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ON THE REALIZATION OF THE DOUBLE LINK AS A BRANCHED SET

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ABSTRACT We construct a family of 3-manifolds by pairwise identifications of faces in the boundary of certain polyhedral 3-balls and prove that all these manifolds are cyclic branched coverings of the 3-sphere over the double link

1. Introduction

A well known result about the realization of closed 3-manifolds says that any closed 3-manifold can be represented as a branched covering of some link in the 3-sphere. So if we consider a link in the 3-sphere, we can construct many classes of closed orientable 3-manifolds by considering its branched coverings. Indeed the description of closed 3-manifolds as polyhedral 3-balls, whose finitely many boundary faces are glued together in pairs, is another standard way to construct 3-manifolds. Many authors have studied the connections between the face identification procedure and the representation of closed 3-manifolds as branched coverings of the 3-sphere. In [9] Helling, Kim, and Mennicke considered a family of polyhedra yielding closed orientable 3-manifolds and proved that these manifolds are n-fold cyclic coverings of the 3-sphere branched over the Whitehead link. More general cases were handled by A. Cavicchioli and L. Paoluzzi [4]. The trefoil knot, the figure-eight knot and 5_2 knot were realized as branched sets of cyclic

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coverings over the 3-sphere in [2], [10] and [12] respectively. In this paper we consider the double link and links $\mathcal{L}_{(1,d)}$ for a positive integer d as shown in Fig. 1, where the index d denotes the number of half twists. We note that $\mathcal{L}_{(1,d)}$ are links of two or three components according as d is even or odd.

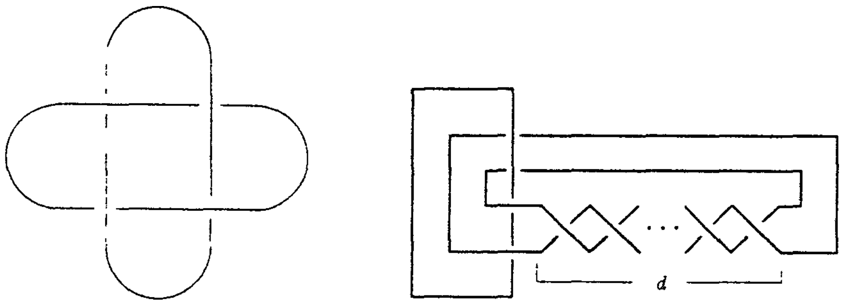

Fig. 1 (a) Double link figure (b) The link $\mathcal{L}_{(1,d)}$

We consider an infinite family polyhedral 3-cell $\mathcal{P}(3,n,k)$ with oriented edges for $n \geq 3, k \geq 2$ and construct an infinite family of 3-manifolds $\mathcal{M}(3,n,k)$ by the identification of oppositely oriented boundary faces of $\mathcal{P}(3,n,k)$. We prove that $\mathcal{M}(3,n,k)$ are cyclic (n/d)-fold coverings of the 3-sphere branched over $\mathcal{L}_{(1,d)}$ where $(n,k) = d$ by two methods. One is the technique of cancelling handles on Heegaard diagrams. The other one is the combinatorial representation of closed 3-manifolds by a special class of edge-colored graphs, called crystallizations. Moreover $\mathcal{M}(3,n,k)$ are cyclic branched n-fold coverings of the double link in the 3-sphere, where the branched indices of its components are n and n/d, respectively.

2. Construction of a family of 3-manifolds $\mathcal{M}(3,n,k)$

We construct an infinite family of 3-manifolds $\mathcal{M}(3,n,k)$ by pairwise identification of the 2-faces of a polyhedron $\mathcal{P}(3,n,k)$ for $n \geq 3, k \geq 2$, which is homeomorphic to a 3-ball, whose boundary complex provides a tessellation of the 2-sphere as shown in Fig. 2. The tessellation consists of two n-gons in the northern and southern hemispheres, and $2n$ triangles in the equatorial zone. Then $\mathcal{P}(3,n,k)$ has $2n+2$ faces,

$4n$ edges and $2n$ vertices.

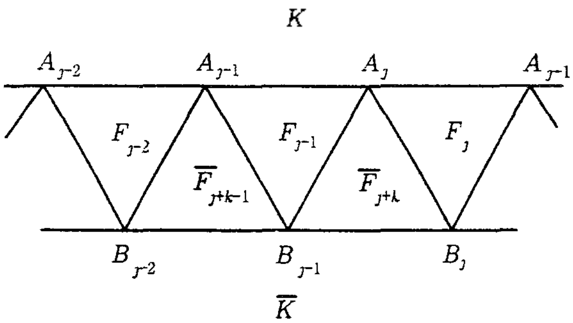

Fig. 2. $\mathcal{P}(3, n, k)$

We define the boundary cycles of two n gons and $2n$ triangles as follow:

$$\begin{aligned} K &: \quad A_1 A_2 \cdots A_n \\ \overline{K} &: \quad B_1 B_2 \cdots B_n \\ F_j &: \quad A_j A_{j+1} B_j \\ \overline{F}_j &: \quad B_{j-k-1} B_{j-k} A_{j-k} \end{aligned}$$

for $j = 1, \ldots, n$ which are depicted in Fig. 2.

Identify faces of the polyhedron:

(1) $$S : \begin{array}{l} K \to \overline{K} \\ A_j \to B_{j-2} \quad \text{for} \quad 1 \leq j \leq n \end{array}$$

and for each $j = 1, \ldots, n$,

(2) $$R_j : \begin{cases} F_j \to \overline{F}_j \\ A_j \to A_{j-k} \\ A_{j+1} \to B_{j-k-1} \\ B_j \to B_{j-k}. \end{cases}$$

Consider the oriented edges

$$x_j = (A_j, A_{j+1}) \text{ and } u_j = (B_j, A_j).$$

Then the identifications (1) and (2) show that each oriented edge u_j has $\frac{n}{(n,k)}$ equivalent edges for $j = 1, 2, \dots, n$. We now calculate the Euler characteristic of a cellular complex K_n induced by the face identification of a polyhedron $\mathcal{P}(3,n,k)$. We note that there is a rotation symmetry T by

$$T : A_j \to A_{j-1} \text{ and } B_j \to B_{j-1} \text{ for } j = 1, 2, \dots, n$$

of the $\mathcal{P}(3,n,k)$. Thus it suffices to consider our case in the quotient space $\mathcal{P}(3,n,k)/T$ and $(n,k) = 1$ in Fig. 3, where N, S are the centers of two n-gons in $\mathcal{P}(3,n,k)$.

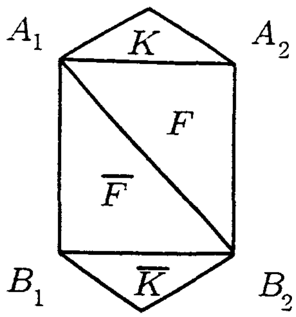

Fig. 3. The quotient space $\mathcal{P}(3,n,k)/T$

Indeed, all edges of two triangles on the equatorial zone except (A_1, B_2) are equivalent under the composition action TRT, and all (N, A_i) and (S, B_i) are equivalent under the composition action RQR, where

$$Q : A_1 \to B_1, A_2 \to B_2, N \to S,$$

$$R : N \to N, S \to S, A_1 \to A_2, B_1 \to B_2,$$

and

$$T : A_1 \to A_1, A_2 \to B_1, B_2 \to B_2.$$

This means that a 3-dimensional cellular complex K_n has d vertices, $n+d$ edges, $n+1$ triangles and one 3-cell and so its Euler characteristic is $\chi(K_n) = d - (n+d) + (n+1) - 1 = 0$. Hence the resulting space

$\mathcal{M}(3,n,k) = |K_n|$ is a closed, oriented, compact 3-manifold, due to H. Seifert and W. Threlfall criterion that a complex, which is formed by identifying the faces of a polyhedron will be a manifold if and only if its Euler characteristic equals zero.

3. Realizations of the Double link

We now realize the double link as a branched set of covering spaces of the 3-sphere.

THEOREM *The closed connected orientable 3-manifolds $\mathcal{M}(3,n,k)$ are cyclic (n/d)-fold coverings of the 3-sphere branched over a link $\mathcal{L}_{(1,d)}$, where $d = (n,k)$. Furthermore, $\mathcal{M}(3,n,k)$ are cyclic branched n-fold coverings of the double link in the 3-sphere, where the branched indices of its components are n and n/d, respectively.*

PROOF The symmetry of $\mathcal{P}(3,n,k)$ enables us to consider a particular case without a loss of generality. The general case can be handled by the same way. The singular set is the image in the quotient of the axis of a rotation symmetry T and of all the edges (A_i, B_i) in Fig. 2. Notice that these edges are identified by the gluing groups of $\frac{n}{(n,k)}$ edges. The polyhedron $\mathcal{P}(3,n,k)$ defines in a natural way a decomposition of $\mathcal{M}(3,n,k)/T$ into handles. We note that the 3-handles are neighborhoods in $\mathcal{M}(3,n,k)/T$ of the images of the vertices of $\mathcal{P}(3,n,k)$, the 2-handles are the neighborhoods of the images of its edges, and the 1-handles are the neighborhoods of the images of its faces. Hence we can express all the information related to the gluing patterns in a planar graph, called a Heegaard diagram. For a Heegaard diagram, we take an arc and an orientation on it. The endpoint of this arc is glued onto another point by the identification of the faces. Then we have another arc starting at this latter point. We continue this procedure until we return to the arc at which we started. If there are arcs without orientation, we pick up one of them and do the same operation as before. Fig. 4 shows a Heegaard diagram of the identification space $\mathcal{M}(3,6,3)/T$, where the dotted line denotes the singular axis.

Eliminate 1-handles X_1, X_2 and X_3 with 2-handles α, β and γ respectively.

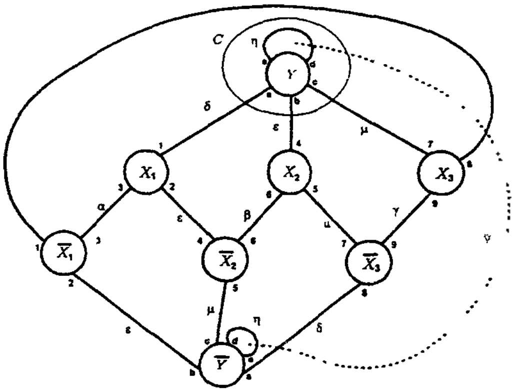

Fig. 4.

We then dig a ball along C to reduce the complexity. Glue 1-handle Y and create 1-handle Y^* to get a deformed graph as shown in Fig. 5 where the broken lines are the singular cores of 2-handles.

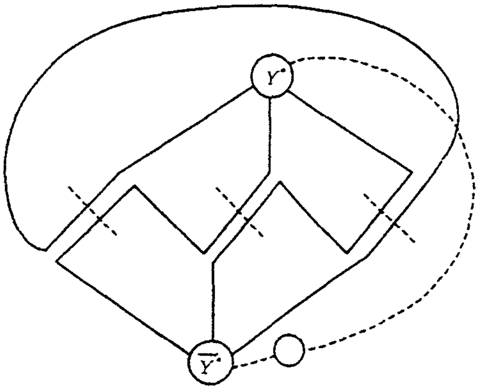

Fig. 5.

Then by cancelling 1-handle Y^* with 2-handle δ and Reidemeister moves, we have the link $\mathcal{L}_{(1,3)}$ as shown in Fig. 6, where all components have a branching index 2. This gives $\mathcal{M}(3,6,3)$ is a 2-fold covering of the 3-sphere branched over $\mathcal{L}_{(1,3)}$. The quotient by a 3-rotation along one trivial component gives that $\mathcal{M}(3,6,3)$ is a 6-fold cyclic covering of the 3-sphere branched over the double link, where the numbers denote branched indices.

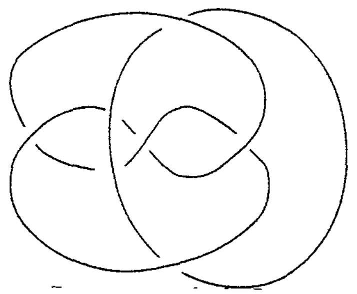

Fig. 6. The link $\mathcal{L}_{(1,3)}$

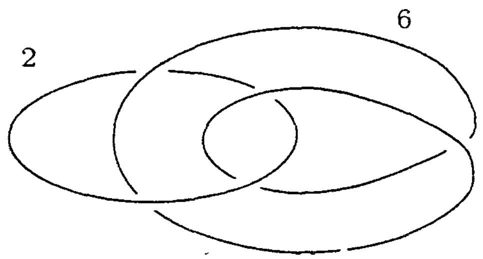

Fig. 7. The double link

For the alternating approach to Theorem, we introduce some basic results on the representation theory of closed triangulated manifolds via colored graph called crystallizations. For the further detail information, refer [6], [7], [8] and [13].

THEOREM ([13]) *Any closed connected piecewise-linear n-manifold can be represented by a crystallization.*

THEOREM ([5], [11]) *Let M, M' be closed 3-manifolds and (Γ,γ), (Γ',γ') two crystallizations of them. Then the following statements are equivalent:*

(1) *M is homeomorphic with M',*
(2) *(Γ,γ) and (Γ',γ') are (I,II)-equivalent,*
(3) *(Γ,γ) and (Γ',γ') are A-equivalent,*
(4) *(Γ,γ) and (Γ',γ') are LCG-equivalent.*

In [7], the algorithm for constructing a crystallization of the 2-fold cyclic covering of the 3-sphere branched over a bridge-presentation of a link and the following main theorem were given.

THEOREM ([7]) *Given a bridge-presentation of a link L, the associated 2-symmetric graph is a crystallization of the 2-fold cyclic covering of the 3-sphere branched over L.*

LEMMA *The closed orientable 3-manifolds $\mathcal{M}(3,2d,d)$ are 2-fold cyclic coverings of the 3-sphere branched over a link $\mathcal{L}_{(1,d)}$. Furthermore, $\mathcal{M}(3,2d,d)$ are 2d-fold cyclic branched coverings of the double link in the 3-sphere, where the branched indices of its components are 2d and 2, respectively.*

PROOF We claim that $\mathcal{M}(3,2d,d) := \mathcal{M}(2d,d)$ can be represented by a 2-symmetric crystallization $\Gamma(2d,d)$. Consider the polyhedral schemata $\mathcal{P}(3,2d,d) := \mathcal{P}(2d,d)$, which defines the closed orientable 3-manifold $\mathcal{M}(2d,d)$ as a quotient of a triangulated 3-ball B^3 by pairwise identification of its boundary 2-cells(see Fig. 2). Triangulate $\mathcal{P}(2d,d)$ into a simplicial complex $K(3,2d,d) := K(2d,d)$ by using stellar subdivisions two different ways according as d is even or odd. For example, see triangulations of $\mathcal{P}(4,2)$ and $\mathcal{P}(6,3)$ in Fig. 8(a) and (b)

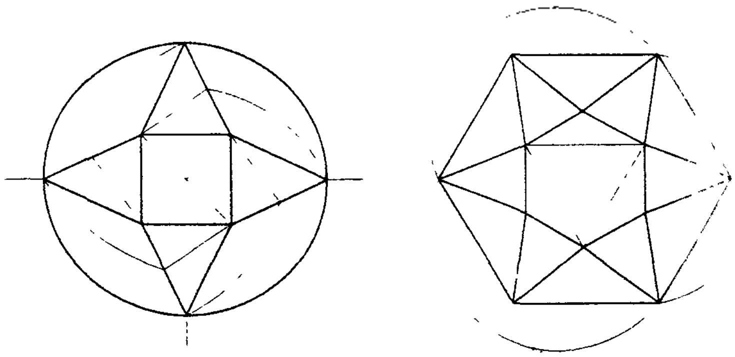

Fig. 8 (a). $K(4.2)$ (b). $K(6,3)$

We only treat the case d odd. Indeed the case d even can be handled by the same way. Moreover we consider the case $d = 3$. That is, we claim that $\mathcal{P}(3,6,3)$ can be represented by a 2-symmetric crystallization $\Gamma(6,3)$. One can immediately extend the construction for the general cases by a simple iteration. Let $\mathcal{P}(6,3)$ be the polyhedral schemata which defines the closed orientable 3-manifold $\mathcal{M}(6,3)$ as a quotient of a triangulated 3-ball B^3 by pairwise identification of its boundary 2-cells. Triangulate $\mathcal{P}(6,3)$ into a colored simplicial complex

$K(6,3)$.

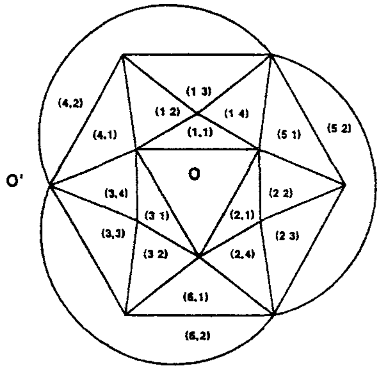

Fig. 9. The colored complex $K(6,3)$

We now construct a crystallization $\Gamma(6,3)$ associated to $K(6,3)$ as follows(see Fig. 9). The vertices of $\Gamma(6,3)$ are the elements of

$$V(6,3) = \{(i,j(i))|i = 1,2,\dots,6\} \cup \{O,O'\}$$

where

$$\begin{cases} 1 \le j(i) \le m+3 & \text{if } \ i = 1,2,3, \\ 1 \le j(i) \le 2 & \text{if } \ i = 4,5,6. \end{cases}$$

The colored edges are defined by means of four fixed-point-free involutions v_0, v_1, v_2 and v_3 on $V(6,3)$. That is, for each $i \in \Delta = \{0,1,2,3\}$ two vertices x and y in $V(6,3)$ are joined by an edge colored i if and only if $y = v_i(x)$. For this purpose, we consider a subset V_1 of $V(6,3)$;

$$V_1 = \{(i,j(i))|i = 1,4\} \cup \{O,O'\}$$

with the following edge-colorations:

$$\begin{cases} v_0(1,1) = (5,2), \ \ v_0(1,2) = (6,1), \\ v_0(1,3) = (2,4), \ \ v_0(1,4) = (3,3), \\ v_0(4,1) = (2,2), \ \ v_0(4,2) = (3,1), \\ v_0(O) = O', \end{cases}$$

$$\begin{cases} v_1(1,j) = (1, j-(-1)^j) \ \text{ where } \ j \in \mathbb{Z}_4, \\ v_1(4,2) = O', \end{cases}$$

$$\begin{cases} v_2(1,1) = O, \ \ v_2(1,2) = (4,1), \\ v_2(1,3) = (4,2), \ \ v_2(1,4) = (5,1), \end{cases}$$

$$\begin{cases} v_3(1,j) = (1, j + (-1)^j) \ \text{ where } \ j \in \mathbb{Z}_4, \\ v_3(4,1) = (4,2). \end{cases}$$

If we consider the action of a permutation $\eta = (1\ 2\ 3)(4\ 5\ 6)$ on V as follows;

$$\eta(v,w) = (\eta(v), w) \text{ for } (v,w) \in V \backslash \{O, O'\}, \text{ and}$$
$$\eta \text{ fixes } O \text{ and } O'.$$

then $V_1 \cup \eta(V_1) \cup \eta^2(V_1) = V$. For the edge-coloration of a crystallization $\Gamma(6,3)$ associated to $K(6,3)$ we use the following rule:

if $y = v_i(x)$ for x, y in V, then we define $\eta(y) = v_{\eta(i)}(\eta(x))$.

Fig. 10 gives a crystallization associated to $K(6,3)$ with an extended 1-dipole $\{O, (2,1)\}$ where the dotted line is the auxiliary line for a LCG move for an extended 1-dipole $\{O, (2,1)\}$.

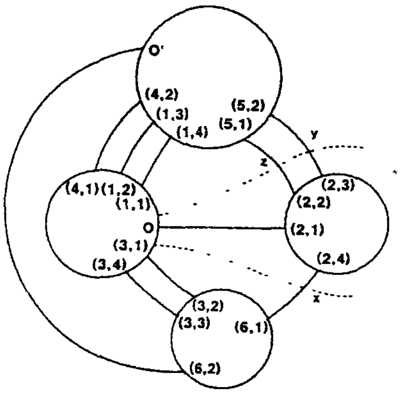

Fig. 10. A crystallization of $\mathcal{M}(6,3)$

We apply a LCG move for an extended 1-dipole $\{O, (2,1)\}$ to get a 2-symmetric crystallization of $\mathcal{M}(6,3)$ as shown in Fig. 11, where the dotted lines denote the axis of a 2-symmetric Heegaard splitting induced by a 2-symmetric crystallization. We note that there may be several extended 1-dipoles. However the result is independent of the choice of an extended 1-dipole for a *LCG* move.

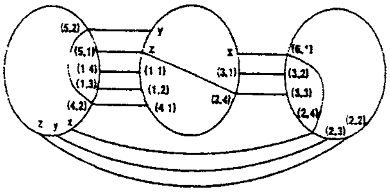

Fig. 11. A 2-symmetric crystallization of $\mathcal{M}(6,3)$

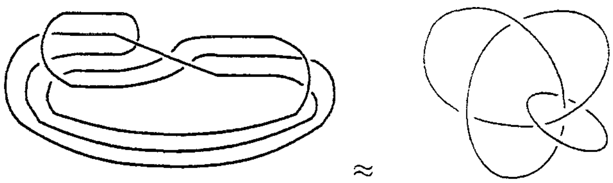

Fig. 12 (a) (b). The link $\mathcal{L}_{(1,3)}$

By applying Reidemeister moves on a 3-bridge link induced by a 2-symmetric crystallization of $\mathcal{M}(6,3)$(see Fig. 12(a)), we have that the link is equivalent to the link $\mathcal{L}_{(1,3)}$ as shown in Fig. 12(b). We note that $\mathcal{M}(2d,d)$ are 2-fold cyclic branched coverings of the 3-sphere over $\mathcal{L}_{(1,d)}$. Furthermore $\mathcal{M}(2d,d)$ are $2d$-fold cyclic branched coverings of the double link in the 3-sphere, where the branched indices of its components are $2d$ and 2, respectively.

PROOF OF THEOREM . We note that $\mathcal{M}(2d,d)$ admit 2-symmetric Heegaard splittings by Lemma. Thus $\mathcal{M}(3,n,k)$ admit (n/d)-symmetric Heegaard splittings by the rotational symmetry, where $d=(n,k)$ as a sense of [1]. This means that $\mathcal{M}(3,n,k)$ are (n/d)-fold cyclic branched coverings of $\mathbb{S}^3$ over the link $\mathcal{L}_{(1,d)}$. The quotient by a d-rotation along one trivial component gives that $\mathcal{M}(3,n,k)$ are n-fold cyclic coverings of the 3-sphere branched over the double link, where the branched indices of its components are n and n/d, respectively. This completes the proof.

We denote by $\mathcal{O}_{n/d}(\mathcal{L}_{(1,d)})$ an orbifold whose underlying space is the 3-sphere and whose singular set is $\mathcal{L}_{(1,d)}$ with branched index n/d. Similarly by $\mathcal{O}_{n,n/d}$(Double link) we denote an orbifold whose underlying space is the 3-sphere and whose singular set is the double link with branching indices of its components are n and n/d, respectively. Then we obtain the following commutative diagram of branched coverings.

COROLLARY

$$\begin{array}{ccc} \mathcal{M}(3,n,k) & \xrightarrow{n/d} & \mathcal{O}_{n/d}(\mathcal{L}_{(1,d)}) \\ \| & & \downarrow d \\ \mathcal{M}(3,n,k) & \xrightarrow{n} & \mathcal{O}_{n,n/d}(Double\ link) \end{array}$$

where the labels of the maps indicate the degree of the covering.

REFERENCES

[1] J.S. Birman and H.M. Hilden, *Heegaard splittings of branched coverings of S^3*, Trans. Amer. Math Soc **213** (1975), 315-352

[2] A. Cavicchioli, F Hegenbarth and A. C Kim, *A geometric study of Sieradski groups*, Algebra Colloquium **5:2,** (1998), 203–217.

[3] A. Cavicchioli, F Hegenbarth and A.C Kim, *On cyclic branched coverings of Torus knots*, J Geom. **64** (1999), 55–66

[4] A. Cavicchioli and L Paoluzzi, *On certain classes of hyperbolic 3-manifolds*, manuscripta math **101** (2000), 457–494.

[5] M. Ferri and C Galiardi, *Crystallization moves*, Pacific J Math. **100** (1982), 85–103

[6] M. Ferri, C Galiardi and L Grasselli, *A graph-theoretical representation of PL-manifolds- A survey on crystallizations*, Aequationes Math **31** (1986), 121–141.

[7] M. Ferri, *Crystallizations of 2-fold branched coverings of* $\mathbb{S}^3$, Proc. Amer. Math. Soc. **73** (1979), 271–276

[8] C. Gagliardi, *A combinatorial characterization of 3-manifold crystallizations*, Boll. Un. Mat Ital **16-A** (1979), 441–449

[9] H. Helling, A.C. Kim and J L Mennicke, *Some honey-combs in hyperbolic 3-space*, Comm. in Algebra **23** (1995), 5169–5206.

[10] H. Helling, A C. Kim, and J. Mennicke, *A geometric study of Fibonacci groups*, J. of Lie Theory **8. no. 1** (1998), 1–23

[11] H Im and S H. Kim, *Heegaard splittings of the brieskorn homology spheres that are equivalent after one stabilization*, Noti di Matematica **21, no. 1** (2001), 65–76

[12] G. Kım, Y Kim and A Vesnin, *The knot* 5_2 *and cyclically presented groups*, J. Korean Math Soc **35, No. 4** (1998), 961–980.
[13] M. Pezzana, *Sulla struttura topologica della varietà compatte*, Attı Sem Mat Fis Unıv Modena **23** (1974), 269–277

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