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NOTE ON VARIOUS METRIC SPACES

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ABSTRACT The purpose of this note is to introduce various metrics and to prove the properties of given metric spaces.

1. Introduction

The Riemannian manifold is a differentiable manifold having a Riemannian structure on it and its differentiability enables us to define an important geometric concepts. But analytically differentiable objects are very special ones. In the view of geometry, is there any general space with a geometric structure? Is the differentiability assumption in Riemannian results really essential? This problem influenced many geometers who tried to prove Riemannian results under weaker differentiability assumption. Now, inspired by a series of striking results by Gromov, many geometers such as S. B. Alexander, R. L. Bishop, W. Ballman, S. Buyalo and M. Brin etc. have studied metric spaces Geometric objects of a manifold such as length, area, angle, curvature and volume etc. are determined completely by a given metric. Hence, we need to survey metrics and study their properties.

In this paper, we will introduce various metrics and prove their properties.

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2. Ultrametric

Metric spaces may have properties which are far from our usual geometric intuition.

2.1 DEFINITION

Let X be a normed space with a metric $|\cdot|$ induced by the norm. If for $x, y \in X$,

$$|x+y| \leq max\{|x|, |y|\},$$

then this metric is called an *ultrametric.*

2.2 EXAMPLE

For a prime p there is a p-adic norm $|\cdot|_p$ on Q, $|0|_p = 0$ and $|x|_p = p^{-n}$ for $x \neq 0$, where $x = (s/t)p^n$ and s, t are not divisible by p. The corresponding metric is $|xy|_p = |x-y|_p$. We can show that $|xy|_p$ is a metric induced by the given norm.

This metric $|xy|_p$ is an ultra metric as follows; $|x+y|_p \leq max\{|x|_p, |y|_p\}$, since $|x+y|_p \leq max\{|x|_p, |y|_p\}$ gives $|x+y|_p \leq |x|_p + |y|_p$. Let $x = bp^n/a$ and $y = dp^{n+\alpha}/c, \alpha \geq 0, n \in Z$. Then

$$\begin{aligned} |x+y|_p &= |bp^n/a + dp^{n+\alpha}/c|_p \\ &= |(bcp^n + adp^{n+\alpha})/ac|_p \\ &= |(bc + adp^{\alpha})/ac \cdot p^n|_p \\ &\leq p^{-n}, \quad \text{since p is not divisor of a,b,c,d} \\ &= |x|_p. \end{aligned}$$

2.3. PROPERTIES OF ULTRAMETRIC.

(1). Any two balls of the metric in Example 2.2. either do not intersect or one is contained in the other.

PROOF. Let $x, y \in Q$ and $\epsilon \geq \delta$. Suppose that $B_\epsilon(x) \cap B_\epsilon(y) \neq \emptyset$. Then there exists a z such that $z \in B_\epsilon(x)$ and $B_\epsilon(y)$.

If $w \in B_\delta(y)$, then

$$\begin{aligned} |w-x|_p &= |w-y+y-x|_p \\ &\leq max\{|w-y|_p, |y-x|_p\} \\ &\leq max\{\delta, max\{|y-z|_p, |z-x|_p\}\} \\ &= max\{\delta, \epsilon\} \\ &= \epsilon. \end{aligned}$$

(2). All points in a ball are centers of the ball.

PROOF Let $|\cdot|_u$ be an ultrametric. Assume that $y \in B_\epsilon(x)$. Then $|y-x|_u < \epsilon$.

If $a \in B_\epsilon(x)$, then $|a-y|_u = |a-x+x-y|_u \leq max\{|a-x|_u, |x-y|_u\} \leq \epsilon$.

Hence, Similary for $a \in B_\epsilon(y)$.

Thus, $B_\epsilon(x) = B_\epsilon(y)$.

(3). Every triangle is an isosceles triangle.

PROOF. Let $|\cdot|_u$ be an ultrametric on X and $x, y \in X$. Suppose that $|x|_u \neq |y|_u$. Then we can assume that $|x|_u < |y|_u$.

$|y|_u = |y-x+x|_u \leq max\{|y-x|_u, |x|_u\} = |y-x|_u$, for above assumption.

Since $|\cdot|_u$ is an ultrametric, $|y-x|_u \leq max\{|y|_u, |x|_u\} = |y|_u$.

Hence, $|y|_u = |y-x|_u$. Therfore, every triangle in X is an isosceles triangle.

3. Interior metric

3 1. DEFINITION

A metric on X associated with metric $|\cdot|$ is said to be *interior* if for every $x, y \in X$ and for each $\epsilon > 0$ there exists an $\epsilon - midpoint$ z between x and y, that is

$$|xz|, |zy| \leq \frac{1}{2}|xy| + \epsilon.$$

In other words, $B_x(\frac{1}{2}+\epsilon) \cap B_y(\frac{1}{2}+\epsilon) \neq \emptyset$.

3 2. DEFINITION.

A metric on X is said to be *strictly interior* if every $x, y \in X$ posses a midpoint z, i.e.,

$$|xz| = |zy| = \frac{1}{2}|xy|.$$

3.3 EXAMPLE.

(1). On R^3, $(S^2, |\cdot|)$ is not interior

PROOF As an euclidean metric, this is trivial.

(2). The euclidean metric on $X = R^2 \setminus 0$ is interior but not strictly interior.

PROOF If we take $x = (1,1)$ and $y = (-1,-1)$, then x does not posses a midpoint.

But, we can take $\epsilon-$midpoint for every two points. Furthermore, X is locally compact for euclidean metric. But is not complete, since we take Cauchy sequence $(x_n) = \frac{1}{n}$.

3 4. THEOREM. *$p-$adic norm $|\cdot|_p$ on Q is not interior.*

PROOF Suppose that $|\cdot|_p$ is an interior metric.

Let $d(x,y) = |x-y|_p = \frac{1}{p^n}, x - y = \frac{sp^n}{t}$, where p is not divisor of s and t.

Then there exists an $\epsilon-$midpoint z such that

$$|x-z|_p \le \frac{1}{2}|x-y|_p + \epsilon, |z-y|_p \le \frac{1}{2}|x-y|_p + \epsilon.$$

But, $|x-y|_p \le max\{|x-z|_p, |z-y|_p\} \le \frac{1}{2}|x-y|_p + \epsilon$.

Hence, $\frac{1}{p^n} \le \frac{1}{2}\frac{1}{p^n} + \epsilon$.

This is a contradiction.

3 5. THEOREM. *Let X be a graph with two vertices and edges $e_n, n \ge 1$, between them such that the length of e_n is equal to $1 + \frac{1}{n}$. This space is called bipartite graph.*

Define the interior metric d on X by $d(a,b) = inf_\gamma L(\gamma)$, where $L(\gamma)$ is the length of γ and the infimum is taken over all graph γ connecting a and b. Then d is interior but not strictly interior.

PROOF. First, we will show that d is interior. Let $x, y \in X$. If $a = b$, then $z = a = b$ is an ϵ-midpoint of a and b. Assume that $a \neq b$ and $a \in e_i, b \in e_j$. If $i \neq j$, then there is an ϵ-midpoint of a and b, since there is a point z such that $\ell(a,z) = \ell(z,b) = \frac{1}{2}$. Now assume that $i = j$. Denote by $\ell_i(a,b)$ the length of graph connecting a and b in e_i. If $\ell_i(a,b) \leq 1 + \frac{1}{2^i}$, then a and b have their midpoint on e_i. Let $\ell_i(a,b) > 1 + \frac{1}{2^i}$. Without loss of generality, we can assume that a is closer to X than b on e_i. Choose a point z_n on $e_n, n \geq i+1$, such that $\ell_n(x, z_n) = \frac{1+1/n+\ell_i(b,y)-\ell_i(a,x)}{2}$. Then $d(a, z_n) = d(z_n, b) = \frac{1+1/n+\ell_i(b,y)+\ell_i(a,x)}{2}$. So for any $\epsilon > 0$ if $n \geq \frac{1}{2\epsilon}$, then we can choose an ϵ-midpoint z_n of a and b. Since

$$\begin{aligned} d(a, z_n) = d(z_n, b) &= \frac{1+\ell_i(b,y)+\ell_i(a,x)}{2} + \frac{1}{2n} \\ &\leq \frac{1+\ell_i(b,y)+\ell_i(a,x)}{2} + \epsilon \\ &= \frac{1}{2}d(a,b) + \epsilon. \end{aligned}$$

But, $d(x,y) = \lim_{n\to\infty}(1 + \frac{1}{n}) = 1$ and there is no midpoint of x and y in X. Therefore d is not strictly interior.

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