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ON FRÉCHET-URYSOHN EXPANSIONS

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ABSTRACT In this paper we study on Fréchet-Urysohn expansions of topological spaces, countable Fréchet-Urysohn spaces and Hausdorff Fréchet-Urysohn spaces.

1. Introduction

Let $(X, \mathcal{T}_c)$ be a topological space endowed with a topology $\mathcal{T}_c$ and let c denote the closure operator on $(X, \mathcal{T}_c)$. Let $\mathbb{N}$ denote the set of all natural numbers and $(x_n | n \in \mathbb{N})$(briefly (x_n)) a sequence of points in a set.

A function $[\cdot]_{seq}$ of the power set $\mathcal{P}(X)$ of X to $\mathcal{P}(X)$ itself defined by for each subset $A \subset X$, $[A]_{seq} = \{x \in X | (x_n)$ converges to x in $(X, \mathcal{T}_c)$ for some sequence (x_n) of points in $A\}$ is called the *sequential closure operator* on $(X, \mathcal{T}_c)$. It is well known that for each subset $A \subset X$, $[A]_{seq} \subset c(A)$ and $[\cdot]_{seq}$ satisfies the Kuratowski closure axioms except for idempotent (see [1]).

Let us recall definitions in a topological space $(X, \mathcal{T}_c)$.

(a) *Fréchet-Urysohn* [1] (also called *Fréchet* [2, 3]): for each subset $A \subset X$, $c(A) \subset [A]_{seq}$.

(b) *countable Fréchet-Urysohn*: for each countable subset $A \subset X$, $c(A) \subset [A]_{seq}$.

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From the definitions and the following example, we have that the following implication holds, but the converse does not hold:

$$\text{Fréchet-Urysohn} \Rightarrow \text{countable Fréchet-Urysohn}.$$

EXAMPLE. *The space of ordinals* $X = [0, \omega_1]$, *where* ω_1 *is the first uncountable ordinal, is a compact Hausdorff space all of whose countable subsets are metrizable. Note that the point* ω_1 *is not a cluster point of each countable subset of* X *not containing* ω_1. *Hence we see that the space* X *is countable Fréchet-Urysohn, but it is not Fréchet-Urysohn(see [3, p.125, Remark]).*

Let $(X, \mathcal{T}_c)$ be a topological space. If $\mathcal{T}_c \subset \mathcal{T}$ and $(X, \mathcal{T})$ is Fréchet-Urysohn, we call $(X, \mathcal{T})$ a *Fréchet-Urysohn expansion* of $(X, \mathcal{T}_c)$.

In this paper we study on Fréchet-Urysohn expansions of topological spaces, countable Fréchet-Urysohn spaces and Hausdorff Fréchet-Urysohn spaces.

Standard notations, not explained below, is the same as in [1].

2. Results

Let $(X, \mathcal{T}_c)$ be a topological space and $\mathcal{A}$ a family of subsets of X. The expansion of $\mathcal{T}_c$ by $\mathcal{A}$ denoted by $\mathcal{T}_c(\mathcal{A})$ is the topology on X with $\mathcal{T}_c \cup \mathcal{A}$ as subbase.

We recall that a family $\mathcal{A}$ of subsets of X is *point finite*[5] if and only if each $x \in X$ belongs to only finitely many $A \in \mathcal{A}$.

THEOREM 1 *Let* $(X, \mathcal{T}_c)$ *be a topological space and let*

$$\mathcal{A} = \{c(A) - [A]_{seq} | A \subset X \text{ and } c(A) - [A]_{seq} \neq \emptyset\}.$$

If $\mathcal{A}$ *is point finite, then* $(X, \mathcal{T}_c(\mathcal{A}))$ *is a Fréchet-Urysohn expansion of* $(X, \mathcal{T}_c)$.

PROOF If $(X, \mathcal{T}_c)$ is a Fréchet-Urysohn space, then clearly $\mathcal{A} = \emptyset$, and so $\mathcal{T}_c(\mathcal{A}) = \mathcal{T}_c$. Hence, it remains to prove the case that $(X, \mathcal{T}_c)$ is not Fréchet-Urysohn. Let $Y \subset X$ and $p \in c_{\mathcal{T}_c(\mathcal{A})}(Y) - Y$, where $c_{\mathcal{T}_c(\mathcal{A})}$ is the topological closure operator on $(X, \mathcal{T}_c(\mathcal{A}))$. Since $\mathcal{A}$ is point finite, $\{A \in \mathcal{A} | p \in A\}$ is finite, say $\{K_1, K_2, ..., K_n\}$. Let $M = \cap\{K_i | i = 1, 2, ..., n\}$. Then, clearly, $Y \cap M \neq \emptyset$ since M is a basic open set in $(X, \mathcal{T}_c(\mathcal{A}))$ containing p. We first show that $p \in c_{\mathcal{T}_c(\mathcal{A})}(Y \cap M)$. Since $\mathcal{T}_c \cup \mathcal{A}$ is a subbase for $\mathcal{T}_c(\mathcal{A})$, by the definition of M, we have that for each basic open set U in $(X, \mathcal{T}_c(\mathcal{A}))$ containing p, $(\cap\{V_j | i \in J\}) \cap M \subset U$ for some finite family $\{V_j | j \in J$ and J is finite $\}$ of open sets V_j in $(X, \mathcal{T}_c)$ containing p, and so $V \cap M \subset U$ for some open set V in $(X, \mathcal{T}_c)$ containing p. Hence, it is sufficient to show that for each open set V in $(X, \mathcal{T}_c)$ containing p, $(Y \cap M) \cap V \neq \emptyset$. Suppose on the contrary that there exists an open set V in $(X, \mathcal{T}_c)$ containing p such that $(Y \cap M) \cap V = \emptyset$. Then, since $M \cap V$ is a basic open set in $(X, \mathcal{T}_c(\mathcal{A}))$ containing p and since $p \in c_{\mathcal{T}_c(\mathcal{A})}(Y)$, $Y \cap (M \cap V) \neq \emptyset$, which is a contradiction. It is easy to see that for each subset Z of X, $c_{\mathcal{T}_c(\mathcal{A})}(Z) \subset [Z]_{seq} \subset c(Z)$. Hence, we have that there exists a sequence (x_n) of points in $Y \cap M$ such that (x_n) converges to p in $(X, \mathcal{T}_c)$. To end the proof, we claim that (x_n) converges to p in $(X, \mathcal{T}_c(\mathcal{A}))$. Suppose that it is not. Then there exists a basic open set U in $(X, \mathcal{T}_c(\mathcal{A}))$ containing p such that (x_n) is not eventually in U. We have already known that $V \cap M \subset U$ for some open set V in $(X, \mathcal{T}_c)$ containing p. It follows that there is an open set V in $(X, \mathcal{T}_c)$ containing p such that (x_n) is not eventaully in $V \cap M$, and hence (x_n) is also not eventually in V because (x_n) is a sequence of points in M, which is a contradiction.

Now we study on Fréchet-Urysohn expansions of countable Fréchet-Urysohn spaces.

THEOREM 2 *If $(X, \mathcal{T}_c)$ is a countable Fréchet-Urysohn space, then $(X, \mathcal{T}_{[\,]_{seq}})$ is a Fréchet-Urysohn expansion of $(X, \mathcal{T}_c)$ and moreover, for each sequence (x_n) of points in X and each $p \in X$, (x_n) converges to p in $(X, \mathcal{T}_c)$ if and only if (x_n) converges to p in $(X, \mathcal{T}_{[\,]_{seq}})$.*

PROOF First we show that $[[A]_{seq}]_{seq} \subset [A]_{seq}$ for each subset $A \subset X$. Let $x \in [[A]_{seq}]_{seq}$ Then, by the definition of $[\cdot]_{seq}$, (x_n) converges to x in $(X, \mathcal{T}_c)$ for some sequence (x_n) of points in $[A]_{seq}$. And since $x_n \in [A]_{seq}$ for each $n \in \mathbb{N}$, there exists a sequence (x_{nm}) converges to x_n in $(X, \mathcal{T}_c)$. So, $x \in c(\{x_{nm} | n, m \in \mathbb{N}\})$. Since $(X, \mathcal{T}_c)$ is a countable

Fréchet-Urysohn space and $\{x_{nm}|n, m \in \mathbb{N}\}$ is a countable subset of A,

$$c(\{x_{nm}|n, m \in \mathbb{N}\}) \subset [\{x_{nm}|n, m \in \mathbb{N}\}]_{seq}$$

and hence

$$x \in [\{x_{nm}|n, m \in \mathbb{N}\}]_{seq} \in [A]_{seq}.$$

Thus we have that $(X, \mathcal{T}_{[\,]_{seq}})$ is a topological sapce with the closure operator $[\cdot]_{seq}$.

Next we show that for each sequence (x_n) of points in X and $x \in X$, (x_n) converges to x in $(X, \mathcal{T}_c)$ if and only if (x_n) converges to x in $(X, \mathcal{T}_{[\,]_{seq}})$. Since $[A]_{seq} \subset c(A)$ for each subset $A \subset X$, it is clear that the topology $\mathcal{T}_{[\cdot]_{seq}}$ on X induced by $[\cdot]_{seq}$ is finer than the topology $\mathcal{T}_c$ and hence we have that if (x_n) converges to x in $(X, \mathcal{T}_{[\,]_{seq}})$, then (x_n) converges to x in $(X, \mathcal{T}_c)$.

Conversely, if (x_n) does not converge to x in $(X, \mathcal{T}_{[\,]_{seq}})$, then (x_n) is not eventually in some neighborhood U of x in $(X, \mathcal{T}_{[\,]_{seq}})$. It is obvious that there exists a subsequence $(x_{\phi(n)})$ of (x_n) such that the range $\{x_{\phi(n)}|n \in \mathbb{N}\}$ of $(x_{\phi(n)})$ and U are disjoint. Hence $x \notin [\{x_{\phi(n)}|n \in \mathbb{N}\}]_{seq}$ and so, by the definition of $[\cdot]_{seq}$, $(x_{\phi(n)})$ does not converge to x in $(X, \mathcal{T}_c)$. Note that if (x_n) converges to x in $(X, \mathcal{T}_c)$, then $(x_{\phi(n)})$ converges to x in $(X, \mathcal{T}_c)$. Thus, by the contraposition of above fact, (x_n) does not converge to x in $(X, \mathcal{T}_c)$.

Finally we show that $(X, \mathcal{T}_{[\,]_{seq}})$ is a Fréchet-Urysohn space. Let $A \subset X$ and $x \in [A]_{seq}$. Then, by the definition of $[A]_{seq}$, (x_n) converges to x in $(X, \mathcal{T}_c)$ for some sequence (x_n) of points in A. By the above fact that (x_n) converges to x in $(X, \mathcal{T}_c)$ if and only if (x_n) converges to x in $(X, \mathcal{T}_{[\,]_{seq}})$, (x_n) converges to x in $(X, \mathcal{T}_{[\,]_{seq}})$ and thus it holds.

The proof is complete.

REMARK *It is clear that if $(X, \mathcal{T}_c)$ is not a Fréchet-Urysohn space, then $\mathcal{T}_c \subsetneqq \mathcal{T}_{[\,]_{seq}}$.*

We finally introduce some results of J. A. Narvarte and J. A. Guthrie for Fréchet-Urysohn expansions of Hausdorff Fréchet-Urysohn spaces.

THEOREM 3[4] *Let $(X, \mathcal{T}_c)$ be a Hausdorff Fréchet-Urysohn space and $A \subset X$. Then $(X, \mathcal{T}_c(\{A\}))$ is a Fréchet-Urysohn expansion of $(X, \mathcal{T}_c)$.*

THEOREM 4[4]. *Let $(X, \mathcal{T}_c)$ be a Hausdorff Fréchet-Urysohn space and $\mathcal{A}$ a family of subsets of X. If $\mathcal{A}$ is point finite, then $(X, \mathcal{T}_c(\mathcal{A}))$ is a Fréchet-Urysohn expansion of $(X, \mathcal{T}_c)$.*

According to Theorem 2, we immediately have the following corollary and hence we omit the proof.

COROLLARY 5. *If $(X, \mathcal{T}_c)$ is a Fréchet-Urysohn space, then $(X, \mathcal{T}_{[\,]_{seq}})$ is a Fréchet-Urysohn expansion of $(X, \mathcal{T}_c)$.*

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