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SOME MAJORIZATION PROBLEMS ASSOCIATED WITH p-VALENTLY STARLIKE AND CONVEX FUNCTIONS OF COMPLEX ORDER

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ABSTRACT The main object of this paper is to investigate several majorization problems involving two subclasses $\mathcal{S}_{p,q}(\gamma)$ and $\mathcal{C}_{p,q}(\gamma)$ of p-valently starlike and p-valently convex functions of complex order $\gamma \neq 0$ in the open unit disk $\mathbb{U}$. Relevant connections of the results presented here with those given by earlier workers on the subject are also indicated

1. Introduction and Definitions

Let the functions $f(z)$ and $g(z)$ be *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Following the pioneering work of MacGregor [6], we say that $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$ and write

$$f(z) \ll g(z) \quad (z \in \mathbb{U}) \tag{1.1}$$

if there exists a function $\varphi(z)$, analytic in $\mathbb{U}$, such that

$$|\varphi(z)| \leqq 1 \text{ and } f(z) = \varphi(z)\, g(z) \quad (z \in \mathbb{U}). \tag{1.2}$$

The majorization (1.1) is closely related to the concept of quasi-subordination between analytic functions in $\mathbb{U}$, which was considered

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recently by (for example) Altintaş and Owa [1]. Altintaş *et al.* [2], on the other hand, investigated several majorization problems involving a number of subclasses of analytic functions in $\mathbb{U}$. In the present sequel to the work of Altintaş *et al.* [2], we propose to investigate the corresponding majorization problems associated with the classes $\mathcal{S}_{p,q}(\gamma)$ and $\mathcal{C}_{p,q}(\gamma)$ of p-valently starlike and p-valently convex functions of *complex* order $\gamma \neq 0$ in $\mathbb{U}$, which are introduced below.

Let $\mathcal{A}_p$ denote the class of functions f *normalized* by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.3}$$

which are analytic *and* p-valent in $\mathbb{U}$. Also let

$$\mathcal{A} := \mathcal{A}_1. \tag{1.4}$$

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_{p,q}(\gamma)$ of *p-valently starlike functions of complex order* $\gamma \neq 0$ in $\mathbb{U}$ if and only if

$$\Re\left\{1 + \frac{1}{\gamma}\left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q\right)\right\} > 0 \tag{1.5}$$

$$(z \in \mathbb{U};\ p \in \mathbb{N};\ q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \gamma \in \mathbb{C} \setminus \{0\};\ |2\gamma - p + q| \leqq p - q),$$

where, as usual, $f^{(q)}(z)$ denotes the derivative of $f(z)$ with respect to z of order $q \in \mathbb{N}_0$. Furthermore, a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{C}_{p,q}(\gamma)$ of *p-valently convex functions of complex order* $\gamma \neq 0$ in $\mathbb{U}$ if and only if

$$\Re\left\{1 + \frac{1}{\gamma}\left(\frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - p + q\right)\right\} > 0 \tag{1.6}$$

$$(z \in \mathbb{U};\ p \in \mathbb{N};\ q \in \mathbb{N}_0;\ \gamma \in \mathbb{C} \setminus \{0\};\ |2\gamma - p + q| \leqq p - q).$$

Clearly, we have the following relationships:

$$\mathcal{S}_{1,0}(\gamma) = \mathcal{S}(\gamma) \quad \text{and} \quad \mathcal{C}_{1,0}(\gamma) = \mathcal{C}(\gamma) \quad (\gamma \in \mathbb{C} \setminus \{0\}), \tag{1.7}$$

where $\mathcal{S}(\gamma)$ and $\mathcal{C}(\gamma)$ are the classes of starlike and convex functions of *complex* order $\gamma \neq 0$ in $\mathbb{U}$, which were considered earlier by Nasr and Aouf [8] and Wiatrowski [12], respectively, and (more recently) by Altintaş *et al.* [2] (see also Aouf *et al.* [3]). Moreover, it is easily seen that

$$\mathcal{S}_{1,0}(1-\alpha) = \mathcal{S}(1-\alpha) = \mathcal{S}^*(\alpha) \quad (0 \leqq \alpha < 1) \tag{1.8}$$

and

$$\mathcal{C}_{1,0}(1-\alpha) = \mathcal{C}(1-\alpha) = \mathcal{K}(\alpha) \quad (0 \leqq \alpha < 1), \tag{1.9}$$

where $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote, respectively, the familiar classes of (normalized) starlike and convex functions of order α in $\mathbb{U}$, which were introduced by Robertson [10] (see also Srivastava and Owa [11]).

2. Majorization Problems for the Class $\mathcal{S}_{p,q}(\gamma)$

We first state and prove

Theorem 1. *Let the function $f(z)$ be in the class $\mathcal{A}_p$ and suppose that $g \in \mathcal{S}_{p,q}(\gamma)$. If $f^{(q)}(z)$ is majorized by $g^{(q)}(z)$ in $\mathbb{U}$ for $q \in \mathbb{N}_0$, then*

$$\left|f^{(q+1)}(z)\right| \leqq \left|g^{(q+1)}(z)\right| \quad (|z| \leqq r_1), \tag{2.1}$$

where

$$r_1 = r_1(p,q;\gamma) := \frac{\kappa - \sqrt{\kappa^2 - 4(p-q)|2\gamma - p + q|}}{2|2\gamma - p + q|} \tag{2.2}$$

$$(\kappa := 2 + p - q + |2\gamma - p + q|;\ p \in \mathbb{N};\ q \in \mathbb{N}_0;\ \gamma \in \mathbb{C}\setminus\{0\}).$$

Proof Since $g \in \mathcal{S}_{p,q}(\gamma)$, we find from (1.5) that, if

$$h(z) := 1 + \frac{1}{\gamma}\left(\frac{zg^{(q+1)}(z)}{g^{(q)}(z)} - p + q\right) \quad (\gamma \in \mathbb{C}\setminus\{0\}), \tag{2.3}$$

then

$$\Re\{h(z)\} > 0 \quad (z \in \mathbb{U}) \tag{2.4}$$

and

$$h(z)=\frac{1+w(z)}{1-w(z)} \quad (w\in\Omega), \tag{2.5}$$

where Ω denotes the well-known class of *bounded* analytic functions in $\mathbb{U}$, which satisfy the conditions (*cf.*, *e.g.*, Goodman [5, p. 58]):

$$w(0)=0 \text{ and } |w(z)|\leqq|z| \quad (z\in\mathbb{U}). \tag{2.6}$$

Making use of (2.3) and (2.5), we readily obtain

$$\frac{zg^{(q+1)}(z)}{g^{(q)}(z)}=\frac{p-q+(2\gamma-p+q)\,w(z)}{1-w(z)}, \tag{2.7}$$

which, in view of (2.6), immediately yields the inequality:

$$\left|g^{(q)}(z)\right|\leqq\frac{(1+|z|)\,|z|}{p-q-|2\gamma-p+q|\cdot|z|}\left|g^{(q+1)}(z)\right| \quad (z\in\mathbb{U}). \tag{2.8}$$

Next, since $f^{(q)}(z)$ is majorized by $g^{(q)}(z)$ in $\mathbb{U}$, from (1.2) we have

$$f^{(q+1)}(z)=\varphi(z)\,g^{(q+1)}(z)+\varphi'(z)\,g^{(q)}(z) \quad (z\in\mathbb{U}). \tag{2.9}$$

Thus, observing that $\varphi\in\Omega$ satisfies the inequality (*cf.* Nehari [9, p. 168]):

$$|\varphi'(z)|\leqq\frac{1-|\varphi(z)|^2}{1-|z|^2} \quad (z\in\mathbb{U}), \tag{2.10}$$

and applying (2.8) and (2.10) in (2.9), we get

$$\left|f^{(q+1)}(z)\right|\leqq\left(|\varphi(z)|+\frac{1-|\varphi(z)|^2}{1-|z|^2}\cdot\frac{(1+|z|)\,|z|}{p-q-|2\gamma-p+q|\cdot|z|}\right)\cdot\left|g^{(q+1)}(z)\right| \quad (z\in\mathbb{U}), \tag{2.11}$$

which, upon setting

$$|z|=r \text{ and } |\varphi(z)|=\rho \quad (0\leqq\rho\leqq 1), \tag{2.12}$$

leads us to the inequality:

$$\left|f^{(q+1)}(z)\right| \leqq \frac{\Theta(\rho)}{(1-r)(p-q-|2\gamma-p+q|\,r)}\left|g^{(q+1)}(z)\right| \quad (z \in \mathbb{U}), \tag{2.13}$$

where the function $\Theta(\rho)$ defined by

$$\Theta(\rho) := -r\rho^2 + (1-r)(p-q-|2\gamma-p+q|\,r)\,\rho + r \quad (0 \leqq \rho \leqq 1) \tag{2.14}$$

takes on its *maximum* value at $\rho = 1$ with

$$r = r_1(p, q; \gamma)$$

given by (2.2). Furthermore, if

$$0 \leqq \sigma \leqq r_1(p, q; \gamma),$$

where $r_1(p, q; \gamma)$ is given by (2.2), then the function $\Lambda(\rho)$ defined by

$$\Lambda(\rho) := -\sigma\rho^2 + (1-\sigma)(p-q-|2\gamma-p+q|\,\sigma)\,\rho + \sigma \tag{2.15}$$

is seen to be an *increasing* function on the interval $0 \leqq \rho \leqq 1$, so that

$$\Lambda(\rho) \leqq \Lambda(1) = (1-\sigma)(p-q-|2\gamma-p+q|\,\sigma)$$

$$\left(0 \leqq \rho \leqq 1;\ 0 \leqq \sigma \leqq r_1(p, q; \gamma)\right).$$

Hence, by setting $\rho = 1$ in (2.13), we conclude that the assertion (2.1) of Theorem 1 holds true for $|z| \leqq r_1(p, q; \gamma)$, where $r_1(p, q; \gamma)$ is given by (2.2). This evidently completes the proof of Theorem 1.

In view of the first relationship in (1.7), a special case of Theorem 1 when $p = 1$ and $q = 0$ yields

Corollary 1 (Altintaş *et al* [2, p. 211, Theorem 1]). *Let the function $f(z)$ be in the class $\mathcal{A}$ and suppose that $g \in \mathcal{S}(\gamma)$. If $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, then*

$$|f'(z)| \leqq |g'(z)| \quad (|z| \leqq R_1), \tag{2.16}$$

where

$$R_1 = R_1(\gamma) := \frac{3 + |2\gamma - 1| - \sqrt{9 + 2|2\gamma - 1| + |2\gamma - 1|^2}}{2|2\gamma - 1|}. \tag{2.17}$$

Several further consequences of Corollary 1, involving such familiar classes as (see, for details, Duren [4] and Goodman [5])

$$\mathcal{S}^* := \mathcal{S}^*(0) \quad \text{and} \quad \mathcal{K} := \mathcal{K}(0) \tag{2.18}$$

were given earlier by MacGregor [6, p. 96, Theorems 1B and 1C] (see also Altintaş *et al.* [2, p. 213, Corollaries 1 and 2]).

3. Majorization Problems for the Class $\mathcal{C}_{p,q}(\gamma)$

The proof of our next result (Theorem 2 below) is based essentially upon the following

Lemma. *If* $f \in \mathcal{C}_{p,q}(\gamma)$ $(\gamma \in \mathbb{C} \setminus \{0\})$, *then* $f \in \mathcal{S}_{p,q}\left(\frac{1}{2}\gamma\right)$, *that is,*

$$\mathcal{C}_{p,q}(\gamma) \subset \mathcal{S}_{p,q}\left(\frac{1}{2}\gamma\right) \quad (\gamma \in \mathbb{C} \setminus \{0\}). \tag{3.1}$$

PROOF. Since (*cf.*, *e.g.*, MacGregor [7, p. 71])

$$f \in \mathcal{K} \Longrightarrow f \in \mathcal{S}^*\left(\frac{1}{2}\right), \tag{3.2}$$

or, equivalently, since

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \Longrightarrow \Re\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \quad (z \in \mathbb{U}), \tag{3.3}$$

for $f(z) \longmapsto f^{(q)}(z)$ $(q \in \mathbb{N}_0)$ with $f \in \mathcal{A}_p$, we have

$$\Re\left\{1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - (p - q - 1)\right\} > 0$$
$$\Longrightarrow \Re\left\{1 + \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - (p - q)\right\} > \frac{1}{2} \quad (z \in \mathbb{U}), \tag{3.4}$$

which readily yields the assertion:

$$1+\frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)}-p+q+1=\frac{1-w(z)}{1+w(z)}$$
$$\Longrightarrow 1+\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}-p+q=\frac{1}{1+w(z)}\quad (w\in\Omega). \tag{3.5}$$

Now, by making use of (3.5) appropriately, it is easily seen that

$$1+\frac{1}{\gamma}\left(1+\frac{zf^{(q+2)}(z)}{f^{(q)}(z)}-p+q\right)=\frac{\gamma+(\gamma-2)w(z)}{\gamma[1+w(z)]}$$
$$\Longrightarrow 1+\frac{2}{\gamma}\left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}-p+q\right)=\frac{\gamma+(\gamma-2)w(z)}{\gamma[1+w(z)]}\quad (w\in\Omega), \tag{3.6}$$

and the desired inclusion property (3.1) follows immediately from (3.6) in view of the characterizations (1.5) and (1.6) for the function classes $\mathcal{S}_{p,q}(\gamma)$ and $\mathcal{C}_{p,q}(\gamma)$, respectively.

Theorem 2. *Let the function $f(z)$ be in the class $\mathcal{A}_p$ and suppose that $g\in\mathcal{C}_{p,q}(\gamma)$. If $f^{(q)}(z)$ is majorized by $g^{(q)}(z)$ in $\mathbb{U}$ for $q\in\mathbb{N}_0$, then*

$$\left|f^{(q+1)}(z)\right|\leqq\left|g^{(q+1)}(z)\right|\quad (|z|\leqq r_2), \tag{3.7}$$

where

$$r_2=r_2(p,q;\gamma):=\frac{\mu-\sqrt{\mu^2-4(p-q)|\gamma-p+q|}}{2|\gamma-p+q|} \tag{3.8}$$

$$(\mu:=2+p-q+|\gamma-p+q|;\ p\in\mathbb{N};\ q\in\mathbb{N}_0;\ \gamma\in\mathbb{C}\backslash\{0\}).$$

PROOF In view of the inclusion property (3.1) asserted by the above Lemma, Theorem 2 can be deduced as a simple consequence of Theorem 1 with $\gamma \longmapsto \frac{1}{2}\gamma$.

By setting $p = 1$ and $q = 0$, Theorem 2 yields

Corollary 2 (Altintaş *et al.* [2, p. 214, Theorem 2]). *Let the function $f(z)$ be in the class $\mathcal{A}$ and suppose that $g \in \mathcal{C}(\gamma)$. If $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, then*

$$|f'(z)| \leqq |g'(z)| \quad (|z| \leqq R_2), \tag{3.9}$$

where

$$R_2 = R_2(\gamma) := \frac{3 + |\gamma - 1| - \sqrt{9 + 2|\gamma - 1| + |\gamma - 1|^2}}{2|\gamma - 1|}. \tag{3.10}$$

Finally, in its limit case when $\gamma \longrightarrow 1$, if we make use of the relationship [*cf.* Equations (1.9) *and* (2.18)]:

$$\mathcal{C}(1) = \mathcal{K}(0) =: \mathcal{K}, \tag{3.11}$$

Corollary 2 *further* yields

Corollary 3 (*cf.* MacGregor [6, p. 96, Theorem 1C]). *Let the function $f(z)$ be in the class $\mathcal{A}$ and suppose that $g \in \mathcal{K}$. If $f(z)$ is majorized by $g(z)$ in $\mathbb{U}$, then*

$$|f'(z)| \leqq |g'(z)| \quad \left(|z| \leqq \frac{1}{3}\right). \tag{3.12}$$

In view of the inclusion property (3.2), Corollary 3 can also be deduced from Corollary 1 by letting $\gamma \longrightarrow \frac{1}{2}$ (see also Altintaş *et al.* [2, p. 213, Corollary 2]).

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