East Asian Math J 17(2001), No 2, pp. 325-337

ON A HYPERSURFACE OF THE FIRST APPROXIMATE MATSUMOTO SPACE

IL-YONG LEE AND DONG-GUM JUN

ABSTRACT We consider the special hypersurface of the first approximate Matsumoto metric with $b_i(x) = \partial_i b$ being the gradient of a scalar function $b(x)$ In this paper, we consider the hypersurface of the first approximate Matsumoto space with the same equation $b(x)$ = constant We are devoted to finding the condition for this hypersurface to be a hyperplane of the first or second kind We show that this hypersurface is not a hyper-plane of third kind

1. The first approximate Matsumoto space

The Matsumoto metric is expressed as the form

$$\frac{\alpha^2}{\alpha-\beta} = \lim_{r\to\infty} \alpha \sum_{k=0}^{r} \left(\frac{\beta}{\alpha}\right)^k \tag{1.1}$$

for $|\beta| < |\alpha|$. We regard $b_i(x)$ as very small numerically. If we neglect all the powers which are greater than r of $b_i(x)$ in (1.1), then (α, β)-metric

$$L = \alpha \sum_{k=0}^{r} \left(\frac{\beta}{\alpha}\right)^k \tag{1.2}$$

Received March 2, 2001 Revised Sepmtember 19, 2001
2000 Mathematics Subject Classification 53B40
Key words and phrases Finsler space, hypersurface, induced Cartan connection, Matsumoto metric

is an approximate metric to the Matsumoto metric. Then we shall call the (α,β)-metric (1.2) *the general approximate Matsumoto metric.* If we put $r = 2$, then L is the first approximate Matsumoto metric. That is to say, we have as follows:

$$L = \alpha + \beta + \frac{\beta^2}{\alpha}. \tag{1.3}$$

Here, by taking a general Riemannian metric α and a general non-zero 1-form β on a general differentiable manifold M^n, Hong-Suh Park, Il-Yong Lee and Chan-Keun Park [8] give as follows:

DEFINITION 1.1. On an n-dimensional differential manifold M^n, an (α,β)-metric L of type (1.3) is called the *first approximate Matsumoto metric* and the Finsler space (M^n, L) is called *the first approximate Matsumoto space.*

The derivatives of the first approximate Matsumoto metric L with respect ot α and β are given by

$$\begin{aligned} &L_\alpha = (\alpha^2 - \beta^2)/\alpha^2, \quad L_\beta = (\alpha + 2\beta)/\alpha, \\ &L_{\alpha\alpha} = 2\beta^2/\alpha^3, \quad L_{\beta\beta} = 2/\alpha, \\ &L_{\alpha\beta} = -2\beta/a^2, \end{aligned} \tag{1.4}$$

where $L_\alpha = \partial L/\partial\alpha$, $L_\beta = \partial L/\partial\beta$.

If in the first approximate Matsumoto space $F^n = (M^n, L)$ where $L = \alpha + \beta + \beta^2/\alpha$, we put

$$\alpha = (a_{ij}(x)y^i y^j)^{\frac{1}{2}}, \quad \beta = b_i(x)y^i,$$

then the normalized element of support $l_i = \dot\partial_i L$ is given by

$$l_i = \alpha^{-1} L_\alpha y_i + L_\beta b_i, \tag{1.5}$$

where $Y_i = a_{ij}y^i$. The angular metric tensor $h_{ij} = L^{-1}\dot\partial_i\dot\partial_j L$ is given by

$$h_{ij} = pa_{ij} + q_0 b_i b_j + q_1(b_i Y_j + b_j Y_i) + q_2 Y_1 Y_j, \tag{1.6}$$

where

$$
(1.7)\qquad
\begin{aligned}
p &= LL_\alpha \alpha^{-1} = \frac{(\alpha^2+\alpha\beta+\beta^2)(\alpha^2-\beta^2)}{\alpha^k},\\
q_0 &= LL_{\beta\beta} = \frac{2(\alpha^2+\alpha\beta+b^2)}{\alpha^2},\\
q_1 &= LL_{\alpha\beta}\alpha^{-1} = -\frac{2\beta(\alpha^2+\alpha\beta+\beta^2)}{\alpha^k},\\
q_2 &= L\alpha^{-2}(L_{\alpha\alpha}-L_\alpha\alpha^{-1}) = \frac{(\alpha^2+\alpha\beta+\beta^2)(3\beta^2-\alpha^2)}{\alpha^6}.
\end{aligned}
$$

The fundamental tensor $g_{ij} = \dfrac{1}{2}\dot\partial_i\dot\partial_j L^2$ is given by

$$
(1.8)\qquad g_{ij} = pa_{ij} + p_0 b_i b_j + p_1(b_i Y_j + b_j Y_i) + q_2 Y_i Y_j,
$$

where

$$
(1.9)\qquad
\begin{aligned}
p_0 &= q_0 + L_\beta^2 = \frac{3(\alpha^2+2\alpha\beta+\beta^2)}{\alpha^2},\\
p_1 &= q_1 + L^{-1}pL_\beta = \frac{\alpha^3+4\alpha^2\beta+\alpha\beta^2}{\alpha^4},\\
p_2 &= q_2 + p^2L^{-2} = \frac{-\alpha^3\beta+3\alpha\beta^3+4\beta^4}{\alpha^6}.
\end{aligned}
$$

Moreover, the reciprocal tensor g^{ij} of g_{ij} is given by

$$
(1.10)\qquad g^{ij} = p^{-1}a^{ij} - S_0 b^i b^j - S_1(b^i y^j + b^j y^i) - S_2 y^i y^j,
$$

where

$$
(1.11)\qquad
\begin{aligned}
& b^i = a^{ij}b_j, \quad S_0 = (pp_0 + (p_0p_2 - p_1^2)\alpha^2)/\zeta p,\\
& S_1 = (pp_1 + (p_0p_2 - p_1^2)\beta)/\zeta p,\\
& S_2 = (pp_2 + (p_0p_2 - p_1^2)b^2)/\zeta p, \quad b^2 = a_{ij}b^ib^j,\\
& \zeta = p(p + p_0b^2 + p_1\beta) + (p_0p_2 - p_1^2)(\alpha^2b^2 - \beta^2).
\end{aligned}
$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by ([9])

$$2pC_{ijk} = p_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma m_i m_j m_k, \tag{1.12}$$

where

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_1 q_0, \quad m_i = b_i - \alpha^{-2}\beta Y_i. \tag{1.13}$$

It is noted that the covariant vector m_i is a non-vanishing one, and is orthogonal to the element of support y^i.

Let $\left\{ {i \atop jk} \right\}$ be the components of Christoffel's symbol of the associated Riemannian space R^n and ∇_k be covariant differentiation with respect to x^k relative to this Christoffel's symbol. We shall use the following tensors.

$$2E_{ij} = b_{ij} + b_{ji}, \quad 2F_{ij} = b_{ij} - b_{ji}, \tag{1.14}$$

where $b_{ij} = \nabla_j b_i$.

If we denote the Cartan's connection $C\Gamma$ as $(\Gamma^{*i}_{j\ k}, \Gamma^{*i}_{0\ k}, C_j{}^i{}_k)$, then the difference tensor $D_j{}^i{}_k = \Gamma^{*i}_{j\ k} - \left\{ {i \atop jk} \right\}$ of the first approximate Matsumoto space is given by ([10]).

(1.15)

$$\begin{aligned} D_j{}^i{}_k = {} & B^i E_{jk} + F^i{}_k B_j + F^i{}_j B_k + B^i{}_j b_{0k} + B^i{}_k b_{0j} \\ & - b_{0m} g^{im} B_{jk} - C_j{}^i{}_m A^m{}_k - C_k{}^i{}_m A^m{}_j + C_{jkm} A^m{}_s g^{is} \\ & + \lambda^s \left(C_j{}^i{}_m C_s{}^m{}_k + C_k{}^i{}_m C_s{}^m{}_j - C_j{}^m{}_k C_m{}^i{}_s \right), \end{aligned}$$

where

$$\begin{aligned} & B_k = p_0 b_k + p_1 Y_k, \quad B^i = g^{ij} B_j, \quad F^k{}_i = g^{kj} F_{ji}, \\ & B_{ij} = \left\{ p_1(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j \right\} \Big/ 2, \\ & B^k{}_i = g^{kj} B_{ji}, \\ & A^m{}_k = B^m{}_k E_{00} + B^m E_{k0} + B_k F^m{}_0 + B_0 F^m{}_k, \\ & \lambda^m = B^m E_{00} + 2B_0 F^m{}_0, \quad B_0 = b_i y^i. \end{aligned} \tag{1.16}$$

Here and in the following we denote 0 as contraction with y^i except for the quantities p_0, q_0 and s_0.

2. Induced Cartan connection

Let F^{n-1} be a hypersurface of F^n given by the equations $x^i = x^i(u^\alpha)$. Suppose that the matrix of the projection factor $B^i{}_\alpha = \partial x^i/\partial u^\alpha$ is of rank $n-1$. The element of support y^i of F^n is to be taken tangential to F^{n-1}, that is,

$$y^i = B^i{}_\alpha(u)v^\alpha. \tag{2.1}$$

Thus v^α is the element of support of F^{n-1} at the point u^α. The metric tensor $g_{\alpha\beta}$ and HV-torsion tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij}B^i{}_\alpha B^j{}_\beta, \quad C_{\alpha\beta\gamma} = C_{ijk}B^i{}_\alpha B^j{}_\beta B^k{}_\gamma. \tag{2.2}$$

At each point u^α of F^{n-1}, a unit normal vector $N^i(u,v)$ is defined by

$$g_{ij}(x(u,v),y(u,v))B^i{}_\alpha N^i = 0, \quad g_{ij}(x(u,v),y(u,v))N^iN^j = 1. \tag{2.3}$$

As for the angular metric tensor h_{ij}, we have

$$h_{\alpha\beta} = h_{ij}B^i{}_\alpha B^j{}_\beta, \quad h_{ij}B^i{}_\alpha N^j = 0, \quad h_{ij}N^iN^j = 1. \tag{2.4}$$

If $(B^\alpha{}_i, N_i)$ denote the inverse of $(B^i{}_\alpha, N^i)$, then we have

$$\begin{aligned} &B^\alpha{}_i = g^{\alpha\beta}g_{ij}B^j{}_\beta, \quad B^i{}_\alpha B^\beta{}_i = \delta^\beta_\alpha, \\ &B^\alpha{}_iN^i = 0, \quad B^i{}_\alpha N_i = 0, \quad N_i = g_{ij}N^j, \\ &B^i{}_\alpha B^\alpha{}_j + N^iN_j = \delta^i_j. \end{aligned} \tag{2.5}$$

The induced connection $IC\Gamma = (\Gamma^{*}_{\beta}{}^{\alpha}{}_{\gamma}, G^\alpha{}_\beta, C_\beta{}^\alpha{}_\gamma)$ of F^{n-1} induced from the Cartan's connection $C\Gamma = (\Gamma^{*}_{j}{}^{i}{}_{k}, \Gamma^{*}_{0}{}^{i}{}_{k}, C_j{}^i{}_k)$ is given by ([6])

$$\Gamma^{*}_{\beta}{}^{\gamma}{}_{\gamma} = B^\alpha{}_i(B_\beta{}^i{}_\gamma + \Gamma^{*}_{j}{}^{i}{}_{k}B^i{}_\beta B^k{}_\gamma) + M^\alpha{}_\beta H_\gamma, \tag{2.6}$$

$$G^\alpha{}_b = B^\alpha{}_i(B_0{}^i{}_\beta + \Gamma^{*}_{0}{}^{i}{}_{k}B^j{}_\beta), \tag{2.7}$$

$$C_\beta{}^\alpha{}_\gamma = B^\alpha{}_iC_j{}^i{}_kB^j{}_\beta B^k{}_\gamma, \tag{2.8}$$

where

$$M_{\beta\gamma} = N_i C_j{}^i{}_k B^j{}_\beta B^k{}_\gamma, \quad M^\alpha{}_\beta = g^{\alpha\gamma} M_{\beta\gamma}, \tag{2.9}$$

$$H_\beta = N_i (B_0{}^i{}_\beta + \Gamma^{*i}_{0\ j} B^j{}_\beta), \tag{2.10}$$

and $B_\beta{}^i{}_\gamma = \partial B^i{}_\beta / \partial u^r$, $B_0{}^i{}_\beta = B_\alpha{}^i{}_\beta v^\alpha$. The quatities $M_{\beta\gamma}$ and H_β are called *second fundamental v-tensor* and *normal curvature vector* respectively ([6]). The second fundamental h-tensor $H_{\beta\gamma}$ is defined as ([6])

$$H_{\beta\gamma} = N_i (B_\beta{}^i{}_\gamma + \Gamma^{*i}_{j\ k} B^j{}_\beta B^k{}_\gamma) + M_\beta H_\gamma, \tag{2.11}$$

where

$$M_\beta = N_i C_j{}^i{}_k B^j{}_\beta N^k. \tag{2.12}$$

The relative h- and v-covariant derivatives of projection factor $B^i{}_\alpha$ with respect to $IC\Gamma$ are given by

$$B^i{}_{\alpha|\beta} = H_{\alpha\beta} N^i, \quad B^i{}_{\alpha|\beta} = M_{\alpha\beta} N^i. \tag{2.13}$$

The equation (2.11) shows that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\alpha\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta. \tag{2.14}$$

Furthermore (2.10), (2.11) and (2.12) yield

$$H_{0\gamma} = H_\gamma, \quad H_{\gamma 0} = H_\gamma + M_\gamma H_0. \tag{2.15}$$

We qnote the following Lemma which is due to Matsumoto [6] as follows:

LEMMA 2.1 ([6]). *The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector H_β vanishes.*

LEMMA 2.2 ([6]). *A hypersurface F^{n-1} is a hyperplane of the first kind if and only if $H_\alpha = 0$.*

LEMMA 2.3 ([6]) *A hypersurface F^{n-1} is a hyperplane of the second kind with respect to the connection $C\Gamma$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

LEMMA 2.4 ([6]) *A hypersurface F^{n-1} is a hyperplane of the third kind with respect to the connection $C\Gamma$ if and only if $H_\alpha = 0$ and $M_{\alpha\beta} = H_{\alpha\beta} = 0$.*

3. Hypersurface $F^{n-1}(c)$ of the first approximate Matsumoto space

Let us consider a special Matsumoto metric with a gradient $b_i(x) = \partial_i b$ for a scalar function $b(x)$ and consider a hypersurface $F^{n-1}(c)$ which is given by the equation $b(x) = c$ (constant). From parametric equations $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$ we get $\partial_\alpha b(x(u)) = 0 = b_i B^i{}_\alpha$, so that $b_i(x)$ are regarded as covariant components of a normal vector field of $F^{n-1}(c)$. Therefore, along the $F^{n-1}(c)$ we have

$$(3.1) \qquad b_i B^i{}_\alpha = 0 \quad \text{and} \quad b_i y^i = 0.$$

In general, the induced metric $L(u, v)$ from the Matsumoto metric is given by

$$L(u,v) = (a_{ij}(x)B^i{}_\alpha B^j{}_\beta v^\alpha v^\beta)^{\frac{1}{2}} + b_i(x)B^i{}_\alpha v^\alpha + \frac{b_i(x)b_j(x)B^i{}_\alpha B^j{}_\beta v^\alpha v^\beta}{\sqrt{a_{ij}(x)B^i{}_\alpha B^j{}_\beta v^\alpha v^\beta}}$$

Therefore, the induced metric of the $F^{n-1}(c)$ becomes

$$(3.2) \qquad L(u,v) = \sqrt{a_{\alpha\beta}(u)v^\alpha v^\beta}, \quad a_{\alpha\beta} = a_{ij}(x)B^i{}_\alpha B^j{}_\beta$$

which is the Riemannian metric.

At the point of $F^{n-1}(c)$, form (1.7), (1.9) and (1.11), we have

(3.3)

$$p = 1, \quad q_0 = 2, \quad q_1 = 0, \quad q_2 = -\alpha^{-2}, \quad p_0 = 2, \quad p_1 = \alpha^{-1}, \quad p_2 = 0,$$
$$\zeta = 1 + 2b^2, \quad S_0 = 2/(1+2b^2), \quad S_1 = \{\alpha(1+2b^2)\}^{-1},$$
$$S_2 = -b^2/\{\alpha^2(1+2b^2)\}.$$

Therefore, from (1.10) we get
(3.4)
$$g^{ij} = a^{ij} - \frac{2}{1+2b^2}b^ib^j - \frac{1}{\alpha(1+2b^2)}(b^iy^j - b^jy^i) + \frac{b^2}{\alpha^2(1+2b^2)}y^iy^j.$$

Thus along F^{n-1}, (3.4) and (3.1) lead to $g^{ij}b_ib_j = \dfrac{b^2}{\alpha^2(1+2b^2)}$.
Therefore, we get

$$b_i(x(u)) = \sqrt{\frac{b^2}{\alpha^2(1+2b^2)}}N_i, \quad b^2 = a^{ij}b_ib_j. \tag{3.5}$$

Again from (3.4) and (3.5) we get

$$b^i = a^{ij}b_j = \sqrt{b^2(1+2b^2)}N^i + b^2\alpha^{-1}y^i. \tag{3.6}$$

Hence, we have the following

THEOREM 3 1 *Let F^n be the first approximate Matsumoto space with a gradient $b_i(x) = \partial_i b(x)$ and let $F^{n-1}(c)$ be a hypersurface of F^n which is given by $b(x) = c$ (constant). Suppose the Riemannian metric $a_{ij}(x)dx^idx^j$ is positive definite and b_i is a non-zero field. Then the induced metric on $F^{n-1}(c)$ is a Riemannian metric given by (3.2) and relation (3.5) and (3.6) hold.*

Along $F^{n-1}(c)$, the angular metric tensor and metric tensor are given by

$$h_{ij} = a_{ij} + 2b_ib_j - \frac{Y_iY_j}{\alpha^2}, \tag{3.7}$$

$$g_{ij} = a_{ij} + 3b_ib_j + \frac{1}{\alpha}(b_iy_j + b_jy_i). \tag{3.8}$$

From (3.1), (3.7) and (2.4) it follows that if $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$, then along $F^{n-1}(c)$ $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$. From (1.11), we get $\dfrac{\partial p_0}{\partial\beta} = 12\alpha^4/(\alpha-\beta)^5$. Thus along $F^{n-1}(c)$,

$\dfrac{\partial p_0}{\partial \beta} = \dfrac{12}{\alpha}$ and therefore (1.13) gives $r_1 = 6/\alpha$, $m_i = b_i$. Therefore, the hv-torsion tensor becomes

$$C_{ijk} = \frac{1}{2\alpha}(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) + \frac{3}{\alpha}b_i b_j b_k. \tag{3.9}$$

Therefore, (2.4), (2.9), (2.12), (3.1) and (3.9) give

$$M_{\alpha\beta} = \frac{1}{2\alpha}\sqrt{\frac{b^2}{1+2b^2}}h_{\alpha\beta}, \quad M_\alpha = 0. \tag{3.10}$$

Hence, from (2.14) it follows that $H_{\alpha\beta}$ is symmetric.

THEOREM 3 2 *The second fundamental v-tensor of* $F^{n-1}(c)$ *is given by (3.10) and the second fundamental h-tensor* $H_{\alpha\beta}$ *is symmetric.*

Next from (3.1) we get $b_{i|\beta}B^i{}_\alpha + b_i B^i{}_{\alpha|\beta} = 0$. Therefore, from (2.13) and the fact that $b_{i|\beta} = b_{i|j}B^j{}_\beta + b_i|_j N^j H_\beta$, we get

$$b_{i|j}B^i{}_\alpha B^j{}_\beta + b_{i|j}B^i{}_\alpha N^j H_\beta + b_i H_{\alpha\beta}N^i = 0. \tag{3.11}$$

Since $b_i|_j = -b_h C_i{}^h{}_j$, from (2.12), (3.5) and (3.10) we get

$$b_i|_j B^i{}_\alpha N^j = \sqrt{\frac{b^2}{\alpha^2(1+2b^2)}}M_\alpha = 0.$$

Thus (3.11) gives

$$\sqrt{\frac{b^2}{1+2b^2}}H_{\alpha\beta} + b_{i|j}B^i{}_\alpha B^j{}_\beta = 0. \tag{3.12}$$

It is noted that $b_{i|j}$ is symmetric. Furthermore, contracting (3.12) with v^β and v^α respectively and using (2.1), (2.15) and (3.10) we get

$$\sqrt{\frac{b^2}{1+2b^2}}H_\alpha + b_{i|j}B^i{}_\alpha y^j = 0, \quad \sqrt{\frac{b^2}{1+2b^2}}H_0 + b_{i|j}y^i y^j = 0. \tag{3.13}$$

In view of Lemmas (2.1), and (2.2), the hypersurface $F^{n-1}(c)$ is hyperplane of the first kind if and only if $H_0 = 0$. Thus from (3.13) it follows that $F^{n-1}(c)$ is a hyperplane of the first kind if and only if $b_{i|j}y^iy^j = 0$. This $b_{i|j}$ being the covariant derivative with respect to $C\Gamma$, of F^n, it may depend on y^i. On the other hand $\nabla_j b_i = b_{ij}$ is the covariant derivative with respect to the Riemannian connection $\left\{ {i \atop jk} \right\}$ constructed from $a_{ij}(x)$, therefore b_{ij} does not depend on y^i. We shall consider the difference $b_{i|j} - b_{ij}$ in the following. The difference tensor $D_j{}^i{}_k = \Gamma^{*i}_j{}_k - \left\{ {i \atop jk} \right\}$ is given by (1.15). Since b_i is a gradient vector, from (1.14) we have $E_{ij} = b_{ij}$, $F_{ij} = 0$, $F^i{}_j = 0$. Thus (1.17) reduces to

$$
\begin{aligned}
D_j{}^i{}_k = {} & B^i b_{jk} + B^i{}_j b_{0k} + B^i{}_k b_{0j} - b_{0m} g^{im} B_{jk} \\
& - C_j{}^i{}_m A^m{}_k - C_k{}^i{}_m A^m{}_j + C_{jkm} A^m{}_s g^{2k} \\
& + \lambda^s (C_j{}^i{}_m C_s{}^m{}_k + C_k{}^i{}_m C_s{}^m{}_j - C_j{}^m{}_k C_m{}^i{}_s).
\end{aligned} \tag{3.14}
$$

But in view of (3.3) and (3.4), the expressions (1.16) reduce to (3.15)

$$
\begin{aligned}
& B_i = 3b_i + \alpha^{-1} y_i, \quad B^i = \frac{2b^i}{1+2b^2} + \frac{y^i}{\alpha(1+2b^2)}, \\
& B_{ij} = \frac{1}{2\alpha}(a_{ij} - \alpha^{-2} y_i y_j + 12 b_i b_j), \\
& B^i{}_j = \frac{1}{2\alpha}(\delta^i_j - \alpha^{-2} y_j y^i) + \frac{5}{\alpha(1+2b^2)} b^i b_j - \frac{1+12b^2}{2\alpha^2(1+2b^2)} y^i b_j, \\
& A^m{}_k = B^m{}_k b_{00} + B^m b_{k0}, \\
& \lambda^m = B^m b_{00}.
\end{aligned}
$$

By virtue of (3.1) we have $B^i{}_0 = 0$, $B_{i0} = 0$ which gives $A^m{}_0 = B^m b_{00}$.

We, therefore, have

$$
D_j{}^i{}_0 = B^i b_{j0} + B^i{}_j b_{00} - B^m C_j{}^i{}_m b_{00}, \tag{3.16}
$$

$$
D_0{}^i{}_0 = B^i b_{00} = \left[\frac{2b^i}{1+2b^2} + \frac{y^i}{\alpha(1+2b^2)} \right] b_{00}. \tag{3.17}
$$

Thus paying attention to (3.1) along the $F^{n-1}(c)$, we finally get

(3.18)
$$b_i D_j{}^i{}_0 = \frac{2b^i}{1+2b^2} b_{j0} + \frac{1+12b^2}{2\alpha(1+2b^2)} b_{00} - 2b^m b_i C_j{}^i{}_m b_{00},$$

(3.19)
$$b_i D_0{}^i{}_0 = \frac{2b^i}{1+2b^2} b_{00}.$$

From (2.12), (3.5), (3.6) and (3.10) it follows that

$$b^m b_i \cdot C_j{}^i{}_m B^j{}_\alpha = b^2 M_\alpha = 0.$$

Therefore, the relation $b_{i|j} = b_{ij} - b_r D_i{}^r{}_j$ and equations (3.18), (3.19) give

$$b_{i|j} y^i y^j = b_{00} - b_r D_0{}^r{}_0 = \frac{1}{1+2b^2} b_{00}.$$

Consequently, (3.13) may be written as

(3.20) $$\sqrt{b^2} H_\alpha + \frac{1}{\sqrt{1+2b^2}} b_{i0} B^i{}_\alpha = 0, \quad \sqrt{b^2} H_0 + \frac{1}{\sqrt{1+2b^2}} b_{00} = 0.$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where b_{ij} does not depend on y^i. Since y^i is to satisfy (3.1), the condition is written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^i)$ for some $c_j(x)$, so that we have

(3.21) $$2b_{ij} = b_i C_j + b_j C_i.$$

From (3.1) and (3.21) it follows that $b_{00} = 0$, $b_{ij} B^i{}_\alpha B^j{}_\beta = 0$, $b_{ij} B^i{}_\alpha y^j = 0$. Hence, (3.20) gives $H_\alpha = 0$. Again from (3.21) and (3.15) we get $b_{i0} b^i = \dfrac{c_0 b^2}{2}$, $\lambda^m = 0$, $A^i{}_j B^j{}_\beta = 0$ and $B_{ij} B^i{}_\alpha B^j{}_\beta = \dfrac{1}{2\alpha} h_{\alpha\beta}$. Thus (2.9), (3.4), (3.5), (3.6), (3.10) and (3.14) give

$$b_r D_i{}^r{}_j B^i{}_\alpha B^j{}_\beta = \frac{-C_0 b^2}{4\alpha(1+2b^2)^2} h_{\alpha\beta}.$$

Therefore, equation (3.12) reduces to

(3.22) $$\sqrt{\frac{b^2}{1+2b^2}} H_{\alpha\beta} + \frac{C_0 b^2}{4\alpha(1+2b^2)^2} h_{\alpha\beta} = 0.$$

Hence the hypersurface $F^{n-1}(c)$ is umbilic.

THEOREM 3 3 *The necessary and sufficient condition for $F^{n-1}(c)$ to be a hyperplane of the first kind is (3.21) and in this case the second fundamental tensor of $F^{n-1}(c)$ is proportional to its angular metric tensor.*

In view of Lemma (2.3), $F^{n-1}(c)$ is hyperplane of second kind if and only if $H_\alpha = 0$, and $H_{\alpha\beta} = 0$. Thus from (3.22) we get $C_0 = C_I(x)y^i = 0$. Therefore, there exist a function $e(x)$ such that $c_i(x) = e(x)b_i(x)$. Thus (3.21) gives

$$b_{ij} = eb_ib_j. \tag{3.23}$$

THEOREM 3 4 *The necessary and sufficient condition for $F^{n-1}(c)$ to be a hyperplane of the second kind is (3.23).*

Finally (3.10) and Lemma 2.4 show that $F^{n-1}(c)$ does not become a hyperplane of the third kind.

THEOREM 3.5 *The hypersurface $F^{n-1}(c)$ is not a hyperplane of the third kind.*

REFERENCES

[1] T Aikou, M Hashiguchi and K Yamaguchi, *On Matsumoto's Finsler space with time measure*, Rep Fac. Sci. Kagoshima Univ (Math. Phys. Chem.) **23** (1990), 1–12

[2] M. Hashiguchi, S Hojo and M Mausumoto, *On Landsberg spaces of two dimensions with (α, β)-metric*, Korean Math. Soc **10** (1973), 17–26

[3] V K Kropina, *On projective Finsler spaces with a metric of some special form*, Naven Doklad Vyas Skolay, Fiz Mat. Nauki **2** (1960), 38–42

[4] V K Kropina, *Projective two-dimensional Finsler spaces with special metric*, Trudy Sem Vektor Tensor Anal **11** (1961), 277–292

[5] M Matsumoto, *On C-reducible Finsler spaces*, Tensor, N. S. **24** (1972), 29–37

[6] M Matsumoto, *The induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry*, J. Math. Kyoto Univ. **25** (1985), 107–144

[7] M Matsumoto, *A slope of a mountain is a Finsler surface with respect to a time measure* **29** (1989), 17–25.

[8] Hong-Suh Park, Il-Yong Lee and Chan Keun Park, *Finsler space with the general approximate matsumoto metric*, Indian J. Pure Appl. Math (to appear).

[9] G Randers, *On an asymmetrical metric in the four-space of general relativity*, Phys, Rev.(2) **59** (1941), 195–199.

[10] C Shibata, *On Finsler spaces with an (α, β)-metric*, J Hokkaido Univ. Edu (Section IIA) **35** (1984), 1–16

Division of Mathematical Sciences
Kyungsung University
Pusan 608-736, Korea
E-mail: iylee@star.kyungsung.ac.kr

Department of Mathematics
Soonchunhyang University
Asan 337-880, Korea