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ON THE WEAK DEPENDENCE STRUCTURE OF MULTIVARIATE PROCESSES AND CORRESPONDING HITTING TIMES

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Abstract. In this paper we introduce a new concept of positive orthant dependence of multivariate stochastic processes. This concept is weaker than the POD but it enjoys most of the properties and preservation results of the POD. Some examples are presented.

1. Introduction

The theory of positive quadrant dependent(PQD) random variables and of its dual notion, negative quadrant dependent(NQD) random variables was initiated by the seminal paper of Lehmann (1966). After this a number of aspects of positive(negative) dependence notions have been studied for several decades. Concepts of this dependence have subsequently been extended to stochastic processes in different directions by many authors. For a bibliography of available results see Ebrahimi and Ramallingam(1989); they are introduced some positive dependence concepts in terms of the finite dimensional of the hitting times of the components of a vector process. Most of the multivariate dependence introduced in the literature are stronger than POD(NOD), so that one

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may face problems that one wishes to investigate a new weakly positive(negative) dependen ce concept weaker than POD(NOD).

In this paper, we are concerned with weakly positive dependence of system reliability for time interval $[0, t]$. The reliability, $\bar{F}(t)$, of a system(component) is the probability that the system functions will preserve its characteristics within specified limits during a specified time interval $[0, t]$. If a system failure is an event in which at least one characteristic of the system shifts outside certain permissible limits, and if T is the time to failure, then

$$\bar{F}(t) = P(T > t).$$

Suppose that the system reliability is determined by a finite number of characteristics. For $i = 1, \cdots, n$, denote the value of the ith characteristic at time t by $X_i(t)$ and assume that it is within permissible limits if $X_i(t) < a_i$, where $a_1, \cdots, a_n$ are fixed and known values. For example, we may look upon a_i as the breaking threshold of total damages $X_i(t)$ by time t. Let the random time $T_i(a_i)$, at which the ith characteristic first crosses its limit is given by

$$T_i(a_i) = \begin{cases} \inf\{t \in \Lambda \mid X_i(t) \geq a_i\} \\ \infty \text{ if } X_i(t) < a_i \text{ for all } t \in \Lambda, \quad i = 1, \cdots, n \end{cases} \tag{1.1}$$

where the index set Λ is a subset of $R_+ = [0, \inf)$. In this setting, the failure time of the system, T, is given by

$$T = min(T_1(a_1), \cdots, T_n(a_n)). \tag{1.2}$$

In view of (1.1) and (1.2),

$$\bar{F}(t) = P(T_1(a_1) > t, \cdots, T_n(a_n) > t). \tag{1.3}$$

Formulation of system reliability by means of (1.1)-(1.3) is relevant to engineering disciplines relating to structural safety, variation of current and voltage, etc.

In general, it is possible to assess the system reliability $\bar{F}(t)$ provided that we can jointly model $X_1(t), \cdots, X_n(t)$ and we can also seek weakly probability inequalities for system reliability. To obtain such weak probability inequalities, information about the dependence structure of $T_1(a_1), \cdots, T_n(a_n)$ is essential.

For example if we know that for some $s_1, \cdots, s_n$,

$$\int_{s_1}^{\infty} \cdots \int_{s_n}^{\infty} (P(\cap_{i=1}^{n} T_i(a_i) > x_i) - \prod_{i=1}^{n} P(T_i(a_i) > x_i))dx_n \cdots dx_1 \geq 0 \tag{1.4}$$

and

$$\int_{0}^{s_1} \cdots \int_{0}^{s_n} (P(\cap_{i=1}^{n} T_i(a_i) > x_i) - \prod_{i=1}^{n} P(T_i(a_i) > x_i))dx_n \cdots dx_1 \geq 0, \tag{1.5}$$

then we can assess (1.4) and (1.5). Besides bounds information about the dependence structure may bring forth new weakly probability inequalities for stochastic processes. These results are of value as they help us to understand in what ways the hitting times for dependence structures of hitting times can be inherited from the corresponding processes. Furthermore, these results sometimes can tell us how to control the reliability of a system by controlling its characteristics.

The importance of this paper lies in the fact that it is weaker than positive orthant dependence and it enjoys most of the properties and theoretical results of weakly positive orthant dependence. In particular, usefulness of weakly positive dependence in applied probability, reliability, and statistical inference such as analysis of variance, multivariate test of hypothesis, sequential testing is well known.

In Section 2 of this paper, some notations, properties, and definitions are presented. In Section 3, we prove some theorems which not only clarify some properties of dependent process, but also help us to identify weakly positive dependence structures both among processes and their corresponding hitting times. Finally, in Section 4, we give some examples of processes and hitting times.

2. Notations and definitions

In this section we present definitions, notations, and properties used throughout the paper. In what follows 'increasing' means 'non-decreasing'. Suppose that $\{X(t) = (X_1(t), \cdots, X_n(t)) \mid t \in \Lambda\}$ is an n-dimensional stochastic processes, where the index set Λ is a subset of $R_+ = [0, \infty]$. The state space of $\{X(t) \mid t \in \Lambda\}$ is the cartesian product $E = E_1 \times E_2 \times \cdots \times E_n$, which will be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. If the index set *Lambda* is $\{0, 1, 2, \cdots\}$, then

$$P(\cap_i = 1^n T_i(a_i) > t_i) = P(\max_{0 \leq j_i \leq [t_i]} X_i(j_i) < a_i, i = 1, \cdots, n), \tag{2.1}$$

where $[r]$ is the largest integer less than or equal to r.

We now present some concepts of positive and weakly positive dependence, $X(t)$ is smaller than $Y(t)$ in the upper(lower) orthant-convex(concave) order, associated, and stochastically increasing for any n-dimensional stochastic process.

definitionDefinition 2.1[Ebrahimi and Ramallingam(1989)] The stochastic processes $X(t)$ is positive orthant dependent(POD) if

$$P(\bigcap_i = 1^n T_i(a_i) > t_i) \geq \prod_i = 1^n P(T_i(a_i) > t_i)$$

and

$$P(\bigcap_i = 1^n T_i(a_i) \leq t_i) \geq \prod_i = 1^n P(T_i(a_i) \leq t_i)$$

for all a_i and t_i, $i = 1, \cdots, n$.

DEFINITION 2.2. The stochastic process $\{X(t) | t \in \Lambda\}$ is weakly positive upper(lower) orthant dependent(WPUOD(WPLOD)) if they satisfy the following both

$$\int_{s_1}^{\infty} \cdots \int_{s_n}^{\infty} (P(\cap_{i=1}^n X_i(t_i) > a_i) - \prod_{i=1}^n P(X_i(t_i) > a_i)) da_n \cdots da_1 \geq 0 (WPUOD1)$$

$$\{\int_{s_1}^{\infty}\cdots\int_{s_n}^{\infty}(P(\cap_{i=1}^{n}X_i(t_i)\le a_i) - \prod_{i=1}^{n}P(X_i(t_i)\le a_i))da_n\cdots da_1\ge 0\}(WPLOD1)$$

and

$$\int_{0}^{s_1}\cdots\int_{0}^{s_n}(P(\cap_{i=1}^{n}X_i(t_i)> a_i) - \prod_{i=1}^{n}P(X_i(t_i)> a_i))da_n\cdots da_1\ge 0(WPUOD2)$$

$$\{\int_{0}^{s_1}\cdots\int_{0}^{s_n}(P(\cap_{i=1}^{n}X_i(t_i)\le a_i) - \prod_{i=1}^{n}P(X_i(t_i)\le a_i))da_n\cdots da_1\ge 0\}(WPLOD2)$$

for all $a_i\in E_i$ and $t_i\in\Lambda$, $i=1,\cdots,n$ and $\{X(t)\,|\,t\in\Lambda\}$ is each (univariate) WPUOD(WPLOD). For $j=1,2,\cdots,n$, we say that a one-dimensional process $X_j(t)$ is WPUOD if for any $0\le s_1< s_2<\cdots<s_n$, $s_i\in\Lambda$ and $a_i\in E_j$, $i=1,2,\cdots,n$,

$$\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}(P(\cap_{i=1}^{n}X_j(s_i)> a_i) - \prod_{i=1}^{n}P(X_j(s_i)> a_i))da_n\cdots da_1\ge 0$$

and

$$\int_{0}^{x_1}\cdots\int_{0}^{x_n}(P(\cap_{i=1}^{n}X_j(s_i)> a_i) - \prod_{i=1}^{n}P(X_j(s_i)> a_i))da_n\cdots da_1\ge 0$$

and WPLOD could be denoted like a Definition 2.2 by $X_j(t)$.

Also, the hitting times $T_1(a_1), \cdots, T_n(a_n)$ are WPUOD if

$$\int_{s_1}^{\infty} \cdots \int_{s_n}^{\infty} (P(\cap_{i=1}^{n} T_i(a_i) > t_i) - \prod_{i=1}^{n} P(T_i(a_i) > t_i))dt_n \cdots dt_1 \geq 0 (WPUOD1)$$

and

$$\int_{0}^{s_1} \cdots \int_{0}^{s_n} (P(\cap_{i=1}^{n} T_i(a_i) > t_i) - \prod_{i=1}^{n} P(T_i(a_i) > t_i))dt_n \cdots dt_1 \geq 0 (WPUOD2)$$

for every $a_i \in E_i$ and $t_i \in \Lambda$, $i = 1, 2, \cdots, n$, and WPLOD could be defined like a Definition 2.2 by $T_i(a_i)$, $i = 1, \cdots, n$. Moreover, $X(t)$ is WPOD if they satisfy both WPUOD and WPLOD.

DEFINITION 2.3. The stochastic process $\{X(t)|t \in Lambda\}$ is associated if

$$Cov(f(X_i(t_i), i = 1, \cdots, n), g(X_i(t_i), i = 1, \cdots, n)) \geq 0$$

for all increasing real valued functions f and g such that the covariance exists and all $t_i \in \Lambda, i = 1, \cdots, n$, and $\{X(t) \,|\, t \in \Lambda\}$ is each(univariate) associated. For $j = 1, 2, \cdots, n$, we say that a one-dimensional process $X_j(t)$ is associated if $0 \leq s_1 < s_2 < \cdots < s_n$,

$$Cov(f(X_j(s_i), i = 1, \cdots, n), g(X_j(s_i), i = 1, \cdots, n)) \geq 0.$$

Also, we say that the hitting times $T_1(a_1), \cdots, T_n(a_n)$ are associated if

$$Cov(f(T_1(a_1), \cdots, T_n(a_n)), g(T_1(a_1), \cdots, T_n(a_n))) \geq 0$$

for all increasing real valued functions f and g for which the covariance exists. As a direct consequence of definitions 2.1, 2.2, and 2.3 we have the following Remark 1.

REMARK 1. $X(t)$ is associated $\Rightarrow E\prod_i = 1^n f_i(X_i(t_i)) \geq \prod_i = 1^n Ef_i(X_i(t_i))$ for all increasing functions $f_i, i = 1, \cdots, n$ $Lefirightarrow$ $X(t)$ is POD $Rightarrow$ $X(t)$ is WPOD.

The following example shows that WPUOD does not imply PUOD.

EXAMPLE 2. Consider a discrete-time process $\{X_1(n) : n \geq\}$ such that $X_1(0), X_1(1)$, and $X_1(2)$ have the following joint distribution:

		$X_1(2)$								
		0			1			2		
		$X_1(1)$			$X_1(1)$			$X_1(1)$		
		0	1	2	0	1	2	0	1	2
	0	0.1	0	0.15	0	0	0	0	0	0
$X_1(0)$	1	0	0	0	0.3	0.15	0.05	0	0	0
	2	0	0	0	0	0	0	0	0.05	0.2

Table 1

It is easy to check that $X_1(n)$ is WPUOD1 but not PUOD.

DEFINITION 2.4. The stochastic process $\{X(t)|t \in \Lambda\}$ is smaller than $\{Y(t) \mid t \in \Lambda\}$ in the upper(lower) orthat-convex(concave) order $X(t) \leq_{uo-cx(lo-cv)} Y(t)$ if

$$\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P(\cap_{i=1}^{n} X_i(t_i) > a_i) da_n \cdots da_1$$
$$\leq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P(\cap_{i=1}^{n} Y_i(t_i) > a_i) da_n \cdots da_1$$
$$\{\int_{0}^{x_1} \cdots \int_{0}^{x_n} P(\cap_{i=1}^{n} X_i(t_i) \leq a_i) da_n \cdots da_1$$
$$\geq \int_{0}^{x_1} \cdots \int_{0}^{x_n} P(\cap_{i=1}^{n} Y_i(t_i) \leq a_i) da_n \cdots da_1\}$$

for all $a_i \in E_i$ and $t_i \in \Lambda$, $i = 1, \cdots, n$ and $\{X(t) \mid t \in \Lambda\}$ is each(univariate) smaller than $Y(t)$ in the upper(lower) orthant-convex(concave) order. For $j = 1, \cdots, n$, we say that a one-dimensional process $X_j(t)$ is smaller than $Y_j(t)$ in the upper(lower) orthant convex(concave) order if for any $0 \leq s_1 < s_2 < \cdots < s_n, s_i \in \Lambda$ and $a_i \in E_i$, $i = 1, \cdots, n$,

$$\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P(\cap_{i=1}^{n} X_j(s_i) > a_i) da_n \cdots da_1 \\ \leq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P(\cap_{i=1}^{n} Y_j(s_i) > a_i) da_n \cdots da_1$$

$$\{\int_{0}^{x_1} \cdots \int_{0}^{x_n} P(\cap_{i=1}^{n} X_j(s_i) \leq a_i) da_n \cdots da_1 \\ \geq \int_{0}^{x_1} \cdots \int_{0}^{x_n} P(\cap_{i=1}^{n} Y_j(s_i) \leq a_i) da_n \cdots da_1\}$$

Also, the hitting time $(T_1(a_1), \cdots, T_n(a_n))$ is smaller than $(S_1(a_1), \cdots, S_n(a_n))$ in the upper(lower) orthant convex(concave) order if

$$\int_{s_1}^{\infty} \cdots \int_{s_n}^{\infty} P(\cap_{i=1}^{n} T_i(a_i) > t_i) dt_n \cdots dt_1 \\ \leq \int_{s_1}^{\infty} \cdots \int_{s_n}^{\infty} P(\cap_{i=1}^{n} S_i(a_i) > t_i) dt_n \cdots dt_1$$

$$\{\int_{0}^{s_1} \cdots \int_{0}^{s_n} P(\cap_{i=1}^{n} T_i(a_i) \leq t_i) dt_n \cdots dt_1 \\ \geq \int_{0}^{t_1} \cdots \int_{0}^{s_n} P(\cap_{i=1}^{n} S_i(a_i) \leq t_i) dt_n \cdots dt_1\}$$

for every $a_i \in E_i$ and $t_i \in \Lambda$, $i = 1, 2, \cdots, n$.

DEFINITION 2.5. The stochastic process $\{X(t) | t \in \Lambda\}$ is stochastically increasing in $\{Y(t) \mid t \in \Lambda\}$ if

$$E(f(X_i(t_i), i = 1, \cdots, n) \mid Y_i(t_i) = b_i, i = 1, \cdots, n)$$

is increasing in $b_1, b_2, \cdots, b_n$ and for every real valued increasing function f. For $j = 1, 2, \cdots, n$, we say that a one-dimensional process $X_j(t)$ is stochastically increasing in $\{Y(t) \mid t \in \Lambda\}$ for any $0 \leq s_1 < \cdots < s_n$, $E(f(X_j(s_i)) \mid Y_i(s_i) = b_i, i = 1, \cdots, n)$. Also, we say that the hitting times $T_1(a_1), \cdots, T_n(a_n)$ are stochastically increasing in $T_1(b_1), \cdots, T_n(b_n)$ if $Ef(T_i(a_i)) \mid T_i(b_i) = t_i, i = 1, \cdots, n)$ is increasing in $t_i \in \Lambda$ and $a_i, b_i \in E_i$ for all $i = 1, \cdots, n$.

Before introducing the main results, let us present some basic properties of WPOD stochastic processes.

(P_0) Any set of independent stochastic processes is WPOD.

(P_1) Any subset of WPOD processes is WPOD.

(P_2) The set consisting of a single process is WPOD.

(P_3) The union of independent set of WPOD processes are WPOD.

3. Theoretical Results

In this section we will assume that the index set $\Lambda = \{1, 2, \cdots\}$.

THEOREM 3.1(A). *Let one-dimensional process $\{X_1(t) \mid t \in \Lambda\}$ be WPOD. Then $T(a_1), \cdots, T(a_n)$ are WPOD, here $T(a_i) = \inf\{n \mid X_1(n) \geq a_i\}$, $i = 1, \cdots, n$.*

Proof. We will prove this theorem for $n = 2$. Suppose $X_1(t)$ is WPOD, then we need to show that for $a_1 \leq a_2$,

$$\int_{x_1}^{\infty} \int_{x_2}^{\infty} P(\cap_{i=1}^{2} T(a_i) > t_i) dt_2 dt_1 \geq \int_{x_1}^{\infty} \int_{x_2}^{\infty} \prod_{i=1}^{2} P(T(a_i) > t_i) dt_i.$$

Now,

$$\begin{aligned}
&\int_{x_1}^{\infty}\int_{x_2}^{\infty} P(\cap_{i=1}^{2} T(a_1) > t_i, T(a_2) > t_2)dt_2dt_1 \\
&= \int_{x_1}^{\infty}\int_{x_2}^{\infty} P(max_{0\le j\le [t_1]} X_1(j) < a_1, \\
&\quad max_{0\le j\le [t_2]} X_1(j) < a_2)dt_2dt_1 \\
&= \int_{x_1}^{\infty}\int_{x_2}^{\infty} [P(X_1(j) < a_1, 0 \le j \le [t_1], X_1(j) < a_2, \\
&\quad [t_1]+1 \le j \le [t_2])I(t_1 \le t_2) \\
&\quad + P(X_1(j) < a_1, 0 <\le j \le [t_1])I(t_1 > t_2)]dt_2dt_1 \\
&\ge \int_{x_1}^{\infty}\int_{x_2}^{\infty} [P(X_1(j) < a_1, 0leqj \le [t_1])P(X_1(j) < a_2, \\
&\quad [t_1]+1 \le j \le [t_2])I(t_1 \le t_2) \\
&\quad + P(X_1(j) < a_1, 0 \le j \le [t_1])I(t_1 > t_2)]dt_2dt_1 \\
&\ge \int_{x_1}^{\infty}\int_{x_2}^{\infty} [P(T(a_1) > t_1)P(T(a_2) > t_2)I(t_1 \le t_2) \\
&\quad + P(T(a_1) > t_1)P(T(a_2) > t_2)I(t_1 > t_2)]dt_2dt_1 \\
&= \int_{x_1}^{\infty}\int_{x_2}^{i} nfty \prod_{i=1}^{2} P(T(a_i) > t_i)dt_i,
\end{aligned}$$

where I is the usual indicator function.

THEOREM 3.2(B). *Let one-dimensional process $\{X_1(t)\,|\,t \in \Lambda\}$ be WPOD and let f_i be increasing functions, $i = 1, 2, \cdots, n$. Then $f_1(T(a_1)), \cdots, f_n(T(a_n))$ are WPOD, here $T(a_i) = \inf\{n|X_1(n) \ge a_i\}$ and $W(a_i) = \inf\{n\,|\,f_i(X_1(n)) \ge a_i\}$, $i = 1, \cdots, n$.*

Proof. We can obtain the result using a method similar to that used in the proof of Theorem 3.1 (a).

In order to obtain Theorem 3.4 we need the following Lemma 3.2.

LEMMA 3.2. *Let $\{X_1(t)\,|\,t \in \Lambda\}$, $\cdots$, $\{X_n(t)\,|\,t \in \Lambda\}$ and $\{Y_1(t)\,|\,t \in \Lambda\}$, $\cdots$, $\{Y_n(t)\,|\,tn\Lambda\}$ be stochastic processes. Then* *the corresponding hitting times,* $T_n(a_n)) \leq_{uo-cx(lo-cv)} (S_1(a_1), \cdots, S_n(a_n))$ *if and only if* $E(\prod_{i=1}^n f_i(T_i(a_i)) \leq E(\prod_{i=1}^n f_i(S_i(a_i))$ *for all non negative(nonpositive) increasing convex(concave) functions* $f_1, \cdots, f_n$.

Proof. Use the similar method which Shaked and Shanthikumar(1994) used proving their Theorem 5.A.14.

THEOREM 3.3. *Suppose that* $\{X_1(t)\,|\,t \in \Lambda\}, \cdots, \{X_n(t)\,|\,t \in \Lambda\}$ *are stochastic processes and* $\{Y_1(t)\,|\,t \in \Lambda\}, \cdots, \{Y_n(t)\,|\,t \in \Lambda\}$ *have independent stochastic processes such that* $P(max_{0 \leq j \leq [t_i]} X_i(j) < a_i) = P(max_{0 \leq j \leq [t_i]} Y_i(j) < a_i)$ *for* $i = 1, \cdots, n$. *Then* $(T_1(a_1), \cdots, T_n(a_n))$ *are WPUOD1(WPLOD2 if and only if* $(T_1(a_1), \cdots, T_n(a_n)) \geq_{uo-cx(lo-cv)} (S_1(a_1), \cdots, S_n(a_n))$.

Proof. We only prove WPUOD1 case.

($\Rightarrow$) Assume $T_1(a_1), \cdots, T_n(a_n)$ are WPUOD1. From that assumption we have

$$\begin{aligned}
&\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P(\cap_{i=1}^n P(T_i(a_i) > t_i) dt_n \cdots dt_1 \\
&\geq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod -i = 1^n P(T_i(a_i) > t_i) dt_n \cdots dt_1 \\
&= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{i=1}^n P(\max_{0 \leq j \leq [t_i]} X_i(j) < a_i) dt_n \cdots dt_1 \\
&= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{i=1}^n P(\max_{0 \leq j \leq [t_i]} Y_i(j) < a_i) dt_n \cdots dt_1 \\
&= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{i=1}^n P(S_i(a_i) > t_i) dt_n \cdots dt_1 \\
&= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{i=1}^n P(\cap_{i=1}^n S_i(a_i) > t_i) dt_n \cdots dt_1
\end{aligned}$$

Hence $T_1(a_1), \cdots, T_n(a_n)) \geq_{uo-cx} (S_1(a_1) \cdots, S_n(a_n))$.

($\Leftarrow$) It follows from assumptions $(T_1(a_1), \cdots, T_n(a_n)) \geq_{uo-cx} (S_1(a_1), \cdots, S_n(a_n))$ and that $P(max_{0 \leq j \leq [t_i]} X_i(j) < a_i)$

$= P(max_{0 \le j \le [t_i]} Y_i(j) < a_i)$

$$\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} (P(\cap_{i=1}^n T_i(a_i) > t_i) - \prod_{i=1}^n P(T_i(a_i) > t_i)) dt_n \cdots dt_1$$
$$\ge \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} (P(\cap_{i=1}^n S_i(a_i) > t_i)$$
$$- \prod_{i=1}^n P(S_i(a_i) > t_i)) dt_n \cdots dt_1 = 0.$$

The zero follows from the assumption that $\{Y_1(t) \,|\, t \in \Lambda\}, \cdots, \{Y_n(t) \,|\, t \in \Lambda\}$ have independent stochastic processes. Hence $(T_1(a_1), \cdots, T_n(a_n))$ are WPUOD1. Similarly, we can prove WPLOD2 case.

From Lemma 3.2 and Theorem 3.3 we obtain the following theorem.

THEOREM 3.4. *Suppose that* $\{X_1(t) \,|\, t \in \Lambda\}, \cdots, \{X_n(t) \,|\, t \in \Lambda\}$ *are stochastic processes. Then the hitting times* $(T_1(a_1), \cdots, T_n(a_n))$ *are WPUOD1(WPLOD2) if and only if* $E(\prod_{i=1}^n f_i(T_i(a_i)) \le (\ge) \prod_{i=1}^n E(f_i(T_i(a_i))$ *for all nonnegative(nonpositive) increasing convex (concave) functions* $f_1, \cdots, f_n$.

Proof. It follows from this that Lemma 3.2 and Theorem 3.3.

In order to prove our next result we need to use the following notations.

Let $X = (X_1, \cdots, X_k)$ be a k-dimensional vector with distribution function F and the marginal distribution functions $F_j, j = 1, 2, \cdots, k$. The dependence function of X(or of F) is defined by

$$D_F(u_1, \cdots, u_k) = P(F_j(X_j) \le u_j, j = 1, 2, \cdots, k). \qquad (3.1)$$

It is clear that D_F is the distribution function on $[0,1]^k$, and it has uniform marginal distributions if the F_j's are continuous. The marginal distributions together with the dependence function determine F, since $F(x_1, \cdots, x_k) = D_F(F_1(x_1), \cdots, F_k(x_k))$. Furthermore, a dependence function D_F is said to be an extreme dependence function if all the marginals are non degenerative, and for each $n \geq 1$,

$$D_F^n(u_1, \cdots, u_k) = D_F(u_1^n, \cdots, u_k^n), (u_1, \cdots, u_k) \in [0,1]^k.$$

Hsing(1987) showed that D_F is extreme dependence function if and only if

$$D_{F^n}(u_1, \cdots, u_k) = D_F(u_1, \cdots, u_k), (u_1, \cdots, u_k) \in [0,1]^k. \quad (3.2)$$

It is clear that if $(X_{i1}, \cdots, X_{ik}), 1 \leq i \leq n$ are independent random vectors all having a distribution of F, then F^n is the distribution function of $(max_{1 \leq i \leq n} X_{i1}, \cdots, max_{1 \leq i \leq n} X_i k)$ and hence (3.2) is equivalent to $D_{F^n}(u_1, \cdots, u_k) = P(G_j(max_{1 \leq i \leq n} X_{ij}) \leq u_j, j = 1, \cdots, k)$, where G_j is the distribution function of $max_{1 \leq i \leq n} X_{ij}$ which is F_j^n.

Now, we will define a concept for k-dimensional processes. For a k-dimensional process $\{X_1(t) \,|\, t \in \Lambda\}, \cdots, \{X_k(t) \,|\, t \in \Lambda\}$, $\{W(t_1, \cdots, t_k) \,|\, t_1, \cdots, t_k \in \Lambda\}$ is said to be a dependence function if for any $t_1, \cdots, t_k \in \Lambda$, $W(t_1, \cdots, t_k) = D_{F_{X_1(t_1), \cdots, X_1(t_k)}}$, where $D_{F_{X_1(t_1), \cdots, X_k(t_k)}}(u_1, \cdots, u_k) = P(F_i(X_i(t_i)) \leq u_i, \; i = 1, \cdots, k)$, where $F_i(x) = P(X_i(t_i) \leq u_i, \; i = 1, \cdots, k)$.

Then we obtain the following theorem.

THEOREM 3.5. *Let (a)* $\{X_1(t) \,|\, t \in \Lambda\}, \cdots, \{X_k(t) \,|\, t \in \Lambda\}$ *be WPOD stochastic processes,*

(b) $X_i(t)$ *is strictly stationary* $i = 1, \cdots, k$ *(The process* $Y(t)$ *is said to be strictly stationary if for* $0 \leq t_1 < t_2 < \cdots < t_k$ *and* $h > 0, (Y(t_1 + h), \cdots, Y(t_k + h) =^d (Y(t_1), \cdots, Y(t_k))$,

(c) $D_{G_{n_1, \cdots, n_k}}(u_1, \cdots, u_k) = D_F(u_1, \cdots, u_k)$ *for all* $n_1, \cdots, n_k \in \{0, 1, \cdots\}$ *and* $(u_1, \cdots, u_k) \in [0,1]^k$*, this condition is equivalent to condition (3.2) for the case that* $X_{ij}'s$ *are not i.i.d, where* $D_{G_{n_1, \cdots, n_k}}(u_1, \cdots, u_k) = P(G_i(max_{1 \leq j \leq n_i} X_{ij}(j) \leq u_i, \; i = 1, \cdots, k)$ *and* $D_F(u_1, \cdots, u_k) = P(F_i(X_i(0)) \leq u_i, \; i = 1, \cdots, k)$*. Then* $(T_1(a_1), \cdots, T_k(a_k))$ *is WPOD.*

Proof. Noting $P(T_1(a_1) > n_1, \cdots, T_k(a_k) > n_k)$ $= P(max_{1 \leq j \leq n_1} X_1(j) \leq a_1, \cdots, max_{1 \leq j \leq n_k} X_k(j) \leq a_k)$ and the fact that $G_1, \cdots, G_k$ are the distribution functions of

$max_{1\le j\le n_1} X_1(j), \cdots, max_{1\le j\le n_k} X_k(j)$, we can obtain result of this theorem.

In order to prove Theorem 3.7, we need the following Lemma 3.6.

LEMMA 3.6. *Let (a) $\{Y(t) \mid t \in \Lambda\}$ be associated, (b) $\{X(t) \mid t \in \Lambda\}$, given $Y(t)$, be conditionally associated, (c) $X(t)$ is stochastically increasing in $Y(t)$. Then $(X(t), Y(t))$ is associated and $X(t)$ is associated. Furthermore, the corresponding hitting time $(T_1(a), T_2(b))$ is associated and $T_1(a)$ is associated, here $T_1(a) = \{n \mid X(n) >= a\}$, $T_2(b) = \inf\{n \mid Y(n) \ge a\}$.*

Proof. It is enough to show that for any increasing functions f and g and $n_1, n_2 \in \{0, 1, 2, \cdots\}$,

$$cov(f(X(n_1), Y(n_2)), g(X(n_1), Y(n_2))) \ge 0.$$

Note that

$$\begin{aligned} &cov(f(X(n_1), Y(n_2)), g(X(n_1), Y(n_2))) \\ &= E(cov((f(X(n_1), Y(n_2)), g(X(n_1), Y(n_2))) \\ &+ cov(E(f(X(n_1), Y(n_2)) \mid Y(n_2)), E(g(X(n_1), Y(n_2)) \mid Y(n_2))) \end{aligned} \tag{3.3}$$

conditioned on $Y(n_2)$, $Y(n_2)$ is associated. Thus the first term on the right side of (3.3) is nonnegative. By assumption, $Y(t)$ is associated. Thus by increasing functions of associated stochastic processes are associated, the covariance of the conditional expectations in the second term is nonnegative. It follows that $cov(f(X(n_1), Y(n_2)), g(X(n_1), Y(n_2))) \ge 0$, so that $(X(t), Y(t))$ is associated. Second, since $(X(t), Y(t))$ is associated by subset of associated is associated $X(t)$ is also associated. Furthermore, we can show to similar method that for $n_{11}, \cdots, n_{1k}, n_{21}, \cdots, n_{2m} \in \{0, 1, 2, \cdots, \}$, $cov(f(X(n_{11}), \cdots, 4X(n_{1k}), Y(n_{21}), \cdots, Y(n_2m)),$ $g(X(n_{11}), \cdots, X(n_1k), Y(n_{21}), \cdots, Y(n_2m)) \ge 0$, and consequently

$$cov(f(T_1(a), T_2(b)), g(T_1(a), T_2(b))) \ge 0.$$

The inequality comes from the fact both f and g are increasing functions of $\{X(n) \mid n \in \Lambda\}$ and $\{Y(n) \mid n \in \Lambda\}$. Thus $(T_1(a), T_2(b))$ is associated, since $(T_1(a), T_2(b))$ is associated $(T_1(a))$ is also associated.

THEOREM 3.7. *Suppose $\{X(t) \mid t \in \Lambda\}$ and $\{Y(t) \mid t \in \Lambda\}$ satisfy a linear regression relationship of the form $X(t) = aY(t) + Z(t)$, where $a > 0$, $Z(t)$ is an independent stochastic process of $Y(t)$ and $Y(t)$ is WPOD. Then $(X(t), Y(t))$ is WPOD . Furthermore, the hitting time $((T_1(a), T_2(b))$ is WPOD, here $T_1(a) = \inf\{n \mid X(n) \geq a\}$, $T_2(b) = \inf\{n \mid Y(n) \geq b\}$.*

Proof. Since $X(t) = aY(t) + Z(t)$ is stochastically increasing in $Z(t)$, $X(t)$ given $Z(t)$, is associated, by Lemma 3.6, $(X(t), Y(t))$ is associated. Thus by Remark 1 $(X(t), Y(t))$ is WPOD. Furthermore, we can know that the corresponding hitting time $(T_1(a), T_2(b))$ is WPOD and $(T_1(a))$ is WPOD.

4. Examples

EXAMPLE 4.1. Consider a one-dimensional process $\{X_1(t) \mid t \in \Lambda\}$ such that $X_1(t)$ is a Brownian motion process (cf. Shaked and Shanthikumar(1994)). Then $T(a_1), \cdots, T(a_n)$ are WPOD, here $T(a_i) = \inf\{n \mid X_1(n) \geq a_i\}$, $i = 1, \cdots n$.

Proof. We will prove this result for WPUOD1.

$$\begin{aligned}
&\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P(\cap_{i=1}^{n} T(a_i) > t_i) dt_n \cdots dt_1 \\
&= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P(max_{1 \leq j \leq [t_1]} X_1(j) < a_1, \cdots, \\
&\qquad max_{1 \leq j \leq [t_n]} X_1(j) < a_n) dt_n \cdots dt_1 \\
&\geq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{i=1}^{n} P(max_{1 \leq j \leq [t_i]} X_1(j) < a_i) dt_i \\
&= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{i=1}^{n} P(T(a_i) > t_i) dt_i.
\end{aligned}$$

The inequality comes from the fact that a Brownian motion process has a continuous path and the results given by Pitt (1982) about multivariate normal distribution.

EXAMPLE 4.2. Consider the uniformly modulated model (see Priestley(1988)) such that non-stationary process $X(t)$ given by

$$X(t) = \alpha(t)Y(t), t \geq 0,$$

where $\alpha(t)$ is a deterministic continuous function such that $\alpha(t) \geq 0$ and $Y(t)$ is nonnegative stationary process. If $Y(t)$ is WPOD, we can know that $X(t)$ is WPOD. Using the Theorem 3.1(b), for f and g are two increasing functions, the corresponding hitting times $f_1(T(a_1)), \cdots, f_n(T(a_n))$ are WPOD, here $T(a_i) = \inf\{n \,|\, X(n) \geq a_i\}$, $i = 1, \cdots, n$.

EXAMPLE 4.3. Birth and death processes which start at 0 are free of positive skips and satisfy conditions of (a) $\{X(t) \,|\, t \geq\}$ be a strong Markov process with state space $\{0, 1, 2, \cdots\}$, (b) $\{X(0) = 0\}$, (c) $\{X(t) \,|\, t \geq\}$ is free of positive skips, that is the sample paths cannot have positive jumps greater than 1. What is more, Keilson(1971) has shown that the density function of $T(a)$ is log-concave. It follows that any such birth and death process is WPOD(WNOD) with itself. This fact leads to the following application.

APPLICATION 4.4. *Let $\{X(t) \,|\, t \geq\}$ be a diffusion process on the interval $[0, y]$ which has a reflecting boundary at 0. Let $\{N_k(t) \,|\, t \geq 0\}$ be a sequence of birth-death processes on $\{0, 1, 2, \cdots\}$ with the assumption $mu_0 = 0$ for its death rates. If $\alpha_k N_k(t) \to^w X(t)$, then $X(t)$ is WPOD(WNOD) and the corresponding hitting time is WPOD(WNOD), $T(a_i) = \inf\{n \,|\, X(n) \geq a_i\}$, $i = 1, 2 \cdots, n$.*

Proof. The result can be proved using the following facts. First, Note $X(t)$ is WPOD(WNOD) with itself. Second, hitting times $T(a)$ of $X(t)$ have log-concave densities, being the limits $(k \to \infty)$ of log-concave densities of hitting times of the processes $\alpha_k N_k(t)$ (see Keilson(1971)).

EXAMPLE 4.5. Leslie(1969) has considered the waiting time until the occurrence of a cluster of size k', $k \geq 2$, in a homogeneous Poisson process. Marshall and Shaked (1983) argue that such a waiting time is the hitting time $T(\frac{k-1}{2})$ of a new better than used (NBU) process $Z(t)$ given in Example 2.4 of their paper. We can verify that $Z(t)$ is WPOD(WNOD) with itself. Consequently, given $2 \leq k_1 < \cdots < k_n$, a bound for the joint distribution of the waiting times for clusters of sizes $k_1, \cdots, k_n$ is provided by

$$\begin{aligned}
&\int_{s_1}^{\infty} \cdots \int_{s_n}^{\infty} P(\cap_{i=1}^{n}(k_i - \frac{1}{2}) > x_i) dx_n \cdots dx_1 \\
&\geq (\leq) \int_{s_1}^{\infty} \cdots \int_{s_n}^{\infty} \prod_{i=1}^{n} P(T(k_i - \frac{1}{2}) > x_i) dx_i \\
&\{\int_{s_1}^{\infty} \cdots \int_{s_n}^{\infty} P(\cap_{i=1}^{n}(k_i - \frac{1}{2}) \leq x_i) dx_n \cdots dx_1 \\
&\geq (\leq) \int_{s_1}^{\infty} \cdots \int_{s_n}^{\infty} \prod_{i=1}^{n} P(T(k_i - \frac{1}{2}) \leq x_i) dx_i\}
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\int_{-\infty}^{s_1} \cdots \int_{-\infty}^{s_n} P(\cap_{i=1}^{n}(k_i - \frac{1}{2}) > x_i) dx_n \cdots dx_1 \\
&\geq (\leq) \int_{-\infty}^{s_1} \cdots \int_{-\infty}^{s_n} \prod_{i=1}^{n} P(T(k_i - \frac{1}{2}) > x_i) dx_i \\
&\{\int_{-\infty}^{s_1} \cdots \int_{-\infty}^{s_n} P(\cap_{i=1}^{n}(k_i - \frac{1}{2}) \leq x_i) dx_n \cdots dx_1 \\
&\geq (\leq) \int_{-\infty}^{s_1} \cdots \int_{-\infty}^{s_n} \prod_{i=1}^{n} P(T(k_i - \frac{1}{2}) \leq x_i) dx_i\}
\end{aligned} \tag{4.2}$$

It should be noted that the terms in the left-hand side of the inequality (4.1) can be computed by the similar methods of Leslie (1969), Section 4.

EXAMPLE 4.6. Consider two different repair policies. The first policy that we replace a failed unit with a new identical unit and we denote the number of replacement up to time t by $N(t)$. The second policy consists of repairing the unit to its condition just prior to failure, that is, a minimal repair and we denote the number of minimal repairs up to time t by $W(t)$. Suppose the sequences $\{X_n \mid n \geq 1\}$, $\{Y_n \mid n \geq 1\}$ denote the interarrival times for a renewal process $N(t)$ and a minimal repair process $W(t)$ respectively. Then, it is clear that, $X_1 = Y_1$,

$$P(X_n > t \mid X_i = Y_i, i = 1, \cdots, n-1) = \bar{F}(t)$$

and

$$\begin{aligned} P(Y_n > t \mid X_1 = Y_1 = y_1, Y_i = y_i, i = 2, \cdots, n-1) \\ = \frac{\bar{F}(t + y_1 + \cdots + y_{n-1})}{\bar{F}(y_1 + \cdots + y_{n-1})}. \end{aligned}$$

Suppose $\bar{F}(t)$ is a new worse than used(NWU)($\bar{F}(t)$ is said to be NWU if $\bar{F}(x+y) \geq= \bar{F}(x)\bar{F}(y)$ for all $x, y \geq 0$), then we can know that for any $n_1, n_2 \in \{1, 2, \cdots\}$, $X_1, \cdots, X_{n1}$, $Y_1, \cdots, Y_{n2}$ are WPOD. Using this fact we can show that $N(t)$ and $W(t)$ are WPOD. If $T_1(a) = \inf\{t \mid N(t) \geq a\}$ and $T_2(b) = \inf\{W(t) \geq 0\}$, then

$$\begin{aligned} &\int_{x_1}^{\infty}\int_{x_2}^{\infty} P(T_1(a) > t_1, T_2(b) > t_2) dt_2 dt_1 \\ &= \int_{x_1}^{\infty}\int_{x_2}^{\infty} P(N(t_1) \leq a, W(t_2) \leq b) dt_2 dt_1 \\ &\geq \int_{x_1}^{\infty}\int_{x_2}^{\infty} P(T_1(a) > t_1) P(T_2(b) > t_2) dt_2 dt_1. \end{aligned}$$

That is hitting times are WPOD.

EXAMPLE 4.7. Consider the following stress-strength model for two systems. Let $Z_i(t)$, $i = 1, 2$, be the strength of system i at time t. We will assume that the two systems receive shocks from a common source. Using a cumulative damage shock model (see Barlow and Proschan (1975)), we now let $N(t)$ denote the number of shocks occurring by time t and U_i are i.i.d. positive random variables denoting the damage to either system due to the ith shock ($i = 1, 2, \cdots$). Hence, the stress experienced by either system at time t is given by the process $W(t) = \sum_{i=1}^{N(t)} U_i$. Then, we can show that $W(t)$ is WPOD with itself. Assuming that $Z_1(t)$ and $Z_2(t)$ are independent processes with non-increasing sample paths and that $Z_1(t)$ and $Z_2(t)$ are independent of $W(t)$

and $X(t) = W(t) - Z_1(t)$ and $Y(t) = W(t) - Z_2(t)$, we obtain using Theorem 3.7 that the bivariate processes $\{(X(t), Y(t)) \mid t \in \{0, 1, \cdots\}\}$ is WPOD processes. Further more, the corresponding hitting times $T_1(a)$ and $T_2(b)$ are WPOD.

EXAMPLE 4.8. Consider a simple form of an econometrical model of investment and capital gain in Theorem 3.7. Let $\{X(t) \mid t \in \Lambda\}$ and $\{Y(t) \mid t \in \Lambda\}$ denote the investment and capital gain at time t, respectively. The model is

$$Y(t) = aX(t) + Z_2(t)$$
$$X(t) = Y(t-1) + Z_1(t),$$

where $a > 0$, $Z_1(t)$ and $Z_2(t)$ are both stochastic processes and $(Z_1(t), Z_2(t))$ are a sequence of independent random vectors. Then we can write that

$$X(n) = \sum_{i=0}^{n} a^{n-i} Z_1(i) + \sum_{i=0}^{n-1} a^{n-i-1} Z_2(i)$$
$$X(0) = Z_2(0)$$

and we can obtain $\{X(n) \mid n \in \{0, 1, 2, \cdots\}\}$ which is WPOD and we can write that as

$$Y(n) = \sum_{i=0}^{n} a^{n-i-1} Z_1(i) + \sum_{i=0}^{n} a^{n-i} Z_2(i)$$
$$Y(0) = Z_2(0),$$

and we can know that $\{Y(n) \mid n \in \{0, 1, 2, \cdots\}\}$ is WPOD. Thus the processes $\{X(n) \mid n \in \{0, 1, 2, \cdots\}\}$ and $\{Y(n) \mid n \in \{0, 1, 2, \cdots\}\}$ are WPOD. Furthermore, the corresponding hitting time $(T_1(a), T_2(b))$is WPOD.

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