Honam Mathematical J. **23**(2001), No. 1,pp. 41–50

SUBORDINATION ON δ–CONVEX FUNCTIONS IN A SECTOR

MARJONO AND D.K. THOMAS

Dept. of Mathematics,
University of Brawijaya, Jl. Hryono 169 Malang 65145, INDONESIA,
E-mail: marjono@indo.net.id.
Dept. of Mathematics,
University of Wales,
Swansea Singleton Park SA2 8PP, UNITED KINGDOM.

Abstract. This paper concerns with the subclass of normalized analytic function f in $D = \{z : |z| < 1\}$, namely a δ-convex function in a sector. This subclass is denoted by $\Delta(\delta)$, where δ is a real positive. Given $0 < \beta \leq 1$ then for $z \in D$, the exact $\alpha(\beta, \delta)$ is found such that $f \in \Delta(\delta)$ implies $f \in S^*(\beta)$, where $S^*(\beta)$ is starlike of order β in a sector. This work is a more general version of the result of Nunokawa and Thomas [11].

1. Introduction

This paper is concerned with an applications of the Clunie-Jack Lemma and the Miller-Mocanu Lemma. The application relates to functions convex and starlike in a sector. We note that for a result on functions convex and starlike in a half plane can be seen (see e.g. Marjono [8]). We will show how the Clunie-Jack Lemma and the Miller-Mocanu Lemma have been used to solve the 'inclusion' problems for these functions and we will present a new result which generalizes the 'inclusion' problem for a convex and starlike functions in a sector. We will need the following definitions :

Received March 2, 2001.
1991 AMS Subject Classification : 30C45.
Key words and phrases : analytic, convex, starlike.

DEFINITION 1.1. [3] Let f be analytic in D with $f(0) = f'(0) - 1 = 0$ and $0 \leq \alpha < 1$. If f satisfies

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$$

for $z \in D$, then f is said to be convex of order α in a half plane. We denote the subclass of all such functions by $K(\alpha)$.

DEFINITION 1.2. [3] Let f be analytic in D with $f(0) = f'(0) - 1 = 0$ and $0 \leq \alpha < 1$. If f satisfies

$$Re\frac{zf'(z)}{f(z)} > \alpha,$$

for $z \in D$, then f is said to be starlike of order α in a half plane. We denote the subclass of all such functions by $St(\alpha)$.

The classes $K(\alpha)$ and $St(\alpha)$ have been extensively studied and now form a standard part of the literature (see e.g. Goodman [3]).

Less studied, and rather more difficult, are the corresponding classes of convex and starlike functions in a sector, defined as follows :

DEFINITION 1.3. Let f be analytic in D with $f(0) = f'(0) - 1 = 0$ and $0 \leq \alpha < 1$. If f satisfies

$$\left|arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2},$$

for $z \in D$, then f is said to be convex of order α in a sector. We denote the subclass of all such functions by $C(\alpha)$.

DEFINITION 1.4. Let f be analytic in D with $f(0) = f'(0) - 1 = 0$ and $0 \leq \alpha < 1$. If f satisfies

$$\left|arg\frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2},$$

for $z \in D$, then f is said to be starlike of order α in a sector. We denote the subclass of all such functions by $S^*(\alpha)$.

For $0 \leq \alpha < 1$, the problem of determining the exact $\beta(\alpha)$ such that $f \in K(\alpha)$ implies $f \in St\,(\beta(\alpha))$ was first considered by Jack [4]. Much earlier, Marx [10] and Strohhäcker [13] showed that $f \in K(0)$ implies $f \in St\,(1/2)$ and that this was best possible. Jack [4], using the Clunie-Jack Lemma, found a $\beta(\alpha)$ such that $f \in K(\alpha)$ implies $f \in St\,(\beta(\alpha))$, but this result was not best possible. Again using the Clunie-Jack Lemma, MacGregor [5] and simultaneously Goel [2] eventually gave the correct solution, but were unable to conclude that the result was best possible. This was subsequently established by Wilken and Feng [14]. Marjono [7] also gave a sufficient condition for starlikeness for analytic functions.

Perhaps surprisingly, the corresponding problem of finding the exact $\alpha(\beta)$ such that $f \in C(\alpha(\beta))$ implies $f \in S^*(\beta)$ was only recently solved by Nunokawa and Thomas [11]. We first state the result of Nunokawa and Thomas, which is in fact a stronger subordination chain version of the one refered to Goel, MacGregor, Wilken and Feng. Their result is the following :

THEOREM 1.5. *Let f be analytic in D, and f normalized such that $f(0) = f'(0) - 1 = 0$. Then for $0 < \beta \leq 1$ and $z \in D$, there exists $\alpha(\beta)$ such that*

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta)}$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta}, \tag{1}$$

in particular we may choose

$$\alpha(\beta) = \frac{2}{\pi}\arctan\left[\tan\frac{\beta\pi}{2} + \frac{\beta}{(1+\beta)^{\frac{1+\beta}{2}}(1-\beta)^{\frac{1-\beta}{2}}\cos\frac{\beta\pi}{2}}\right]. \tag{2}$$

Furthermore, $\alpha(\beta)$ given by (2) is the largest number such that (1) holds.

COROLLARY 1.6. *$C(\alpha) \subset S^*(\beta)$ for $\alpha(\beta)$ given by (2) and $\alpha(\beta)$ is the largest number such that the inclusion holds.*

We now introduce the notation of a δ-convex function in a sector as follows :

DEFINITION 1.7. Let f be analytic in D, with $f(0) = f'(0) - 1 = 0$ and $f(z)f'(z)/z \neq 0$ in D. For $\delta > 0$, write

$$J(\delta, f) = (1-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right). \tag{3}$$

Then if, for $z \in D$,

$$|argJ(\delta, f)| < \frac{\pi\delta}{2},$$

f is called a δ-convex function in a sector. We denote the class of all such functions by $\Delta(\delta)$.

2. Result

We now present our generalization of the work of Nunokawa and Thomas.

THEOREM 2.1. *Let f be analytic in $D = \{z : |z| < 1\}$, with $f(0) = f'(0) - 1 = 0$ and $J(\delta, f)$ be defined by (3) for $\delta > 0$. Let $0 < \beta \leq 1$ be given, then for $z \in D$, there exists $\alpha(\beta, \delta)$ such that*

$$J(\delta, f) \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta,\delta)}$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta}, \tag{4}$$

in particular we may choose

$$\alpha(\beta, \delta) = \frac{2}{\pi}\arctan\left[\tan\frac{\beta\pi}{2} + \frac{\delta\beta}{(1+\beta)^{\frac{1+\beta}{2}}(1-\beta)^{\frac{1-\beta}{2}}\cos\frac{\beta\pi}{2}}\right], \tag{5}$$

and $\alpha(\beta, \delta)$ given by (5) is the largest number such that (4) holds.

The proof of Theorem 2.1. requires the Miller-Mocanu Lemma.

LEMMA 2.2 (MILLER-MOCANU LEMMA). [6, 9] *Let p be analytic in D and q be analytic and univalent in $\bar{D}$, with $p(0) = q(0)$. If $p \not\prec q$, then there is a point $z_0 \in D$ and $\zeta_0 \in \partial D$ such that $p(|z| < |z_0|) \subset q(D)$, and $p(z_0) = q(\zeta_0)$ and*

$$z_0 p'(z_0) = k\zeta_0 q'(\zeta_0), \qquad \text{for} \qquad k \geq 1.$$

Proof of Theorem 2.1.

Write $p(z) = zf'(z)/f(z)$, so that p is analytic in D and $p(0) = 1$. Differentiating we obtain

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}.$$

Thus for $\delta > 0$

$$J(\delta, f) = p(z) + \frac{\delta z p'(z)}{p(z)},$$

and we need to show that

$$p(z) + \frac{\delta z p'(z)}{p(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha(\beta,\delta)} \qquad \text{implies} \qquad p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}.$$

We now write

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha(\beta,\delta)} \qquad \text{and} \qquad q(z) = \left(\frac{1+z}{1-z}\right)^{\beta},$$

so that $|\arg h(z)| < \alpha(\beta,\delta)\pi/2$ and $\arg q(z)| < \beta\pi/2$.

Suppose that $p \not\prec q$. Then from the Miller-Mocanu Lemma [9], there is a point $z_0 \in D$ and $\zeta_0 \in \partial D$ such that $p(|z| < |z_0|) \subset q(D)$, and $p(z_0) = q(\zeta_0)$ and

$$z_0 p'(z_0) = k\zeta_0 q'(\zeta_0), \qquad \text{for} \qquad k \geq 1.$$

We next note that $p(z) \neq 0$ for $z \in D$, since otherwise we can write

$$p(z) = (z - z_0)^k p_1(z), \qquad \text{for some} \qquad k \geq 1,$$

where p_1 is analytic in D with $p_1(z_0) \neq 0$. Then

$$p(z) + \frac{\delta z p'(z)}{p(z)} = \frac{\delta z p_1'(z)}{p_1(z)} + \frac{\delta k z}{z - z_0} + (z - z_0)^k p_1(z). \tag{6}$$

Thus choosing $z \to z_0$ suitably, the argument of the right-hand side of (6) can take any value between 0 and 2π, which contradicts the hypotheses of the theorem.

Since $p(z_0) = q(\zeta_0) \neq 0$, it follows that $\zeta_0 \neq \pm 1$. Thus we can write

$$ri = (1 + \zeta_0)/(1 - \zeta_0), \qquad \text{for} \qquad r \neq 0, \tag{7}$$

i.e.

$$\zeta_0 = \frac{ri - 1}{ri + 1}. \tag{8}$$

The Miller-Mocanu Lemma now gives

$$p(z_0) + \frac{\delta z_0 p'(z_0)}{p(z_0)} = q(\zeta_0) + \frac{\delta k \zeta_0 q'(\zeta_0)}{q(\zeta_0}, \qquad \text{for} \qquad k \geq 1. \tag{9}$$

Differentiating $q(z)$ we obtain

$$\frac{q'(z)}{q(z)} = \frac{2\beta}{1 - z^2},$$

and so from (8) we have

$$\frac{\zeta_0 q'(\zeta_0)}{q(\zeta_0)} = \frac{\beta(1 - r^2)i}{2r}. \tag{10}$$

Using (7) and (10) in (9), we obtain

$$\begin{aligned} p(z_0) + \frac{\delta z_0 p'(z_0)}{p(z_0)} &= (ri)^\beta + \frac{\delta k \beta \left(1 + r^2\right) i}{2r}, \\ &= r^\beta \cos \frac{\beta \pi}{2} + i \left[r^\beta \sin \frac{\beta \pi}{2} + \frac{\delta k \beta \left(1 + r^2\right)}{2r} \right]. \end{aligned}$$

Next assume that $r > 0$, (if $r < 0$, the proof is similar). Since $k \geq 1$, an elementary argument shows that

(11)
$$\arg\left(p(z_0) + \frac{\delta z_0 p'(z_0)}{p(z_0)}\right) = \arctan\left[\tan\frac{\beta\pi}{2} + \frac{\delta k\beta\left(1+r^2\right)}{2r^{\beta+1}\cos\frac{\beta\pi}{2}}\right],$$
$$\geq \arctan\left[\tan\frac{\beta\pi}{2} + \frac{\delta\beta\left(1+r^2\right)}{2r^{\beta+1}\cos\frac{\beta\pi}{2}}\right].$$

Write $\eta(r) = \left(1+r^2\right)/r^{\beta+1}$, then $\eta'(r) = \frac{\left[2r^2-(\beta+1)(1+r^2)\right]}{r^{\beta+2}}$. Therefore, $\eta'(r) = 0$, if $r^2 = (1+\beta)/(1-\beta)$, or

$$1 + r^2 = \frac{2}{1-\beta}. \tag{12}$$

Thus the minimum value will be obtained when r satisfies (12) and so

$$\frac{\beta(1+r^2)}{2r^{\beta+1}\cos\frac{\beta\pi}{2}} \geq \frac{\beta}{(1+\beta)^{\frac{1+\beta}{2}}(1-\beta)^{\frac{1-\beta}{2}}\cos\frac{\beta\pi}{2}}. \tag{13}$$

Using (13)in (11), we have

$$\arg\left(p(z_0) + \frac{\delta z_0 p'(z_0)}{p(z_0)}\right)$$
$$\geq \arctan\left[\tan\frac{\beta\pi}{2} + \frac{\delta\beta}{(1+\beta)^{\frac{1+\beta}{2}}(1-\beta)^{\frac{1-\beta}{2}}\cos\frac{\beta\pi}{2}}\right]$$
$$= \frac{\alpha(\beta,\delta)\pi}{2}.$$

Hence
$$\frac{\alpha(\beta,\delta)\pi}{2} \leq \arg\left(p(z_0) + \frac{\delta z_0 p'(z_0)}{p(z_0)}\right) \leq \frac{\pi}{2},$$

which contradicts the fact that

$$|\arg h(z)| < \frac{\alpha(\beta,\delta)\pi}{2},$$

provided that (5) holds.

To show that $\alpha(\beta,\delta)$ given by (5) is not the largest number such that (4) holds, let $p(z) = [(1+z)/(1-z)]^{\beta}$. Then from the minimum modulus principle for harmonic functions, it follows that

$$\inf_{|z|<1} \arg\left(p(z) + \frac{\delta z p'(z)}{p(z)}\right)$$

is attained at some point $z = e^{i\theta}$, for $0 < \theta < 2\pi$. Thus

$$p(z) + \frac{\delta z p'(z)}{p(z)} = \left(\frac{1+z}{1-z}\right)^{\beta} + \frac{2\delta\beta z}{1-z^2}, \tag{14}$$

$$= \left(\frac{\sin\theta}{1-\cos\theta}\right)^{\beta} \exp\left(\frac{\beta\pi i}{2}\right) + \frac{i\delta\beta}{\sin\theta}. \tag{15}$$

To see this write $z = e^{i\theta}$, then $(1+z)/(1-z) = i\sin\theta/(1-\cos\theta)$ and so

$$\left(\frac{1+z}{1-z}\right)^{\beta} = \left(\frac{\sin\theta}{1-\cos\theta}\right)^{\beta} \exp\left(\frac{\beta\pi i}{2}\right). \tag{16}$$

Also $z/(1-z^2) = i\sin\theta/(1-\cos 2\theta)$, and since $\cos 2\theta = 1 - 2\sin^2\theta$, we have

$$\frac{z}{1-z^2} = \frac{i}{2\sin\theta},$$

and so

$$\frac{2\delta\beta z}{1-z^2} = \frac{i\delta\beta}{2\sin\theta}. \tag{17}$$

Using (16) and (17) in (14), we obtain

$$p(z) + \frac{\delta z p'(z)}{p(z)} = \left(\frac{\sin\theta}{1-cos\theta}\right)^{\beta} \exp\left(\frac{\beta\pi i}{2}\right) + \frac{i\delta\beta}{\sin\theta}.$$

which is (15).

Next write $t = \cos\theta$ in (15), so that

$$p(z) + \frac{\delta z p'(z)}{p(z)} = \left(\frac{1+t}{1-t}\right)^{\beta/2} \exp\left(\frac{\beta\pi}{2}\right) - \frac{i\delta\beta}{\sqrt{1-t^2}}$$

$$= \left(\frac{1+t}{1-t}\right)^{\beta/2} \cos\frac{\beta\pi}{2} + i\left[\left(\frac{1+t}{1-t}\right)^{\beta/2} \cos\frac{\beta\pi}{2}\sin\frac{\beta\pi}{2} + \frac{\delta\beta}{\sqrt{1-t^2}}\right]$$

and so taking arguments, we obtain

$$\begin{aligned}
&\arg\left(p(z)+\frac{\delta zp'(z)}{p(z)}\right)\\
&=\arctan\left[\left\{\left(\frac{1+t}{1-t}\right)^{\beta/2}\sin\frac{\beta\pi}{2}+\frac{\delta\beta}{\sqrt{1-t^2}}\right\}/\left(\frac{1+t}{1-t}\right)^{\beta/2}\cos\frac{\beta\pi}{2}\right]\\
&=\arctan\left[\tan\frac{\beta\pi}{2}+\frac{\delta\beta}{(1+t)^{\frac{1+\beta}{2}}(1-t)^{\frac{1-\beta}{2}}\cos\frac{\beta\pi}{2}}\right].
\end{aligned}$$

An elementary calculation shows that the minimum expression is attained when $t=\beta$. To see this, let $\nu(t)=1/[(1+t)^{\frac{1+\beta}{2}}(1-t)^{\frac{1-\beta}{2}}]$. Differentiate this function, we have

$$\nu'(t)=(1+t)^{-\frac{\beta+3}{2}}(1-t)^{\frac{\beta-3}{2}}\left[\left(\frac{1-\beta}{2}\right)(1+t)-\left(\frac{1+\beta}{2}\right)(1-t)\right].$$

Then $\nu'(t)=0$, if $(1-\beta)(1+t)=(1+\beta)(1-t)$, and so $t=0$. Hence

$$\begin{aligned}
&\arg\left(p(z)+\frac{\delta zp'(z)}{p(z)}\right)\\
&\qquad\geq\arctan\left[\tan\frac{\beta\pi}{2}+\frac{\delta\beta}{(1+t)^{\frac{1+\beta}{2}}(1-t)^{\frac{1-\beta}{2}}\cos\frac{\beta\pi}{2}}\right]\\
&\qquad=\alpha(\beta,\delta)\pi/2.
\end{aligned}$$

Thus $\alpha(\beta,\delta)$ is the exact value such that (2) holds and this complete the proof of Theorem 2.1 .

REMARK. We note that Obradovic, Fukui and Sekine [12] proved the following theorem, which is a partal solution to our result.

THEOREM 2.3. [12] *Let f be analytic for z in D, and $J(\delta,f)$ be defined by (3) for $\delta>0$, and let $0<\beta\leq 1$ be given. If*

$$|\arg J(\delta,f)|<\frac{\alpha(\beta,\delta)\pi}{2},\qquad \textit{then for } z\in D, \left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\beta\pi}{2},$$

where

$$\alpha(\beta,\delta)=\frac{2}{\pi}\arctan\left[\tan\frac{\beta\pi}{2}+\frac{\delta\beta}{2\cos\frac{\beta\pi}{2}}\right].$$

References

1. Duren, P.L., *Univalent Functions.*, Springer-Verlag, 1983.
2. Goel, R.M., *Functions Starlike and Concex of Order* α, J. London Math. Soc. **2** (1974), 128-130.
3. Goodman, A.W., *Univalent Functions*, Mariner Publishing Company, Inc., Tampa, Florida, 1983.
4. Jack, I.S., *Functions starlike and convex of order* α, J. London Math. Soc.(2) **3** (1971), 469-474.
5. MacGregor, T.H., *A Subordination for convex functions of order* α, J. London Math. Soc. (2) **23** (1975), 530-536.
6. Marjono, *An Other Version of the Miller-Mocanu Lemma*, Proceeding of the ISSM; Germany (1998), 251-253.
7. Marjono, *A Sufficient Condition for Starlikeness for Analytic Functions*, Math. Japonica **51** no. 3 (2000), 423-428.
8. Marjono, *Functions convex and starlike in a half plane*, J. of the University of Parahyangan, Integral **51** no. 2 (2000), 63-66.
9. S.S. Miller and P.T. Mocanu, *Differential subordinations and univalence functions*, Michigan Math. J. **28** (1981), 157-171.
10. Marx, A., *Untersuchungen über Schlichte Abbildung*, Math. Ann. **107** (1932/33), 40-67.
11. Nunokawa, M. and D.K. Thomas, *On convex and starlike functions in a sector*, J. Austral. Math. Soc. Ser. **A 60** no. 3 (1996), 363-368.
12. Obradovic, M., Fukui, S and T. Sekine, *On* α*-convexity and starlikeness*, preprint.
13. Strohhäcker, E., *Beitrage zur Theorie der Schlichten Funktionen*, Math. Z **37** (1933), 356-380.
14. Wilken, D.R. and J. Feng, *A remark on convex and starlike functions*, J.London Math Soc. (2) **21** (1980), 287-290.