hom(A,B)$. Hence $H^0(\mathbf{C}) \cong Hom_\Lambda(A,B)$. On the other hand, $H^1(\mathbf{C}) = ker\ \delta^1 \ /\ Im\ \delta^0 = C^1 \ /\ Im\ \delta^0 = Hom_\Lambda(R,B) \ /\ Im\ \mu^* = Ext_\Lambda(A,B)$ from Definition 2.5.

THEOREM 3 2 *If* $0 \longrightarrow \mathbf{A} \xrightarrow{\varphi} \mathbf{B} \xrightarrow{\psi} \mathbf{C} \longrightarrow 0$ *is an exact sequence of cochain complexes, then the sequence*

$$\cdots \xrightarrow{\omega^{n-1}} H^n(\mathbf{A}) \xrightarrow{\varphi^*} H^n(\mathbf{B}) \xrightarrow{\psi^*} H^n(\mathbf{C}) \xrightarrow{\omega^n} H^{n+1}(\mathbf{A}) \longrightarrow \cdots$$

is exact.

PROOF From the exact sequence $0 \longrightarrow \mathbf{A} \xrightarrow{\varphi} \mathbf{B} \xrightarrow{\psi} \mathbf{C} \longrightarrow 0$, we have a morphism of degree $+1$ of graded modules $\omega : H(\mathbf{C}) \to H(\mathbf{A})$ such that the diagram

$$\begin{array}{ccc} H(\mathbf{A}) & \xrightarrow{\varphi^*} & H(\mathbf{B}) \\ & & \downarrow \psi^* \\ & & H(\mathbf{C}) \end{array}$$

is exact, from the result of P. J. Hilton and U. Stammbach [2]. This implies that the long sequence is exact.

DEFINITION 3.3. Let X be a simply connected domain in the complex plane $\mathbb{C}$, $\mathcal{O}_X$ be the sheaf on X of holomorphic functions. Let $A = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subset X$ and $\mathfrak{J}_A$ is the ideal generated by $\prod_{j=1}^n (x - \alpha_j)$

in $\mathcal{O}_X$. Put $A_j = \{\alpha_j\}$, $j = 1, 2, \cdots n$. The *first algebraic local cohomology group* with support in A is defined as the inductive limit of extension groups

$$\mathcal{H}^1_{[A]}(\mathcal{O}_X) = \varinjlim_{l \to \infty} \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X/\mathfrak{I}^l_A, \mathcal{O}_X).$$

The extension group and the algebraic local cohomology group can be explained as follows. We have a short exact sequence :
By Theorem 2.6(2), we have a long exact sequence :

LEMMA 3 4 *For an exact sequence*

$$0 \longrightarrow \mathfrak{I}^l_{A_j} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathfrak{I}^l_{A_j} \longrightarrow 0,$$

we have $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathfrak{I}^l_{A_j}, \mathcal{O}_X) = 0$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X$ *and* $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = 0$.

PROOF. We have an exact sequence $0 \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathfrak{I}^l_{A_j}, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathfrak{I}^l_{A_j}, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X/\mathfrak{I}^l_{A_j}, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_X}(\mathfrak{I}^l_{A_j}, \mathcal{O}_X)$.

For $\phi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathfrak{I}^l_{A_j}, \mathcal{O}_X)$ and $\phi(1) = f \in \mathcal{O}_X$, since $(x - \alpha_j)^l \in \mathfrak{I}^l_{A_j}$, we have $(x - \alpha_j)^l = 0$ in $\mathcal{O}_X/\mathfrak{I}^l_{A_j}$ from S. Tajima and Y. Nakamura [6]. Since ϕ is an $\mathcal{O}_X$-module homomorphism, $0 = \phi((x - \alpha_j)^l) = (x - \alpha_j)^l \phi(1) = (x - \alpha_j)^l f$. Thus $f = \phi(1) = 0$ and for any $g \in \mathcal{O}_X/\mathfrak{I}^l_{A_j}, \phi(g) = g\phi(1) = 0$. Hence ϕ is zero homomorphism.

(2) If we define a map F: $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{O}_X$ by $F(\phi) = \phi(1)$ where $\phi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$, then F is an isomorphism.

(3) Let $\Lambda = \mathcal{O}_X$ be the sheaf of rings. Then $\mathcal{O}_X$ is an $\mathcal{O}_X$-module. Since $\mathcal{O}_X$ is a free module, by T. W. Hungerford [3], $\mathcal{O}_X$ is a projective module. By P. J. Hilton and U. Stammbach [2], $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = 0$.

From Lemma 3.4, we get the following exact sequence :

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathfrak{I}^l_{A_j}, \mathcal{O}_X) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X/\mathfrak{I}^l_{A_j}, \mathcal{O}_X) \longrightarrow 0.$$

This implies the following isomorphism :

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X/\mathfrak{I}^l_{A_j}, \mathcal{O}_X) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathfrak{I}^l_{A_j}, \mathcal{O}_X)/\mathcal{O}_X.$$

Since $\mathcal{H}om_{\mathcal{O}_X}(\mathfrak{I}^l_{A_j}, \mathcal{O}_X)$ can be understood as the sheaf of meromorphic functions with poles of order at most l at A_j, by Definition 2.8, we have

$$\mathcal{H}^1_{[A]}(\mathcal{O}_X) \cong \mathcal{O}_X\langle *A\rangle/\mathcal{O}_X,$$

where $\mathcal{O}_X\langle *A\rangle$ is the sheaf of meromorphic functions on X with poles at most at A.

Let $\mathcal{D}_X$ be the sheaf of rings on X of linear differential operators of finite order with holomorphic coefficients. This means that if U is an open set in X, a section in $\mathcal{D}_X(U)$ is a differential operator $D = \sum_{j=0}^n \alpha_j(z)\frac{\partial^j}{\partial z^j}$, where $\alpha_j(z) \in \mathcal{O}_X(U)$. Then $\mathcal{D}_X$ is coherent as a sheaf of rings.

THEOREM 3.5 *(1) $\mathcal{H}^1_{[A]}(\mathcal{O}_X)$ is a coherent left $\mathcal{D}_X$-module.*

(2) The first algebraic local cohomology group with support at a single point is simple as $\mathcal{D}_X$-module.

PROOF. (1) Let $P = \sum_j \alpha_j(z)\frac{\partial^j}{\partial z^j} \in \mathcal{D}_X$ and $[r] = r \bmod \mathcal{O}_X \in \mathcal{H}^1_{[A]}(\mathcal{O}_X)$ where $r \in \mathcal{O}_X\langle *A\rangle$. Define a map $\mathcal{D}_X \times \mathcal{H}^1_{[A]}(\mathcal{O}_X) \to \mathcal{H}^1_{[A]}(\mathcal{O}_X)$ by $(P, [r]) \mapsto [Pr]$. Since $Pr \in \mathcal{O}_X\langle *A\rangle$, $[Pr] \in \mathcal{O}_X\langle *A\rangle/\mathcal{O}_X = \mathcal{H}^1_{[A]}(\mathcal{O}_X)$ from S. Tajima and Y. Nakamura [6]. Thus $\mathcal{H}^1_{[A]}(\mathcal{O}_X)$ is a left $\mathcal{D}_X$-module.

(2) Let $\delta = [\frac{1}{z-\alpha_j}] \in \mathcal{H}^1_{[\alpha_j]}(\mathcal{O}_X)$. Then δ generates $\mathcal{H}^1_{[\alpha_j]}(\mathcal{O}_X)$. Let $N \subset \mathcal{H}^1_{[\alpha_j]}(\mathcal{O}_X)$ be an ideal as $\mathcal{D}_X$-module with $N \neq \{0\}$ and let $\eta(\neq 0) \in N$. Then $\eta = \sum_{k=1}^n [\frac{C_k}{(z-\alpha_j)^k}]$, $C_k \in \mathbb{C}$ and $\delta = \frac{(z-\alpha_j)^{n-1}}{C_n}\eta \in \mathcal{D}_X\eta$. Thus $\mathcal{D}_X\eta = \mathcal{D}_X\delta = \mathcal{H}^1_{[\alpha_j]}(\mathcal{O}_X)$. Since $\mathcal{H}^1_{[\alpha_j]}(\mathcal{O}_X) \supset N \supset \mathcal{D}_X\eta = \mathcal{H}^1_{[\alpha_j]}(\mathcal{O}_X)$, $N = \mathcal{H}^1_{[\alpha_j]}(\mathcal{O}_X)$. Hence $\mathcal{H}^1_{[\alpha_j]}(\mathcal{O}_X)$ is simple.

Now, we introduce Gröbner basis which is useful not only for computing residues but for solving other problems such as solving a system of polynomial equations, etc.

One of the simplest monomial orders is *lexicographic*(*lex* for short), which is defined by

$$x_1^{a_1} \cdots x_n^{a_n} >_{lex} x_1^{b_1} \cdots x_n^{b_n} \Leftrightarrow a_1 > b_1, \text{ or } a_1 = b_1 \text{ and } a_2 > b_2, \text{etc.}$$

DEFINITION 3 6 If $f = \sum_\alpha c_\alpha x^\alpha \in K[x_1, \cdots, x_n]$, $c_\alpha \in K$ and $>$ is a monomial order, then a *term* of f is $c_\alpha x^\alpha$ for $c_\alpha \neq 0$ and the *leading term* of f is $LT(f) = max_>\{c_\alpha x^\alpha \mid c_\alpha \neq 0\}$ where $max_>$ means the maximum with respect to $>$.

THEOREM 3 8 (THE GENERAL DIVISION ALGORITHM) *We assume that a monomial order $>$ on $K[x_1, \cdots, x_n]$ is given. If we divide $f \in K[x_1, \cdots, x_n]$ by $f_1, \cdots, f_s \in K[x_1, \ldots, x_n]$, we can look for an expression of the form*

$$f = q_1 f_1 + \cdots + q_s f_s + r,$$

where the remainder r should satisfy that no term of r is divisible by any of $LT(f_1), \cdots, LT(f_s)$. Furthermore, $LT(f) \geq LT(q_i f_i)$, $1 \leq i \leq s$.

The algorithm gives different remainders by changing the order of $f_1, \cdots, f_s$.

DEFINITION 3 9. Let $f_1, \cdots, f_s \in K[x_1, \cdots, x_n]$.

(1) $\langle f_1, \cdots, f_s \rangle = \{\sum_{i=1}^s h_i f_i \mid h_i \in K[x_1, \cdots, x_n]\}$ is the deal generated by $f_1, \cdots, f_s$.

(2) $\mathbf{V}(f_1, \cdots, f_s) = \{p \in K^n \mid f_1(p) = \cdots = f_s(p) = 0\} \subset K^n$ is called *the affine variety.*

(3) Given any ideal $I \subset K[x_1, \cdots, x_n]$, we define $\mathbf{V}(I) = \{p \in K^n \mid f(p) = 0 \text{ for all } f \in I\} \subset K^n$.

THEOREM 3 10 (HILBERT BASIS THEOREM) *If $I \subset K[x_1, \cdots, x_n]$ is an ideal, then we can find $f_1, \cdots, f_s \in K[x_1, \cdots, x_n]$ such that $I = \langle f_1, \cdots, f_s \rangle$.*

This asserts that every $\mathbf{V}(I)$ can be written in the form $\mathbf{V}(f_1, \cdots, f_s)$.

DEFINITION 3.11 $\langle LT(I) \rangle$ is called the *ideal of leading terms* which is generated by the leading terms $LT(f)$ for all $f \in I \backslash \{0\}$.

An important observation is that if $I = \langle f_1, \cdots, f_s \rangle$, then

$$\langle LT(f_1), \cdots, LT(f_s) \rangle \subset \langle LT(I) \rangle,$$

but equality need not occur. A Gröbner basis occurs when we get equality $\langle LT(f_1), \cdots, LT(f_s) \rangle = \langle LT(I) \rangle$.

EXAMPLE 3.12 If $f_1 = x^3 - 2xy$ and $f_2 = x^2y - x - 2y^2$, then $x^2 = y(x^3 - 2xy) - x(x^2y - x - 2y^2) = yf_1 - xf_2 \in \langle f_1, f_2 \rangle$. Using lex order with $x > y$, we have $LT(f_1) = x^3$ and $LT(f_2) = x^2y$. Since $LT(x^2) = x^2 \notin \langle x^3, x^2y \rangle = \langle LT(f_1), LT(f_2) \rangle$, we see that the ideal of leading terms can be strictly larger than the ideal generated by the leading terms of the generators.

DEFINITION 3.13 Given a monomial order $>$ and an ideal $I \subset K[x_1, \cdots, x_n]$, we say that $\{g_1, \cdots, g_t\} \subset I$ is a *Gröbner basis* of I if

$$\langle LT(g_1), \cdots, LT(g_t) \rangle = \langle LT(I) \rangle.$$

More concretely, $\{g_1, \cdots, g_t\} \subset I$ is a *Gröbner basis* if the leading term of every nonzero element of I is divisible by some $LT(g_i)$.

THEOREM 3 14[1]. *Fix a monomial order $>$ on $K[x_1, \cdots, x_n]$ and let $I \subset K[x_1, \cdots, x_n]$ be an ideal. Then I has a Gröbner basis, and furthermore, any Gröbner basis of I is a basis of I.*

PROPOSITION 3 15[1] *If $g_1, \cdots, g_t$ is a Gröbner basis for I and $f \in K[x_1, \cdots, x_n]$, then f can be written uniquely in the form $f = g + r$, where $g \in I$ and no term of r is divisible by any $LT(g_i)$.*

This proposition implies that the remainder on division by a Gröbner basis is unique. If we let $G = \{g_1, \cdots, g_t\}$ be the Gröbner basis, then the remainder of f on division by G will be denoted $r = \overline{f}^G$.

From now, we compute Gröbner bases. *Buchberger* (see D. A. Cox [1]) provided algorithms for determining whether a given basis of an ideal is a Gröbner basis and computing Gröbner bases. The key tool is the *S-polynomial* of $f_1, f_2 \in K[x_1, \cdots, x_n]$, which is defined to be

$$S(f_1, f_2) = \frac{x^\gamma}{LT(f_1)} f_1 - \frac{x^\gamma}{LT(f_2)} f_2,$$

where $x^\gamma = LCM(LM(f_1), LM(f_2))$ and $LM(f_i)$ is the *leading monomial* of f_i (the leading term with the coefficient removed). The basic idea of S-polynomial is that it is the simplest combination of f_1 and f_2 which cancels leading terms. We recall the Buchberger's criterion. A basis $\{g_i, \ldots, g_t\} \subset I$ is a Gröbner basis of I if and only if for all $i < j$, we have $\overline{S(g_i, g_j)}^G = 0$. Here, $\overline{S(g_i, g_j)}^G$ denotes the remainder of $S(g_i, g_j)$ on division by G.

EXAMPLE 3 16 Let $F = \{f_1, f_2\} = \{x^3 - 2xy,\ x^2y - x - 2y^2\}$. We know $\overline{S(f_1, f_2)}^F = x^2 = f_3$, so that setting $F_1 = \{f_1, f_2, f_3\}$, we compute :

$$\begin{aligned} \overline{S(f_1, f_2)}^{F_1} &= 0, \\ \overline{S(f_1, f_3)}^{F_1} &= -2xy = f_4, \\ \overline{S(f_2, f_3)}^{F_1} &= -x - 2y^2 = f_5. \end{aligned}$$

Adding the nonzero remainders to F_1 gives $F_2 = \{f_1, f_2, f_3, f_4, f_5\}$, and then we compute :

$$\begin{aligned} \overline{S(f_1, f_5)}^{F_2} &= -4y^3, \\ \overline{S(f_4, f_5)}^{F_2} &= -2y^3, \\ \overline{S(f_i, f_j)}^{F_2} &= 0 \quad \text{for all other } i < j. \end{aligned}$$

It suffices to add $f_6 = y^3$ to F_2, giving $F_3 = \{f_1, f_2, f_3, f_4, f_5, f_6\}$. This time we get $\overline{S(f_i, f_j)}^{F_3} = 0$, $1 \le i, j \le 6$, so that a Gröbner basis of $\langle x^3 - 2xy,\ x^2y - x - 2y^2\rangle$ for lex order with $x > y$ is $F_3 = \{x^3 - 2xy,\ x^2y - x - 2y^2,\ x^2,\ -2xy,\ -x - 2y^2,\ y^3\}$.

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