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SOME SUMMATION FORMULAS FOR THE APPELL'S FUNCTION F_1

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ABSTRACT The authors aim at presenting summation formulas of Appell's function F_1

$$F_1\left(a,\, b,\, b',\, 1+a+b-b'+i,\, 1,\, -1\right) \quad (i=0,\, \pm1,\, \pm2,\, \pm3,\, \pm4,\, \pm5),$$

which, for $i=0$, yields a known result due to Srivastava.

1. Introduction and Results Required

We start with a known identity

$$\begin{aligned} &F_1\left(a;\, b,\, b';\, 1+a+b-b';\, 1,\, -1\right) \\ &\quad = \frac{\Gamma(1-b')\,\Gamma\left(1+\frac{1}{2}a\right)\,\Gamma(1+a+b-b')}{\Gamma(1+a)\,\Gamma(1+b-b')\,\Gamma\left(1+\frac{1}{2}a-b'\right)}, \end{aligned} \tag{1.1}$$

which is in the work of Srivastava [3] who pointed out that the formula (1.1) can be proved fairly easily by expressing the Appell's function F_1 as an infinite series involving Gauss's ${}_2F_1$ and then employing, in turn, the known results:

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Gauss's summation theorem [1]

$$ {}_2F_1(a,\, b;\, c;\, 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \tag{1.2} $$

provided $\Re(c-a-b) > 0$ and $c \neq 0, -1, -2, \ldots$;

Kummer's theorem [1]

$$ {}_2F_1(a,\, b;\, 1+a-b;\, -1) = \frac{\Gamma(1+a-b)\,\Gamma\left(1+\frac{1}{2}a\right)}{\Gamma(1+a)\,\Gamma\left(1+\frac{1}{2}a-b\right)} \tag{1.3} $$

provided $\Re(b) < 1$ and $1+a-b \neq 0, -1, -2, \ldots$.

Recently Lavoie, Grondin and Rathie [2] have obtained the following extension of (1.3) in the form:

$$ \begin{aligned} {}_2F_1(a,\, b;\, 1+a-b+i;\, -1) &= \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma(1-b)\,\Gamma(1+a-b+i)}{2^a\,\Gamma\left(1-b+\frac{1}{2}(i+|i|)\right)} \\ &\times \Bigg\{ \frac{\alpha_i(a,b)}{\Gamma\left(1+\frac{1}{2}a-b+\frac{1}{2}i\right)\,\Gamma\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}i-\left[\frac{i+1}{2}\right]\right)} \\ &\quad + \frac{\beta_i(a,b)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a-b+\frac{1}{2}i\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}i-\left[\frac{1}{2}i\right]\right)} \Bigg\} \end{aligned} \tag{1.4} $$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. Also $|x|$ denotes the absolute value of x and $[x]$ is the greatest integer less than or equal to x. The values of $\alpha_i(a,b)$ and $\beta_i(a,b)$ are given in the following table.

Table of $\alpha_i(a,b)$ and $\beta_i(a,b)$

i	$\alpha_i(a,b)$	$\beta_i(a,b)$
5	$-[4(6+a-b)^2 - 2b(a-b+6)$	$4(a-b+6)^2 + 2b(a-b+6)$
	$-b^2 - 22(a-b+6) + 13b + 20]$	$-b^2 - 34(a-b+6) - b + 62$
4	$2(3+a-b)(1+a-b) - (b-1)(b-4)$	$-4(2+a-b)$
3	$3b-2a-5$	$2a-b+1$
2	$a-b+1$	-2

1	-1	1
0	1	0
-1	1	1
-2	$a-b-1$	2
-3	$2a-3b-4$	$2a-b-2$
-4	$2(a-b-3)(a-b-1)-b(b+3)$	$4(a-b-2)$
-5	$4(a-b-4)^2-2b(a-b-4)$ $-b^2+8(a-b-4)-7b$	$4(a-b-4)^2+2b(a-b-4)$ $-b^2+16(a-b-4)-b+12$

Here we aim at presenting eleven results contiguous to (1.1) in a single form by using the same technique as in Srıvastava [3].

2. Main Summation Formulas

The results to be proved are

$$
\begin{aligned}
&F_1(a;\, b,\, b';\, 1+a+b-b'+i;\, 1,\, -1)\\
&\quad= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1-b')\,\Gamma(1-b'+i)\,\Gamma(1+a+b-b'+i)}{2^a\,\Gamma(1+b-b'+i)\,\Gamma\left(1-b'+\frac{1}{2}(i+|i|)\right)}\\
&\quad\times\left\{\frac{\alpha_i(a,b')}{\Gamma\left(\frac{1}{2}a-b'+\frac{1}{2}i+1\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}i-\left[\frac{i+1}{2}\right]\right)}\right.\\
&\quad\quad\left.+\frac{\beta_i(a,b')}{\Gamma\left(\frac{1}{2}a-b'+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}i-\left[\frac{1}{2}i\right]\right)}\right\}
\end{aligned}
\tag{2.1}
$$

for $i=0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5$. Also $|x|$ denotes the absolute value of x and $[x]$ is the greatest integer less than or equal to x. Here the values of $\alpha_i(a,b')$ and $\beta_i(a,b')$ are the same as given in the table of $\alpha_i(a,b)$ and $\beta_i(a,b)$ by simply replacing b by b'.

To prove (2.1), denote the left-hand side of (2.1) by I and express the Appell's function F_1 in series as follows:

$$\begin{aligned} I &= F_1(a;\, b,\, b';\, 1+a+b-b'+i;\, 1,\, -1) \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m\,(b')_n}{(1+a+b-b'+i)_{m+n}}\,\frac{(-1)^n}{m!\,n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\,(b')_n}{n!} \sum_{m=0}^{\infty} \frac{(a)_{m+n}\,(b)_m}{(1+a+b-b'+i)_{m+n}\,m!}. \end{aligned}$$

By using $(\alpha)_{m+n} = (\alpha+n)_m\,(\alpha)_n$, we have

$$\begin{aligned} I &= \sum_{n=0}^{\infty} \frac{(-1)^n\,(b')_n\,(a)_n}{(1+a+b-b'+i)_n\,n!} \sum_{m=0}^{\infty} \frac{(a+n)_m\,(b)_m}{(1+a+b-b'+i+n)_m\,m!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\,(b')_n\,(a)_n}{(1+a+b-b'+i)_n\,n!}\,{}_2F_1\left(\begin{matrix} a+n, \quad b\,; \\ 1+a+b-b'+i+n\,; \end{matrix}\; 1\right). \end{aligned}$$

If we use the result (1.2), we get, after a little simplification

$$\begin{aligned} I &= \sum_{n=0}^{\infty} \frac{(-1)^n\,(b')_n\,(a)_n}{(1+a+b-b'+i)_n\,n!}\,\frac{\Gamma(1+a+b-b'+i+n)\,\Gamma(1-b'+i)}{\Gamma(1+a-b'+i+n)\,\Gamma(1+b-b'+i)} \\ &= \frac{\Gamma(1+a+b-b'+i)\,\Gamma(1-b'+i)}{\Gamma(1+a-b'+i)\,\Gamma(1+b-b'+i)} \sum_{n=0}^{\infty} \frac{(a)_n\,(b')_n\,(-1)^n}{(1+a-b'+i)_n\,n!} \\ &= \frac{\Gamma(1+a+b-b'+i)\,\Gamma(1-b'+i)}{\Gamma(1+a-b'+i)\,\Gamma(1+b-b'+i)}\,{}_2F_1\left(\begin{matrix} a, \quad b'\,; \\ 1+a-b'+i\,; \end{matrix}\; -1\right). \end{aligned}$$

Now, if we use (1.4), after a little simplification, we arrive at the right-hand side of (2.1). This completes the proof of (2.1). Clearly, the case $i = 0$ of (2.1) yields (1.1).

References

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