East Asian Math J 17(2001), No. 2, pp 303-308

MONOTONICITY OF EUCLIDEAN CURVATURE UNDER LOCALLY UNIVALENT FUNCTIONS

TAI SUNG SONG

ABSTRACT Let $K(z,\gamma)$ denote the euclidean curvature of the curve γ at the point z. Flinn and Osgood proved that if f is a univalent mapping of the open unit disk $D = \{z : |z| < 1\}$ into itself with $f(0) = 0$ and $|f'(0)| < 1$, then $K(0,\gamma) \le K(0, f\circ\gamma)$ for any C^2 curve γ on D through the origin with $K(0,\gamma) \ge 4$. In this paper we establish a generalization of the Flinn-Osgood Monotonicity Theorem

1. Introduction

Let $\gamma : z = z(t)$, $t \in [a,b]$, be a C^2 curve in the complex plane with $z'(t) \neq 0$ for $t \in [a,b]$. The euclidean curvature $K(z,\gamma)$ of the curve γ at the point $z = z(t)$ is the rate of change of the angle θ that the tangent vector makes with the positive real axis respect to arc length :

$$K(z,\gamma) = \frac{d\theta}{ds} = \frac{d\theta}{dt}\frac{dt}{ds} = \frac{1}{|z'(t)|}\mathrm{Im}\left\{\frac{z''(t)}{z'(t)}\right\}.$$

If f is holomorphic and locally univalent in a neighborhood of γ, then $f\circ\gamma$ is also a C^2 curve with nonvanishing tangent. Let $w = f(z)$ and $\sigma = f\circ\gamma$. Then $w = w(t) = f(z(t))$, $t \in [a,b]$ is a parametrization of σ. We have

Received February 23, 2001. Revised October 2, 2001
2000 Mathematics Subject Classification: 30C45.
Key words and phrases hyperbolic distance, uniform local univalence.

$$K(w,\sigma) = \frac{1}{|w'(t)|}\mathrm{Im}\left\{\frac{w''(t)}{w'(t)}\right\}$$
$$= \frac{1}{|f'(z)|\,|z'(t)|}\mathrm{Im}\left\{\frac{f''(z)z'(t)^2 + f'(z)z''(t)}{f'(z)z'(t)}\right\}.$$

This yields the formula for the change of the euclidean curvature under the locally univalent function f :

$$K(f(z), f\circ\gamma)\,|f'(z)| = K(z,\gamma) + \mathrm{Im}\left\{\frac{f''(z)}{f'(z)}\frac{z'(t)}{|z'(t)|}\right\}.$$

Euclidean curvature is closely related to the concepts of local univalence and uniform local univalence. The notion of uniform local univalence is defined relative to hyperbolic geometry on the open unit disk $D = \{z : |z| < 1\}$ in the complex plane. The connection between the derivative of a locally univalent function and euclidean curvature has been used in a number of ways([2], [4], [7]). Flinn and Osgood [2] established a monotonicity property of euclidean curvature. They proved that if f is a univalent mapping of D into itself with $f(0) = 0$ and $|f'(0)| < 1$, then

$$K(0,\gamma) \le K(0, f\circ\gamma) \tag{1}$$

for any C^2 curve γ on D through the origin with $K(0,\gamma) \ge 4$.

In this paper we establish a generalization of the Flinn-Osgood Monotonicity Theorem. We give a sufficient condition for the monotonicity property (1) to hold for any locally univalent function f of D into itself with $f(0) = 0$ and $|f'(0)| < 1$. The simple condition is that

$$K(0,\gamma) \ge \frac{4}{\tanh(\rho(f)/2)},$$

where $\rho(f)$ is the hyperbolic radius of uniform univalence.

2. Monotonicity of euclidean curvature

We begin this section with a brief introduction to hyperbolic geometry on the open unit disk $D = \{z : |z| < 1\}$. For a general discussion of hyperbolic geometry on D we refer the reader to [1], [3], and [5]. The hyperbolic distance on D induced by the hyperbolic metric $\lambda_D(z)|dz| = 2|dz| / \left(1 - |z|^2\right)$ is

$$d_h(a, b) = 2 \tanh^{-1} \left| \frac{a - b}{1 - a\bar{b}} \right|.$$

The hyperbolic disk in D with center $a \in D$ and hyperbolic radius ρ, $0 < \rho \leq \infty$, is defined by

$$D_h(a, \rho) = \{z : d_h(z, a) < \rho\}.$$

Suppose f is a holomorphic function on D. For $z \in D$, let $\rho(z, f)$ be the hyperbolic radius of the largest hyperbolic disk in D centered at z in which f is univalent. Set

$$\rho(f) = \inf\{\rho(z, f) : z \in D\}.$$

A holomorphic function f on D is called *uniformly locally univalent* (in the hyperbolic sense) if $\rho(f) > 0$. The quantity $\rho(f)$ is called *the hyperbolic radius of uniform univalence for f*.

We note that

$$z \in D_h(a, \rho) \Leftrightarrow \left| \frac{z - a}{1 - \bar{a}z} \right| < \tanh \frac{\rho}{2}.$$

Let $R = \tanh \frac{\rho}{2}$. Then

$$z \in D_h(a, \rho) \Leftrightarrow |z - a|^2 < R^2 |1 - \bar{a}z|^2.$$

From the inequality $|z - a|^2 < R^2 |1 - \bar{a}z|^2$, we obtain

$$\left| z - \frac{\left(1 - R^2\right) a}{1 - R^2 |a|^2} \right| < \frac{R^2 \left(1 - |a|^2\right)^2}{\left(1 - R^2 |a|^2\right)^2}.$$

Thus, the hyperbolic disk $D_h(a,\rho)$ is a euclidean disk $D(c,r) = \{z : |z-c| < r\}$, where

$$c = \frac{1-\left(\tanh\frac{\rho}{2}\right)^2}{1-\left(\tanh\frac{\rho}{2}\right)^2|a|^2}a, \ r = \tanh\frac{\rho}{2}\frac{1-|a|^2}{1-\left(\tanh\frac{\rho}{2}\right)^2|a|^2}.$$

In particular, we obtain the following result.

LEMMA 1 $D_h(0,\rho) = D\left(0,\tanh\frac{\rho}{2}\right)$.

If $g(z) = a_1 z + a_2 z^2 + \cdots$ is a univalent mapping of D into itself, then the Schiffer-Tammi inequality states that

$$|a_2| \le 2|a_1|(1-|a_1|).$$

Liu and Minda [6] proved that if f is a locally univalent function of D into itself. then the number $2/\tanh[\rho(f)/2]$ is an upper bound of the hyperbolic linear invariant norm of f. The following result is a slight modification of the result of Liu and Minda.

LEMMA 2. *If f is a locally univalent function of D into itself with $f(0)=0$ and $|f'(0)| < 1$, then*

$$\frac{|f''(0)|}{|f'(0)|(1-|f'(0)|)} \le \frac{4}{\tanh[\rho(f)/2]}.$$

PROOF. If $\rho(f)=0$, then the inequality is obvious. Suppose $\rho(f) > 0$. Then, by Lemma 1, f is univalent in the euclidean disk $D(0,R)$, where $R = \tanh[\rho(f)/2]$. Let $g(z) = f(Rz)/R$. Then g is a univalent mapping of D into itself with $g(0) = 0$. By the Schiffer-Tammi inequality, we have

$$|g''(0)| \le 4|g'(0)|(1-|g'(0)|). \tag{2}$$

Note that $|g'(0)| = |f'(0)| < 1$ and $|g''(0)| = R|f''(0)|$. Therefore, the inequality (2) yields

$$\frac{|f''(0)|}{|f'(0)|(1-|f'(0)|)} \leq \frac{4}{R} = \frac{4}{\tanh[\rho(f)/2]}.$$

We now establish an inequality for the change of euclidean curvature at the origin under a locally univalent function of the unit disk D into itself that fixes the origin.

THEOREM 3 *Suppose f is a locally univalent function of D into itself with $f(0) = 0$ and $|f'(0)| < 1$. If γ is a C^2 curve on D through the origin, then*

$$K(0,\gamma) \leq K(0, f \circ \gamma)$$

provided that $K(0,\gamma) \geq 4/\tanh[\rho(f)/2]$.

PROOF Let $\alpha = |f''(0)| / |f'(0)|(1-|f'(0)|)$. Then the transformation law for euclidean curvature produces

$$\begin{aligned} K(0, f\circ\gamma)|f'(0)| &\geq K(0,\gamma) - \frac{|f''(0)|}{|f'(0)|} \\ &= K(0,\gamma) - \alpha + \alpha|f'(0)|. \end{aligned}$$

The above inequality yields

$$|f'(0)|(K(0, f\circ\gamma) - \alpha) \geq K(0,\gamma) - \alpha. \tag{3}$$

From Lemma 2, we obtain $K(0,\gamma) - \alpha \geq 0$. Since $|f'(0)| < 1$, the the inequality (3) implies that $K(0,\gamma) \leq K(0, f\circ\gamma)$ whenever $K(0,\gamma) \geq 4/\tanh[\rho(f)/2]$.

REMARK If f is a univalent mapping of D into iteslf, then Lemma 1 yields $\tanh[\rho(f)/2]$=1. This shows that Theorem 3 is a generalization of the Flinn-Osgood Monotonicity Theorem.

REFERENCES

[1] L.V. Ahlfors, *Conformal invariants. Topics in geometric function theory*, McGraw-Hill, New York, 1973

[2] B Flinn and B. Osgood, *Hyperbolic curvature and conformal mapping*, Bull. London Math. Soc. **18** (1986), 272–276.

[3] G.M Goluzin, *Geometric theory of functions of a complex variable, Translations of Mathematical Monographs*, **26**, Amer. Math. Soc., Providence, 1969.

[4] A.W Goodman, *Curvature under an analytic transformation*, J London Math Soc. **43** (1968), 527–533

[5] I. Kra, *Automorphic functions and Kleinian groups*, W. A. Bejamin, Reading, MA, 1980.

[6] X. Liu and D. Minda, *Monotonicity of hyperbolic curvature under univalent mappings*, Annales Academiæ Scientiarum Fennicæ Series A I. Mathematica **16** (1991), 227–242

[7] D Minda, *Hyperbolic curvature on Riemann surfaces*, Complex Variables Theory Appl. **12** (1989), 1–8

Department of Mathematics Education
Pusan National University
Pusan 609-735, Korea
E-mail: tssong@pusan.ac.kr