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ON THE RELATION BETWEEN COMPACTNESS AND STRUCTURE OF CERTAIN OPERATORS ON SPACES OF ANALYTIC FUNCTIONS

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Abstract. Let $\mathcal{B}$ be a Banach space of analytic functions defined on the open unit disk. Assume S is a bounded operator defined on $\mathcal{B}$ such that S is in the commutant of M_{z^n} or $SM_{z^n} = -M_{z^n}S$ for some positive integer n. We give necessary and sufficient condition between compactness of $SM_z + cM_zS$ where $c = 1, -1, i, -i$, and the structure of S. Also we characterize the commutant of M_{z^n} for some positive integer n.

1. Introduction

In this section we give some notation and definitions which we use later. Let $\mathcal{B}$ be a Banach space consisting of complex valued analytic functions defined on the open unit disk $\mathbf{D}$ in the plane such that $1 \in \mathcal{B}, z\mathcal{B} \subset \mathcal{B}$ and for every $\lambda \in \mathbf{D}$ the evaluation functional at λ, $e_\lambda : \mathcal{B} \to \mathbf{C}$, given by $f \mapsto f(\lambda)$, is bounded. Let $\varphi : \mathbf{D} \to \mathbf{D}$ be a rotation we assume that for every $f \in \mathcal{B}$, $f \circ \varphi$ is in $\mathcal{B}$ and the composition operator C_φ defined by $C_\varphi(f) = f \circ \varphi$ is bounded. We define $\tilde{f}(\lambda) = f(-\lambda)$ and $\hat{f}(\lambda) = f(i\lambda)$ for every $\lambda \in \mathbf{D}$. Also we assume that the set of all analytic polynomials $\mathcal{P}$ is dense in $\mathcal{B}$. We set $\mathcal{B}_i =$ closed linear span $\{z^{nk+i} : k \geq 0\}$ for $i = 0, 1, 2, ..., n-1$. Let p be a polynomial having decomposition

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$p = p_0 + p_1 + ... + p_{n-1}$ where $p_i \in \mathcal{B}_i$ for $i = 0, 1, 2, ..., n-1$. Furthermore we assume that there are positive constants c_n and d_n such that $\|p_i\| \le c_n\|p\| \le d_n \max\{\|p_i\| : \ i = 0, 1, 2, \cdots, n-1\}$. Therefore the projection operator $P_{n,i} : \mathcal{P} \to \mathcal{B}$ defined by $P_{n,i}(p) = p_i$ has a unique extension to $\mathcal{B}$, and every $f \in \mathcal{B}$ has a unique decomposition $f = f_0 + f_1 + f_2 + \cdots + f_{n-1}$ where $f_i \in \mathcal{B}_i$ for $i = 0, 1, 2, \cdots, n-1$.

Throughout this article by a Banach space of *analytic functions* we mean one satisfying the above conditions. A complex valued function φ defined on $\mathbf{D}$ for which $\varphi f \in \mathcal{B}$ for all f in $\mathcal{B}$ is called a *multiplier* of $\mathcal{B}$ and the collection of all such multipliers is denoted by $\mathcal{M}(\mathcal{B})$. Each multiplier determines a multiplication operator M_φ on $\mathcal{B}$ defined by $M_\varphi(f) = \varphi f$. By the closed graph theorem it is easy to see that M_φ is bounded. Let $L(\mathcal{B})$ denote the algebra of all bounded operators on $\mathcal{B}$. It is well known that if $X \in L(\mathcal{B})$ and $XM_z = M_zX$, then $S = M_\varphi$ for some function $\varphi \in \mathcal{M}(\mathcal{B})$.

Throughout this article $\{M_\varphi\}'$ denotes the set of all bounded linear operators X on $\mathcal{B}$ such that $M_\varphi X = XM_\varphi$, i.e., the commutant of M_φ. Given $A \subset \mathcal{B}$ by the $\vee A$ we mean the closed linear span of A in $\mathcal{B}$ and for a fixed positive integer n we set $\mathcal{B}_i = \vee\{z^{nk+i} : k \ge 0\}$ for $i = 0, 1, 2, ..., n-1$. We define $\tilde{M}_\varphi : \mathcal{B} \to \mathcal{B}$ by $\tilde{M}_\varphi(f) = \varphi\tilde{f}$; and define $\hat{M}_\varphi : \mathcal{B} \to \mathcal{B}$ by $\hat{M}_\varphi(f) = \varphi\hat{f}$; by the closed graph theorem $\tilde{M}_\varphi$ and $\hat{M}_\varphi$ are bounded.

In what follows we present some examples of such spaces.

EXAMPLES.

a)The spaces D_α of all functions $f(z) = \sum \hat{f}(n)z^n$, holomorphic in $\mathbf{D}$, for which

$$\|f\|_\alpha^2 = \sum (n+1)^\alpha \mid \hat{f}(n) \mid^2 < \infty$$

for every $\alpha \in R$.

b)Let $1 < p < \infty$ and let $\{\alpha(n)\}$ be a sequence of positive numbers with $\alpha(0) = 1$. We consider the space of sequences $f = \{\hat{f}(n)\}$ such that

$$||f||_p^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p[\alpha(n)]^p < \infty.$$

We shall use the formal notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ for $z \in \mathbf{D}$ (See Shields [5] for $p = 2$). Let $\mathcal{B}^p(\alpha) = \{f| \ f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n; ||f||_p < \infty\}$ and

$$\mathcal{B}^p_a(\alpha) = \{f \in \mathcal{B}^p(\alpha)| \ f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \text{ is convergent in } \mathbf{D}\}.$$

Let $\{1/\alpha(n)\} \in \ell^q$. If $f \in \mathcal{B}^p(\alpha)$ and $\lambda \in \mathbf{D}$, we have

$$|f(\lambda)| = |\sum_{n=0}^{\infty} \hat{f}(n)\lambda^n| \leq (\sum_{n=0}^{\infty} |\hat{f}(n)|^p[\alpha(n)]^p)^{1/p}(\sum_{n=0}^{\infty} \frac{|\lambda|^{nq}}{\alpha(n)^q})^{1/q}.$$

Therefore, f is analytic and $||f||_{\mathbf{D}} \leq ||\{\frac{1}{\alpha(n)}\}||_q||f||_p$. Also $\mathcal{B}^p_a(\alpha) = \mathcal{B}^p(\alpha) \subset \mathcal{H}^{\infty}$. Furthermore, each point of $\mathbf{D}$ is a bounded point evaluation for $\mathcal{B}^p(\alpha)$ and also convergence in $\mathcal{B}^p(\alpha)$ implies uniform convergence on $\mathbf{D}$. For more information on these spaces see [3], [6], [7]

Cuckovic in [1] investigates the commutant of M_{z^n} on the Bergman space $L^2{}_a(\mathbf{D})$, Seddighi and Vaezpour [4] have shown that under certain conditions on the reproducing kernels of a functional Hilbert spaces every operator S essentially commuting with M_z and commuting with M_{z^n} for some $n > 1$ is a multiplication operator. In [2] the author characterized the commutant of M_{z^2} on various spaces of analytic functions. In section two we investigate the relation between compactness and the structure of operator S such that $S \in \{M_{z^n}\}'$ or $SM_{z^n} = -M_{z^n}S$ for some positive integer n. We give necessary and sufficient condition under which S gets the form M_{φ} or $\tilde{M}_{\varphi}$ or $\hat{M}_{\varphi}$ for some $\varphi \in \mathcal{M}(\mathcal{B})$.

2. The Main Results

In the next lemma we show that if $SM_z = -M_zS$, then $S = \tilde{M}_{\varphi}$ for some multiplication function φ.

LEMMA 2.1. *Let $\mathcal{B}$ be a Banach space of analytic functions and let $S : \mathcal{B} \to \mathcal{B}$ be an operator such that $SM_z = -M_zS$. Then there exists a $\varphi \in \mathcal{M}(\mathcal{B})$ such that $S(f) = \varphi\tilde{f}$ for every $f \in \mathcal{B}$.*

Proof. Let $S(1) = \varphi$. We have $S(z) = -z\varphi$ and by induction $S(z^n) = (-1)^n z^n\varphi$ for every positive integer n. Hence for a polynomial $p, S(p) = \varphi\tilde{p}$. Now assume that $f \in \mathcal{B}$ and let $\{p_n\}$ be a sequence of polynomials such that $p_n \to f$ in $\mathcal{B}$. According to the properties of $\mathcal{B}$, $\tilde{p}_n \to \tilde{f}$ in $\mathcal{B}$ and so pointwise. Therefore $\varphi\tilde{p}_n \to \varphi\tilde{f}$ pointwise, since $S(p_n) = \varphi\tilde{p}_n$ converges to $S(f)$ pointwise we have $S(f) = \varphi\tilde{f}$.

LEMMA 2.2. *Suppose $\mathcal{B}$ is a Banach space of analytic functions, $S \in L(\mathcal{B})$, $SM_{z^n} = -M_{z^n}S$ for some positive odd integer $n > 1$, and $\mathcal{B}_i = \vee\{z^{2nk+i} : k \geq 0\}$ for $i = 0, 1, 2, \cdots, 2n-1$. Suppose $f \in \mathcal{B}$ has decomposition $f = f_0 + f_1 + \cdots + f_{2n-1}$ where $f_i \in \mathcal{B}_i$ for $i = 0, 1, 2, \cdots, 2n-1$. Then*

$$S(f) = \varphi_0\tilde{f} + \varphi_1(\frac{f_1 - f_{n+1}}{z}) + \varphi_2(\frac{f_2 - f_{n+2}}{z^2}) + \cdots + \varphi_{n-1}(\frac{f_{n-1} - f_{2n-1}}{z^{n-1}}),$$

where $\varphi_0 = S(1), \varphi_i = (SM_{z^i} - M_{z^i}S)(1)$ for positive even integers $i < n$ and $\varphi_i = (SM_{z^i} + M_{z^i}S)(1)$ for positive odd integers $i < n$. Also:

$$\begin{aligned}(SM_z + M_zS)(f) = {} & f_0\varphi_1 + f_1(\frac{\varphi_2}{z} + \varphi_1) + f_2(\frac{\varphi_3}{z^2} + \frac{\varphi_2}{z}) + \cdots \\ & + f_{n-1}\frac{\varphi_{n-1}}{z^{n-2}} - f_n\varphi_1 - f_{n+1}(\frac{\varphi_2}{z} + \varphi_1) \\ & - f_{n+2}(\frac{\varphi_3}{z^2} + \frac{\varphi_2}{z}) - \cdots - f_{2n-1}(\frac{\varphi_{n-1}}{z^{n-2}}).\end{aligned}$$

Proof. Let p be a polynomial having decomposition $p = p_0 + p_1 + p_2 + \cdots + p_{2n-1}$ with respect to $\mathcal{B}_i$ for $i = 0, 1, 2, \cdots, 2n-1$. Since $\varphi_0 = S(1), \varphi_i = (SM_{z^i} + M_{z^i}S)(1)$ for i odd and $\varphi_i = (SM_{z^i} - M_{z^i}S)(1)$ for i even and $1 \leq i \leq n-1$, we have $S(z^i) = \varphi_i - z^i\varphi_0$ for i odd and $S(z^i) = \varphi_i + z^i\varphi_0$ for i even and $1 \leq i \leq n-1$. Also $S(z^{2nk}) = \varphi_0 z^{2nk}$ and $S(z^{2nk+n}) = -\varphi_0 z^{2nk+n}$. Therefore an easy calculation shows that

$$\begin{aligned}S(p_0) &= \varphi_0 p_0, S(p_1) = \frac{p_1}{z}\varphi_1 - p_1\varphi_0, S(p_2) \\ &= \frac{p_2}{z^2}\varphi_2 + p_2\varphi_0, \cdots, S(p_n) = -p_n\varphi_0.\end{aligned}$$

$S(p_{n+1}) = -\frac{p_{n+1}}{z}\varphi_1 + p_{n+1}\varphi_0, \cdots, S(p_{2n-1}) = -\frac{p_{2n-1}}{z^{n-1}}\varphi_{n-1} - p_{2n-1}\varphi_0$. Since $S(p) = S(p_0) + S(p_1) + S(p_2) + \cdots + S(p_{2n-1})$ we can see that $S(p)$ has the same form stated in the theorem. Now since polynomials are dense in $\mathcal{B}$ and convergence in norm implies pointwise convergence, the proof is completed.

The second part follows easily by considering

$$\begin{aligned}(SM_z + M_zS)(f) &= S(zf) + zS(f)\\ &= -z\hat{f}\varphi_0 + (f_0 - f_n)\varphi_1 + \frac{(f_1 - f_{n+1})}{z}\varphi_2\\ &+ \cdots + (\frac{f_{n-2} - f_{2n-2}}{z^{n-2}})\varphi_{n-1} + z\hat{f}\varphi_0 + (f_1 - f_{n+1})\varphi_1\\ &+ (\frac{f_2 - f_{n+2}}{z})\varphi_2 + \cdots + (\frac{f_{n-1} - f_{2n-1}}{z^{n-2}})\varphi_{n-1}.\end{aligned}$$

Note that if $S \in \{M_z\}'$ or $SM_z = -M_zS$, then $S \in \{M_{z^n}\}'$ for all positive even integers n. Hence if $SM_{z^n} = -SM_{z^n}$ for n even, then S does not have the form of M_ϕ or $\tilde{M}_\phi$. However we show that if n is an odd integer and $SM_z + M_zS$ is compact, then $S = \tilde{M}_\varphi$ for some function $\varphi \in \mathcal{M}(\mathcal{B})$.

In the next theorem, with a slight refinement we will use the idea of the proof of the Theorem 1.4 in [1].

THEOREM 2.3. *Suppose $\mathcal{B}$ is a Banach space of analytic functions, $S \in L(\mathcal{B})$ and $SM_{z^n} = -M_{z^n}S$ for some odd integer $n > 1$. There is a function $\phi \in \mathcal{M}(\mathcal{B})$ such that $S = \tilde{M}_\phi$ if and only if $SM_z + M_zS$ is compact.*

Proof. Let $SM_z + M_zS$ be a compact operator and $\mathcal{B}_i = \vee\{z^{2nk+i} : k \geq 0\}$ for $i = 0, 1, 2, \cdots, 2n-1$. suppose $f \in \mathcal{B}$ having decomposition $f = f_0 + f_1 + ... + f_{2n-1}$ where $f_i \in \mathcal{B}_i$, $i = 0, 1, ..., 2n-1$. By Lemma 2.2 we have:

$$\begin{aligned}(SM_z + M_zS)(f) &= f_0\varphi_1 + f_1(\frac{\varphi_2}{z} + \varphi_1) + f_2(\frac{\varphi_3}{z^2} + \frac{\varphi_2}{z}) + \cdots\\ &+ f_{n-1}\frac{\varphi_{n-1}}{z^{n-2}} - f_n\varphi_1 - f_{n+1}(\frac{\varphi_2}{z} + \varphi_1)\\ &- f_{n+2}(\frac{\varphi_3}{z^2} + \frac{\varphi_2}{z}) - \cdots - f_{2n-1}(\frac{\varphi_{n-1}}{z^{n-2}}).\end{aligned}$$

Since $(SM_z + M_zS)|_{\mathcal{B}_0} = M_{\varphi_1}|_{\mathcal{B}_0}$ is compact, it follows that $M_{\varphi_1 z^{2n}}|_{\mathcal{B}_0} = M_{z^{2n}}M_{\varphi_1}|_{\mathcal{B}_0}$ is compact. Also we see that $M_{\varphi_1 z^{2n}}|_{\mathcal{B}_i} = M_{z^i}M_{\varphi_1}|_{\mathcal{B}_0}M_{z^{2n-i}}|_{\mathcal{B}_i}$ is compact for $i = 1, 2, \cdots, 2n-1$. Since there are positive constants c_{2n} and d_{2n} such that $\|f_i\| \leq c_{2n}\|f\| \leq d_{2n}$ max $\{\|f_i\| : \; i = 0, 1, 2, \cdots, 2n-1\}$ we have $M_{z^{2n}\varphi_1}$ is compact on $\mathcal{B}$ and by the Fredholm alternative theorem, $z^{2n}\varphi_1 = 0$ so $\varphi_1 = 0$. Now $(SM_z + M_zS) = M_{\frac{\varphi_2}{z}}$ on $\mathcal{B}_1$ and it is a compact operator. Hence $M_{\frac{\varphi_2}{z}}(M_z|_{\mathcal{B}_0}) = M_{\varphi_2}$ is compact on $\mathcal{B}_0$. By a similar argument, we have $\varphi_2 = 0$, and repeating this method we conclude that $\varphi_1 = \varphi_2 = ... = \varphi_{2n-1} = 0$. Hence by Lemma 2.2, $S(f) = \varphi_0\tilde{f}$ for each $f \in \mathcal{B}$.

The converse is obvious.

COROLLARY 2.4. *Let $\mathcal{B}$ be a Banach space of analytic functions and let $S \in \{M_{z^{2n}}\}'$ for some positive odd integer n. Then $S \in \{M_{z^n}\}'$ if and only if $SM_{z^n} - M_{z^n}S$ is compact.*

Proof. Assume that $SM_{z^n} - M_{z^n}S$ is compact. We have $(SM_{z^n} - M_{z^n}S)M_{z^n} = SM_{z^{2n}} - M_{z^n}SM_{z^n} = -M_{z^n}(SM_{z^n} - M_{z^n}S)$. Hence by Theorem 2.3, $SM_{z^n} - M_{z^n}S = \tilde{M}_\phi$ for some function $\phi \in \mathcal{M}(\mathcal{B})$. Now we show that M_ϕ is compact. Let $\{f_n\}$ be a sequence of functions in $\mathcal{B}$ such that $\|f_n\| < 1$ so $\{\tilde{f}_n\}$ is a bounded sequence in $\mathcal{B}$. Since $\tilde{M}_\varphi$ is compact, there is a subsequence $\{f_{n_k}\}$ such that $\tilde{M}_\phi(\tilde{f}_{n_k})$ converges to a function g in $\mathcal{B}$. But $\tilde{M}_\phi(\tilde{f}_{n_k}) = \phi\tilde{\tilde{f}}_{n_k} = \phi f_{n_k}$, so M_ϕ is compact and by the Fredholm alternative theorem $\phi = 0$. This implies that $M_{z^n}S = SM_{z^n}$, and hence $S \in \{M_{z^n}\}'$.

THEOREM 2.5. *Let $\mathcal{B}$ be a Banach space of analytic functions and let $S \in L(\mathcal{B})$ such that $SM_{z^n} = -M_{z^n}S$ for some odd integer $n \geq 1$. If $SM_z - M_zS$ is compact, then $S = 0$.*

Proof. Suppose $n = 1$. We have $(SM_z - M_zS)M_z = -M_z(SM_z - M_zS)$ and hence by Lemma 2.1, $(SM_z - M_zS) = \tilde{M}_\phi$. But $\tilde{M}_\phi$ is compact and so $\phi = 0$. Therefore $SM_z = M_zS$ and $S = 0$. Now let $n \geq 1$, and p be a polynomial having decomposition $p = p_0 + p_1 + \cdots + p_{2n-1}$ such that $p_i \in B_i$ for $0 \leq i \leq 2n-1$. By

Lemma 2.2 we have:

$$\begin{aligned}(SM_z - M_zS)(p) &= S(zp) - zS(p)\\ &= -z\varphi_0\tilde{p} + \varphi_1(p_0 - p_n) + \varphi_2(\frac{p_1 - p_{n+1}}{z}) + \cdots\\ &+ \varphi_{n-1}(\frac{p_{n-2} - p_{2n-2}}{z^{n-2}}) - z(\varphi_0\tilde{p} + \varphi_1(\frac{p_1 - p_{n+1}}{z})\\ &+ \varphi_2(\frac{p_2 - p_{n+2}}{z^2}) + \cdots + \varphi_{n-1}(\frac{p_{n-1} - p_{2n-1}}{z^{n-1}}))\\ &= p_0(-2z\varphi_0 + \varphi_1) + p_1(+2z\varphi_0 + \frac{\varphi_2}{z} - \varphi_1)\\ &+ p_2(-2z\varphi_0 + \frac{\varphi_3}{z^2} - \frac{\varphi_2}{z}) + \cdots\\ &+ p_{2n-1}(+2z\varphi_0 + \frac{\varphi_{n-1}}{z^{n-2}})\end{aligned}$$

By a similar argument as in Theorem 2.3, we have $SM_z - M_zS = 0$ which is a contradiction or directly we obtain

$$\begin{aligned}&-2z\varphi_0 + \frac{\varphi_i}{z^{i-1}} = 0 \qquad \text{for} \quad 1 \le i \le n-1 \text{ and } i \text{ odd},\\ &\frac{\varphi_i}{z^{i-1}} = 0 \;;\quad \text{for} \quad 1 \le i \le n-1 \text{ and } i \text{ even},\\ &2z\varphi_0 + \frac{\varphi_{n-1}}{z^{n-2}} = 0\end{aligned}$$

which implies that each $\varphi_i = 0$ for $1 \le i \le n-1$.

The proofs of Lemma 2.6 and Theorem 2.7 are similar to the proof of Lemma 2.2 and Theorem 2.3. Also see [2] for the certain Hilbert spaces of functions.

LEMMA 2.6. *Let $\mathcal{B}$ be a Banach space of analytic functions and let $S \in \{M_{z^n}\}'$ for some positive integer n. Suppose that $\mathcal{B}_i = \vee\{z^{nk+i} : k \ge 0\}$ for $i = 0, 1, 2, \cdots, n-1$. If $f \in \mathcal{B}$ having decomposition $f = f_0 + f_1 + \cdots + f_{n-1}$, where $f_i \in \mathcal{B}_i$ for $i = 0, 1, \cdots, n-1$, then*

$$S(f) = f\varphi_0 + \frac{f_1}{z}\varphi_1 + \frac{f_2}{z^2}\varphi_2 + \ldots + \frac{f_{n-1}}{z^{n-1}}\varphi_{n-1},$$

where $\varphi_0 = S(1)$ *and* $\varphi_i = (SM_{z^i} - M_{z^i}S)(1)$ *for* $i = 1, 2, ..., n-1$, *and hence*

$$(SM_z - M_zS)(f) = \varphi_1 f_0 + f_1(\frac{\varphi_2}{z} - \varphi_1) + \cdots$$
$$+ f_{n-2}(\frac{\varphi_{n-1}}{z^{n-2}} - \frac{\varphi_{n-2}}{z^{n-3}}) + f_{n-1}(\frac{\varphi_{n-1}}{z^{n-2}}).$$

THEOREM 2.7. *Under the conditions of Lemma 2.6 for* $\mathcal{B}$, S, n, *and* $\mathcal{B}_i$. *There is a function* $\varphi \in \mathcal{M}(\mathcal{B})$ *such that* $S = M_\varphi$ *if and only if* $SM_z - M_zS$ *is a compact operator.*

COROLLARY 2.8. *Let* $\mathcal{B}$ *be a Banach space of analytic functions and let* $S \in \{M_{z^{2n}}\}'$. *Then* $SM_{z^n} = -M_{z^n}S$ *if and only if* $SM_{z^n} + M_{z^n}S$ *is compact.*

Proof. Assume that $SM_{z^n} + M_{z^n}S$ is compact. Since $(SM_{z^n} + M_{z^n}S) \in \{M_{z^n}\}'$ by Theorem 2.7, there is a function $\varphi \in \mathcal{M}(\mathcal{B})$ such that $SM_{z^n} + M_{z^n}S = M_\varphi$. Now since M_φ is compact, we have $\varphi = 0$.

THEOREM 2.9. *Let* $\mathcal{B}$ *be a Banach space of functions and let* $S \in \{M_{z^n}\}'$ *for some positive integer* $n \geq 1$. *If* n *is an odd integer and* $SM_z + M_zS$ *is compact, then* $S = 0$. *If* n *is an even integer and* $SM_z + M_zS$ *is compact, then* $S = \tilde{M}_\phi$ *for some function* $\phi \in \mathcal{M}(\mathcal{B})$.

Proof. Let p be a polynomial having decomposition $p = p_0 + p_1 + \cdots p_{n-1}$ where $p_i \in B_i$ for $i = 0, 1, 2, \cdots, n-1$. By Lemma 2.6 we have:

$$(SM_z + M_zS)(p) = S(zp) + zS(p)$$
$$= zp\varphi_0 + p_0\varphi_1 + \frac{p_1}{z}\varphi_2 + \cdots$$
$$+ \frac{p_{n-2}}{z^{n-2}}\varphi_{n-1} + zp\varphi_0 + p_1\varphi_1$$
$$+ \frac{p_2}{z}\varphi_2 + \cdots + \frac{p_{n-1}}{z^{n-2}}\varphi_{n-1}$$
$$= p_0(\varphi_1 + 2z\varphi_0) + p_1(\frac{\varphi_2}{z} + \varphi_1 + 2z\varphi_0) + \cdots$$
$$+ p_{n-2}(\frac{\varphi_{n-1}}{z^{n-2}} + \frac{\varphi_{n-2}}{z^{n-3}} + 2z\varphi_0) + p_{n-1}(\frac{\varphi_{n-1}}{z^{n-2}} + 2z\varphi_0).$$

Since $SM_z + M_zS$ is compact as in Theorem 2.3 we can see that

$$2z\varphi_0 + \frac{\varphi_i}{z^{i-1}} = 0; \quad \text{for} \quad 1 \le i \le n-1 \quad \text{and} \quad i \quad \text{odd} ,$$
$$\frac{\varphi_i}{z^{i-1}} = 0 ; \quad \text{for} \quad 1 \le i \le n-1 \quad \text{and} \quad i \quad \text{even} ,$$
$$2z\varphi_0 + \frac{\varphi_{n-1}}{z^{n-2}} = 0.$$

Now if n is an odd number $\frac{\varphi_{n-1}}{z^{n-2}} = 0$ and so $2z\varphi_0 = 0$ that implies $\varphi_0 = 0$. Hence each $\varphi_i = 0$ for $i = 1, 2, \cdots, n-1$ and $S = 0$. If n is an even number, then $\varphi_i = -2z^i\varphi_0$, for i odd and $1 \le i \le n-1$ and $\varphi_i = 0$ for i even and $1 \le i \le n-1$. Hence

$$S(f) = f\varphi_0 + \frac{f_1}{z}\varphi_1 + \frac{f_2}{z^2}\varphi_2 + \cdots + \frac{f_{n-1}}{z^{n-1}}\varphi_{n-1} = \varphi_0\tilde{f} = \tilde{M}_{\varphi_0}(f).$$

THEOREM 2.10. *Let $\mathcal{B}$ be a Banach space of analytic functions and let $S \in \{M_{z^{4n}}\}'$ for some positive integer n. There is a function $\varphi \in \mathcal{M}(\mathcal{B})$ such that $S = \hat{M}_\varphi$, if and only if $SM_z - iM_zS$ is compact.*

Proof. We set $m = 4n, S(1) = \varphi_0$ and:

$$\varphi_j = (SM_{z^j} - iM_{z^j}S)(1) \quad \text{for} \quad 1 \le j < m \quad \text{and} \quad j = 4k+1 \quad \text{for some integer} \quad k$$
$$\varphi_j = (SM_{z^j} + M_{z^j}S)(1) \quad \text{for} \quad 1 \le j < m \quad \text{and} \quad j = 4k+2 \quad \text{for some integer} \quad k$$
$$\varphi_j = (SM_{z^j} + iM_{z^j}S)(1) \quad \text{for} \quad 1 \le j < m \quad \text{and} \quad j = 4k+3 \quad \text{for some integer} \quad k$$
$$\varphi_j = (SM_{z^j} - M_{z^j}S)(1) \quad \text{for} \quad 1 \le j < m \quad \text{and} \quad j = 4k \quad \text{for some integer} \quad k.$$

Suppose $B_j = \vee\{z^{mk+j} : k \ge 0\}$ for $j = 0, 1, 2, \cdots, m-1$ and $f \in B$ has a decomposition $f = f_0 + f_1 + \cdots + f_{m-1}$, where $f_j \in B_j$ for $j = 1, 2, \cdots, m-1$. As in Lemma 2.2 we can show that:

$$S(f) = \hat{f}\varphi_0 + \frac{f_1}{z}\varphi_1 + \frac{f_2}{z^2}\varphi_2 + \cdots + \frac{f_{m-1}}{z^{m-1}}\varphi_{m-1}$$

and

$$(SM_z - iM_zS)(f) = \varphi_1 f_0 + f_1(\frac{\varphi_2}{z} - i\varphi_1) + \cdots$$
$$+ f_{m-2}(\frac{\varphi_{m-1}}{z^{m-2}} - i\frac{\varphi_{m-2}}{z^{m-3}}) - if_{m-1}(\frac{\varphi_{m-1}}{z^{m-2}}).$$

Now if $SM_z - iM_zS$ is a compact operator by a similar argument as in Theorem 2.3 we can show that $S = \hat{M}_{\varphi_0}$. The converse is obvious.

THEOREM 2.11. *Suppose $\mathcal{B}$ is a Banach space of analytic functions, $S \in L(\mathcal{B})$ and $SM_{z^{4n}} = -M_{z^{4n}}S$ for some positive number n. There is a function φ in $\mathcal{M}(\mathcal{B})$ such that $S = \hat{M}_\varphi$ if and only if $SM_z + iM_zS$ is compact.*

REMARK. Let $\phi \in \mathcal{M}(\mathcal{B})$ be a one to one map of $\mathbf{D}$ onto $\mathbf{D}$ such that $f \circ \phi$ and $f \circ \phi^{-1}$ are in $\mathcal{B}$ for every $f \in \mathcal{B}$. Let $S \in \{M_{\phi^n}\}'$. We define $T : \mathcal{B} \to \mathcal{B}$ by $T(f) = f \circ \phi^{-1}$. Clearly $T \in L(\mathcal{B})$ with inverse $T^{-1}(f) = f \circ \phi$. Then $M_zT = TM_\phi$ and hence by induction $M_{z^n}T = TM_{\phi^n}$. Since $SM_{\phi^n} = M_{\phi^n}S$, it follows that $ST^{-1}M_{z^n}T = T^{-1}M_{z^n}TS$ and so $TST^{-1} \in \{M_{z^n}\}'$. Now let $SM_\phi + cM_\phi S$ be a compact operator for some constant c=1,-1,i,-i, then $T(SM_\phi + cM_\phi S)T^{-1} = TST^{-1}M_z + cM_zTST^{-1}$ is compact. Hence we can use from these facts, for study of the structure of S. For example let $S \in \{M_{\phi^n}\}'$ and let $SM_\phi - M_\phi S$ be a compact operator. By Theorem 2.7, $TST^{-1} = M_\psi$ for some function $\psi \in \mathcal{M}(\mathcal{B})$, so $S = M_{\psi\circ\phi}$.

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